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EXISTENCE OF STATIONARY EQUILIBRIUM IN AN INCOMPLETE-MARKET MODEL WITH ENDOGENOUS LABOR SUPPLY*

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In this article, I first study an income fluctuation problem with endogenous labor supply. Let β be the agent's time discount factor and $R > 0$ be the constant gross rate of return on assets. For $\beta R = 1$, I show that the agent's wealth either approaches infinity almost surely or converges to a finite level almost surely. For $\beta R < 1$, I prove the existence, uniqueness, and stability of the stationary distribution of state variables. I then show the existence of the stationary general equilibrium in an incomplete-market model with endogenous labor supply.

1. INTRODUCTION

The aim of this article is to show the existence of the stationary general equilibrium in an incomplete-market model with endogenous labor supply. There is a continuum of households with measure 1 in the economy. Households have uninsurable idiosyncratic labor efficiency shocks. Each household faces an income fluctuation problem with endogenous labor supply. The labor efficiency shock follows a Markov chain along time and is independent and identically distributed (i.i.d.) across households.

Aiyagari and McGrattan (1998) use an incomplete-market heterogeneous agents model with endogenous labor supply to study the optimum quantity of government debt. Marcet et al. (2007) show that incomplete insurance to idiosyncratic employment shocks introduce an ex post wealth effect, which reduces labor supply. The ex post wealth effect on labor supply runs counter to the precautionary savings motive.[2] These articles did not show the existence of the stationary general equilibrium. This article fills this gap.

I first study an income fluctuation problem with endogenous labor supply. Let β be the agent's time discount factor and $R > 0$ be the constant gross rate of return on assets. The net rate of return is $r = R - 1$. For $\beta R = 1$, I show that the agent's wealth either approaches infinity almost surely or converges to a finite level almost surely as $t \to \infty$. If wealth converges to a finite level almost surely, then the agent's labor supply approaches zero almost surely as $t \to \infty$. As long as the agent does not stop working, income shocks always exist. Precautionary savings cause wealth accumulation to eventually reach infinity. If the agent stops working due to the income effect, then wealth accumulation stops. Moreover, there exists upper bounds of wealth accumulation. This general result holds as long as both consumption and leisure are normal goods. Therefore, I extend the well-known result of Chamberlain and Wilson (2000) to situations with endogenous labor supply.

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[2] In their numerical examples, the wealth effect often dominates the precautionary saving effects. Thus, the output and savings in incomplete markets are lower than those in complete markets.

I find sufficient conditions guaranteeing that wealth accumulation has upper bounds for cases of $\beta R = 1$ and $\beta R < 1$. I also find that the ratio between marginal utility functions of consumption in different shock states plays an important role in determining precautionary savings. To confine this ratio, we can always obtain a lower bound of the consumption policy function for $\beta R < 1$ in a model with endogenous labor supply. Recent works by Acikgöz (2018) and Stachurski and Toda (2019) find the lower bound of the consumption policy function for $\beta R < 1$ in models with exogenous labor supply. I provide a general framework to investigate income fluctuation problems with exogenous labor supply and with endogenous labor supply. The unified framework brings us more insight into research on incomplete-market models.

I prove the existence, uniqueness, and stability of the stationary distribution of state variables for $\beta R < 1$.[3] Aiyagari (1993), Huggett (1993), and Marcet et al. (2007) employ the monotone-Markov-process method of Hopenhayn and Prescott (1992) to show this. Kamihigashi and Stachurski (2014) extend this method to the unbounded state space. In contrast, I use a new method to show the existence, uniqueness, and stability of the stationary distribution, and I do not need the monotonicity assumption of the Markov chain. The crucial observation is that the lower bound of the state space for $\beta R < 1$ is an accessible atom. Starting from any asset level, the state variables have positive probability to hit the lower bound in finitely many periods. That the borrowing constraint is binding infinitely often in an income fluctuation problem implies that the lower bound of the state space is an accessible atom. The new result highlights the impact of borrowing constraints and precautionary savings on the stationary wealth distribution.

To show the existence of the stationary equilibrium, I find the intersection of the "supply" and "demand" curves for the capital–labor ratio in the economy. The aggregate capital supply is the total wealth of households in the stationary distribution of state variables. The aggregate labor supply is the total labor supply in the stationary distribution. The "supply" curve for the capital–labor ratio is the ratio of the aggregate capital supply to the aggregate labor supply. I show that the "supply" curve is a continuous function of the interest rate r and tends to infinity as r approaches $\bar{r} = \frac{1}{\beta} - 1$ from below. However, the infinity limit could be due to infinite wealth accumulation or zero labor supply. From the firm's profit-maximization problem the "demand" curve for the ratio is derived, which approaches infinity as r tends to $-\delta$. Following Aiyagari (1994), I show the existence of the stationary equilibrium by finding the intersection of these two continuous curves. Simply replacing capital by the capital–labor ratio, I extend the idea of Aiyagari (1994) to models with endogenous labor supply. Thus I provide a general framework to show the existence of the stationary equilibrium in incomplete-market models with exogenous labor supply and with endogenous labor supply.

My existence proof of the stationary equilibrium also provides new insight into income fluctuation problems. If the agent's wealth approaches infinity almost surely as $t \to \infty$ for the case of $\beta R = 1$, then the aggregate capital supply converges to infinity as $r \uparrow \bar{r}$. If the agent's wealth converges to a finite level almost surely as $t \to \infty$ for the case of $\beta R = 1$, then aggregate labor supply approaches zero as $r \uparrow \bar{r}$. These limit results are due to the continuity of optimal policy functions with respect to parameters, including interest rate r.

After I weaken the monotonicity of the Markov chain shocks, these results are more applicable in simulation exercises. My existence proof of the stationary equilibrium also shows that a bisection algorithm can find a stationary general equilibrium. Therefore, my article offers guidance to simulation works on incomplete-market models with endogenous labor supply.

Using the concept of "tightness" of a collection of probability measures, I provide a new probability-limit-theory tool, which extends the frequently used Theorem 12.13 by Stokey and Lucas (1989), to investigate the parametric continuity of stationary distributions. Specifically, I use tightness to replace compactness in the previous famous theorem. Therefore, I relax the assumption and extend the application scope of the theorem. Equipped with this new tool, I investigate how stationary distributions move when model parameters change and find

[3] The stability here means that, starting from any initial distribution of state variables, the stochastic process converges to the unique stationary distribution.

the connection between parametric continuity of stationary distributions and that of optimal policy functions.

1.1. *Related Literature.* Marcet et al. (2007) show that the agent's wealth converges to a finite level almost surely and labor supply approaches zero almost surely for $\beta R = 1$. This article employs a more general utility function form than the work by Marcet et al. (2007), which uses a separable and homogeneous utility function. For the labor efficiency shocks, Marcet et al. (2007) use a two-state Markov chain that should also satisfy the monotonicity assumption. My results can be applied to multiple-state Markov chains and I do not need the Markov chain to be monotone.

Chamberlain and Wilson (2000) study an income fluctuation problem with stochastic interest rates and labor earnings. They show that if the labor earnings and interest rate processes are sufficiently volatile and the long-run average rate of interest is greater than or equal to the time discount rate, then the household's consumption eventually grows without bound with probability 1. Thus, the agent's asset also converges to infinity almost surely. Note that $\beta R = 1$ is a special case of their model. However, Chamberlain and Wilson (2000) do not investigate the general equilibrium. They assume that the interest rate process is exogenous.

Schechtman and Escudero (1977) investigate optimal policy functions of an income fluctuation problem with $\beta R < 1$. They also study the long-run properties of the wealth accumulation process under $\beta R < 1$.[4] They assume that the labor earnings shock is i.i.d. along time and its support is bounded. They show that the wealth accumulation process is bounded if the coefficient of relative risk aversion is bounded as consumption goes to infinity.[5]

Aiyagari (1993) proves the existence of the stationary general equilibrium in an incomplete-market model with inelastic labor supply and i.i.d. labor efficiency shocks. Aiyagari (1994) uses this model to quantitatively show that precautionary savings is not important for aggregate capital accumulation. Huggett (1993) shows the existence, uniqueness, and stability of the stationary distribution of state variables in an incomplete-market model with serially correlated income shocks. Huggett (1993) assumes that the income shock follows a two-state Markov chain and the Markov chain is monotone.[6]

Miao (2002) shows the existence of the stationary general equilibrium in an incomplete-market model with serially correlated income shocks. However, he assumes that the transition function of the income shock is monotone and satisfies a smoothness condition.

Kuhn (2013) introduces permanent earnings shocks into the Aiyagari model. In Kuhn (2013)'s work, households have inelastic labor supply and their labor efficiency shocks have random growth components. They have a utility function of constant relative risk aversion (CRRA). To obtain the stationary general equilibrium, Kuhn (2013) assumes that households have a death rate. Moreover, Kuhn (2013) proves the existence of the stationary general equilibrium in the model. However, Aiyagari (1993), Huggett (1993), Miao (2002), and Kuhn (2013) do not discuss endogenous labor supply.

Acemoglu and Jensen (2015) study comparative statics of a class of heterogeneous agents models, including the Bewley–Aiyagari–Huggett model. They assume that the idiosyncratic shock follows a Markov process with the Feller property. The support of the idiosyncratic shock is compact. They show the existence of the stationary general equilibrium. Acemoglu and Jensen (2015) argue that their proof applies to an Aiyagari model with endogenous labor supply. However, by applying the proof in the present article, one can easily develop a bisection algorithm similar to that presented by Aiyagari (1994), which can then be used to find the stationary general equilibrium.

[4] Rabault (2002) investigates an income fluctuation problem in which the lowest possible level of earnings is zero and the marginal utility of consumption is infinite when consumption is zero.

[5] For research on income fluctuation problems, see also Laitner (1979, 1992), Mendelson and Amihud (1982), Sotomayor (1984), and Clarida (1987, 1990).

[6] One difference between Aiyagari (1994) and Huggett (1993) is that Aiyagari (1994)'s work has aggregate production while the model of Huggett (1993) is an endowment economy.

Acikgöz (2018) shows the existence of the stationary general equilibrium in an incomplete-market model with production.[7] Acikgöz (2018) assumes that the earnings shock follows a multiple-state Markov chain. Furthermore, the utility function could be unbounded. Acikgöz (2018) shows that the wealth accumulation process of an income fluctuation problem with $\beta R < 1$ is bounded if the coefficient of absolute risk aversion tends to zero as consumption goes to infinity. Foss et al. (2018) show the boundedness result for an income fluctuation problem with a multiple-state Markov chain and the CRRA utility function. Following the working paper version of the present paper, Acikgöz (2018) shows the existence, uniqueness, and stability of the stationary distribution of state variables in his model. Different from Acikgöz (2018) and Foss et al. (2018), my model has endogenous labor supply.

The rest of this article is organized as follows. In Section 2, I investigate the household's policy functions under three cases: (i) $\beta R > 1$, (ii) $\beta R = 1$, and (iii) $\beta R < 1$. I characterize the stationary general equilibrium and prove its existence in Section 3. Section 4 concludes the article. Proofs are in the online appendix of Zhu (2019).

2. AN INCOME FLUCTUATION PROBLEM WITH ENDOGENOUS LABOR SUPPLY

There is a continuum of households with measure 1 in the economy. Each household faces an income fluctuation problem with endogenous labor supply. The household has an instantaneous utility function $u(c, h)$ of consumption c and leisure h. The utility function $u(c, h)$ satisfies

ASSUMPTION 1. $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ *is twice continuously differentiable.*

ASSUMPTION 2. *Case (i) $u(c, h)$ is strictly increasing and strictly concave in c and h. $u_{11}u_{22} - u_{21}u_{12} > 0$, $u_{21}u_1 - u_{11}u_2 > 0$, and $u_{12}u_2 - u_{22}u_1 > 0$. In addition,*

$$\lim_{c\to 0} u_1(c, h) = \infty, \forall h \in (0, 1] \text{ and } \lim_{h\to 0} u_2(c, h) = \infty, \forall c > 0.$$

Case (ii) $u(c, h) = U(c)$ is strictly increasing, and strictly concave in c. Furthermore,

$$\lim_{c\to 0} U'(c) = \infty.$$

Case (ii) is not included in Case (i), which requires $u(c, h)$ to be strictly increasing and strictly concave in c and h. However, I develop a unified framework in the present article to investigate income fluctuation problems with exogenous labor supply and with endogenous labor supply.

ASSUMPTION 3. $u(c, h) \in [0, M]$, $M > 0$.

Each household has the preference

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t),$$

where $\beta \in (0, 1)$ is the time discount factor. Each household is endowed with one unit of time. The household faces idiosyncratic labor efficiency shocks. It is assumed that the labor efficiency process $\{e_t\}_{t=0}^{\infty}$ follows a Markov chain with a transition probability $\pi(e'|e)$.

ASSUMPTION 4. $e_t \in E \equiv \{e^1, e^2, \ldots, e^n\}$, *with* $0 < e^1 < e^2 < \cdots < e^n$. ${}_{e'}\pi(e'|e) = 1$ *for all $e \in E$ and $\pi(e'|e) > 0$ for all $e, e' \in E$.*

[7] Acikgöz (2018) also gives an example that has multiple equilibria.

There is only one risk-free asset in the economy. The constant gross rate of return on assets is $R > 0$. The wage rate of the labor efficiency unit is $w > 0$.[8] The household's budget constraint is

$$c_t + a_{t+1} = Ra_t + (1 - h_t)e_t w, \tag{1}$$

where a_t is the household's asset. The household cannot borrow assets from others and thus

$$a_{t+1} \geq 0. \tag{2}$$

The household's state can be described by $(a, e) \in [0, \infty) \times E$.

I study the household's problem in two steps.[9] Step 1 is an intratemporal problem. The household chooses consumption and leisure to maximize the current period's utility with respect to the given expenditure. This step is a static maximization exercise and is irrelevant to the dynamic optimization. I derive the indirect utility function $J(y, ew)$ in the intratemporal problem. Step 2 is an intertemporal problem that determines the optimal expenditure in every period. I use the indirect utility function to transform the original dynamic programming problem with two control variables into a derived dynamic programming problem with only one control variable.[10]

I define the indirect utility function $J(y, q)$ of the intratemporal problem as

$$J(y, q) = \max_{c,h} u(c, h)$$

$$s.t.\ c + hq = y,\ \ h \in [0, 1],$$

where y is the expenditure on consumption c and leisure h, and q is the price of leisure.

The first-order condition of the intratemporal problem is

$$\begin{array}{ccc} \frac{u_2(c,h)}{u_1(c,h)} < q & \Rightarrow & h = 0 \\ \frac{u_2(c,h)}{u_1(c,h)} > q & \Rightarrow & h = 1 \\ h \in (0, 1) & \Rightarrow & \frac{u_2(c,h)}{u_1(c,h)} = q \end{array}. \tag{3}$$

$c^s(y, q)$ and $h^s(y, q)$ are used to denote optimal solutions of the intratemporal problem (the static problem).

PROPOSITION 1. *Under Assumptions 1–4, we have*
(1) $J(y, q)$ is bounded.
(2) $J(y, q)$ is strictly increasing and strictly concave in y.
(3) $c^s(y, q)$ and $h^s(y, q)$ are continuous and increasing in y.
(4) $J(y, q)$ is continuously differentiable in y. $J_1(y, q) = u_1[c^s(y, q), h^s(y, q)]$ for all $y \in (0, \infty)$.

From part (3) of Proposition 1, it is known that the demand for consumption c and leisure h increases with an increase in income. Thus, both consumption and leisure are normal goods.

[8] I assume that $w > 0$ for two reasons. First, we have $w > 0$ in the stationary equilibrium (see comments after Theorem 8). Second, I need the bound $V_1(a, e) < V_1(0, e) < \infty$ for $a > 0$ and all $e \in E$, as in Proof of Lemma 5. Moreover, the case of $w = 0$ apparently implies that $V_1(0, e) = \infty$.

[9] I thank Charles Wilson for the suggestions of this two-step procedure.

[10] Foley and Hellwig (1975) employ this two-step procedure to study the income fluctuation problem with endogenous labor supply. The difference between their model and my model is that the asset in their model is money. Therefore, the return to the asset is zero in their model. In my model, I study the general case of the return to asset. The interest rate in my model could be positive, zero, or negative.

After I solve the utility maximization problem within the period, the original dynamic utility maximization problem becomes

$$\max_{\{y_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t J(y_t, e_t w)$$

$$s.t. y_t + a_{t+1} = Ra_t + e_t w, \;\; y_t \geq 0.$$

I study the household's problem by using the standard dynamic programming technique. Let $V(a, e)$ be the optimal value function of the household's intertemporal problem. The Bellman equation of the household's problem is

$$V(a, e) = \max_{a' \in \Gamma(a,e)} \left\{ J(Ra + ew - a', ew) + \beta E[V(a', e')|e] \right\},$$

where

$$\Gamma(a, e) = \{a' : 0 \leq a' \leq Ra + ew\}.$$

Let $a'(a, e)$ be the optimal asset for the next period and $y(a, e)$ be the optimal total expenditure for the current period. Proposition 2 characterizes the value function $V(a, e)$ and the policy functions $a'(a, e)$ and $y(a, e)$.

PROPOSITION 2. *Under Assumptions 1–4, we have*
(1) $V(a, e)$ is continuous, strictly increasing, and strictly concave in a.
(2) $V(a, e)$ is continuously differentiable in a, and $V_1(a, e) = RJ_1[y(a, e), ew]$ for all $a \in [0, \infty)$.[11]
(3) $a'(a, e)$ is continuous and increasing in a.
(4) $y(a, e)$ is continuous and strictly increasing in a.

Equation (A.2) in Appendix A of the Supporting Information provides us with the Euler equation of the intertemporal problem,

$$V_1(a, e) \geq \beta RE[V_1(a', e')|e], \text{ with equality if } a' > 0. \tag{4}$$

We define

$$c(a, e) = c^s[y(a, e), ew],$$

and

$$h(a, e) = h^s[y(a, e), ew].$$

Thus, $c(a, e)$, $h(a, e)$, and $a'(a, e)$ are the policy functions of the original dynamic utility maximization problem. From Assumption 2, we know that $c(0, e) > 0$ and $h(0, e) < 1$ for all $e \in E$. Thus, $y(a, e) \geq c(a, e) \geq c(0, e) > 0$ for all $a \geq 0$. Therefore, we have $y > 0$.

To show the existence of the stationary equilibrium, stronger conditions are needed.

[11] $V_1(0, e)$ is used to represent $V_1^+(0, e)$.

ASSUMPTION 5. *Case (A) $u(c,h)$ satisfies*

$$u_{12}(c,h) \geq 0 \text{ and } \exists c > 0 \text{ such that } u_2(c,1) > 0,$$

Case (B) $u(c,h)$ satisfies

$$\limsup_{c\to\infty} \Psi(c,\Delta) \leq 1, \forall \Delta \geq 0,$$

where

$$\Psi(c,\Delta) = \max_{h,h' \in [0,1]} \left\{ \frac{u_1(c,h')}{u_1(c+\Delta,h)} \right\}.$$

Case (A) of Assumption 5 implies that consumption and leisure are complementary. Case (B) of Assumption 5 extends Assumption 3 posited by Acikgöz (2018), $\liminf_{c\to\infty} \frac{U''(c)}{U'(c)} = 0$, to models with endogenous labor supply. If Case (ii) of Assumption 2 holds, then Case (B) of Assumption 5 reduces to $\limsup_{c\to\infty} \frac{U'(c)}{U'(c+\Delta)} \leq 1$ for all $\Delta \geq 0$, which is equivalent to

$$\lim_{c\to\infty} \frac{U'(c+\Delta)}{U'(c)} = 1, \forall \Delta \geq 0, \tag{5}$$

since $\frac{U'(c)}{U'(c+\Delta)} \geq 1$. Thus, if labor supply is exogenous, Case (B) of my Assumption 5 is exactly the same as Acikgöz (2018)'s Assumption 3.[12]

PROPOSITION 3. *Under Assumptions 1–4, we have*
(1) $c(a,e)$ and $h(a,e)$ are continuous and increasing in a. For $e \in E$,

$$\lim_{a\to\infty} h(a,e) = \bar{h}(e),$$

and

$$\lim_{a\to\infty} c(a,e) = \infty.$$

(2) Case (A) of Assumption 5 implies that $h(a,e) = 1$ for sufficiently large a and all $e \in E$. Thus, $\bar{h}(e) = 1$ for all $e \in E$.

We define

$$\bar{k} = \begin{cases} \infty, & \text{if } \Sigma \text{ is empty} \\ \inf \Sigma, & \text{if } \Sigma \text{ is not empty} \end{cases}, \tag{6}$$

where $\Sigma = \{a \geq 0 : h(a,e) = 1, \forall e \in E\}$. From Assumption 2, we know that $c(0,e) > 0$ and $h(0,e) < 1$ for all $e \in E$. Thus, $\bar{k} > 0$.[13] Through part (1) of Proposition 3, we know that $h(a,e)$ is a continuous function of a. Case (A) of Assumption 5 implies that $\bar{k} < \infty$. Thus, $h(a,e) = 1$ for $a \geq \bar{k}$ and all $e \in E$.

[12] Given Lemma D.1 in Appendix D of the work by Acikgöz (2018), we know that Assumption 3 of the same work implies Equation (5). On the other hand, from the Taylor expansion, we have, for $\Delta > 0$, $U'(c+\Delta) = U'(c) + U''(c+\tilde{\Delta})\Delta$, where $\tilde{\Delta} \in [0,\Delta]$. Thus, Equation (5) implies that $\lim_{c\to\infty} \frac{U''(c+\tilde{\Delta})}{U'(c+\tilde{\Delta})} = 0$. Consequently, we have $\liminf_{c\to\infty} \frac{U''(c)}{U'(c)} = \lim_{c\to\infty} \frac{U''(c)}{U'(c)} = 0$.

[13] Note that $\bar{k}$ depends on w and r since $h(a,e)$ depends on w and r.

Case (B) of Assumption 5 seems strong. However, this case includes income fluctuation problems with exogenous labor supply and those with endogenous labor supply, and highlights the connection between these two situations. Case (ii) of Assumption 2, which permits exogenous labor supply, only satisfies Case (B) of Assumption 5.

2.1. *The Case of $\beta R > 1$.* The dynamics of the agent's asset for $\beta R > 1$ is first investigated. The time subscript for each variable in the Euler equation (4) is expressed explicitly:

$$V_1(a_t, e_t) \geq \beta R E[V_1(a_{t+1}, e_{t+1})|e_t], \text{ with equality if } a_{t+1} > 0.$$

Assumption 2 implies that consumption is strictly positive. Thus, $V_1(a_t, e_t) = Ru_1(c_t, h_t)$ by part (4) of Proposition 1 and part (2) of Proposition 2. This is the "envelope" condition of the household's problem.

THEOREM 1. *If $\beta R > 1$ and Assumptions 1–4 hold, then $\lim_{t\to\infty} a_t = \infty$ almost surely.*

The agent's asset grows without bound as long as the interest rate exceeds the time discount rate. Theorem 1 does not depend on Assumption 5. The proof of Theorem 1 uses the Supermartingale Convergence Theorem, which is widely used in studies of the income fluctuation problem, such as those by Schechtman (1976), Mendelson and Amihud (1982), Sotomayor (1984), and Chamberlain and Wilson (2000).[14] The asset accumulation approaches infinity since the agent is too patient and/or the interest rate is too high.

2.2. *The Case of $\beta R = 1$.* There are two purposes to investigating the properties of a household's policy functions for $\beta R = 1$. First, these properties of optimal policies help us to understand the household's long-run behavior. It is found that the household's wealth either approaches infinity almost surely or converges to a finite level almost surely. Second, these properties give us the limit of the capital–labor ratio in the stationary distribution as $R \uparrow \frac{1}{\beta}$ from below.

Lemma 1 describes the long-run property of the marginal return on assets. Following Chamberlain and Wilson (2000), I use the Supermartingale Convergence Theorem to prove it.

LEMMA 1. *If $\beta R = 1$ and Assumptions 1–4 hold, then $\lim_{t\to\infty} V_1(a_t, e_t)$ exists and is finite almost surely.*

Lemma 1 implies that the process $\{V_1(a_t, e_t)\}_{t=0}^{\infty}$ has a finite limit almost surely. The asset accumulation and labor supply when $V_1(a_t, e_t)$ is given. Thus, $u_1(c_t, h_t)$ is fixed since $V_1(a_t, e_t) = Ru_1(c_t, h_t)$. Let

$$u_1(c, h) = \lambda, \tag{7}$$

where $\lambda \in (0, \infty)$ is a constant. From Equation (3), we know that

$$u_2(c, h) = \lambda e w, \tag{8}$$

if $h \in (0, 1)$.

If Case (i) of Assumption 2 holds, we have $u_{11}u_{22} - u_{21}u_{12} > 0$. Thus, using the Implicit Function Theorem, we know that Equations (7) and (8) imply that there exist functions $\xi(\lambda, e)$ and $v(\lambda, e)$ such that

$$u_1[\kappa(\lambda, e), v(\lambda, e)] = \lambda,$$

[14] For the Supermartingale Convergence Theorem, see (2012, p. 498).

and

$$u_2[\kappa(\lambda, e), v(\lambda, e)] = \lambda e w,$$

For $e \in E$, let

$$\lambda_1(e) = \begin{cases} 0, & \text{if } \Lambda_1(e) \text{ is empty} \\ \sup \Lambda_1(e), & \text{if } \Lambda_1(e) \text{ is not empty} \end{cases},$$

where $\Lambda_1(e) = \{\lambda > 0 : v(\lambda, e) \geq 1\}$. From the Implicit Function Theorem, it is also known that $\frac{\partial v(\lambda,e)}{\partial \lambda} = -\frac{u_{21}u_1 - u_{11}u_2}{u_{11}u_{22} - u_{21}u_{12}} < 0$ for $\lambda \in (0, \infty)$. Therefore, we define

$$h = g(\lambda, e) = \begin{cases} 1, & \lambda \in (0, \lambda_1(e)] \\ v(\lambda, e), & \lambda \in (\lambda_1(e), \infty) \end{cases},$$

and

$$c = \xi(\lambda, e) = \begin{cases} \vartheta^{-1}(\lambda), & \lambda \in (0, \lambda_1(e)] \\ \kappa(\lambda, e), & \lambda \in (\lambda_1(e), \infty) \end{cases},$$

where $\vartheta(c) = u_1(c, 1)$. If Case (ii) of Assumption 2 holds, we define $g(\lambda, e) = 0$ and $\xi(\lambda, e) = (U')^{-1}(\lambda)$ for all $e \in E$. In this case, we let $\lambda_1(e) = 0$ for all $e \in E$. Thus, $\xi(\lambda, e)$ and $g(\lambda, e)$ are the consumption and leisure when the marginal utility of consumption is fixed as λ.

PROPOSITION 4. *If $\beta R = 1$ and Assumptions 1–4 hold, $g(\lambda, e)$ is decreasing in $\lambda \in (0, \infty)$ for $e \in E$.*

We define

$$\bar{\lambda} = \min_{e \in E} \{\lambda_1(e)\}. \tag{9}$$

For $\lambda > \bar{\lambda}$, there exists $e \in E$ such that $g(\lambda, e) < 1$. If $\bar{\lambda} > 0$, then we have

$$g(\lambda, e) = 1, \forall e \in E,$$

for $\lambda \in (0, \bar{\lambda}]$. If Case (ii) of Assumption 2 holds, we have $\bar{\lambda} = 0$. This case corresponds to that of Chamberlain and Wilson (2000) with exogenous labor supply.

Following Chamberlain and Wilson (2000), I investigate the asset accumulation for $\beta R = 1$ in a situation with endogenous labor supply. The crucial tool that I use here is the Implicit Function Theorem. Using this theorem, it is found that consumption and leisure are determined by the marginal utility function. By Lemma 1, the marginal utility function is fixed almost surely, as $t \to \infty$. I thus develop a unified framework to investigate the income fluctuation problems with endogenous labor supply and those with exogenous labor supply.

Iterating the budget constraint (1), we have

$$\frac{a_t}{R^{t-\tau}} = a_\tau - \sum_{j=1}^{t-\tau} [c_{\tau+j-1} - (1 - h_{\tau+j-1}) e_{\tau+j-1} w] R^{-j}, \tag{10}$$

for $\tau \leq t$. We define

$$\chi(\phi, e) = \xi(\phi, e) - [1 - g(\phi, e)] e w$$

as consumption minus labor income when the marginal utility of consumption is fixed as ϕ. We know that E is a finite set.

LEMMA 2. *If $\beta R = 1$ and Assumptions 1–4 hold, then we have*

$$\chi(\phi, e^1) \geq \chi(\phi, e^2) \geq \cdots \geq \chi(\phi, e^n),$$

for $\phi > 0$. Furthermore, $\chi(\phi, e^1) = \chi(\phi, e^n)$ for $\phi \in (0, \bar{\lambda}]$ and $\chi(\phi, e^1) > \chi(\phi, e^n)$ for $\phi \in (\bar{\lambda}, \infty)$.

Lemma 2 implies that $\chi(\phi, e)$ cannot be a constant for all $e \in E$ if $\phi \in (0, \bar{\lambda}]$. Lemma 3 then suggests that $\sum_{j=1}^{\infty} \chi(\phi, e_{t+j-1})\beta^j$ is sufficiently volatile if $\phi \in (0, \bar{\lambda}]$.

LEMMA 3. *If $\beta R = 1$ and Assumptions 1–4 hold, then there exists $\varepsilon_\phi > 0$ for each $\phi > \bar{\lambda}$ such that*

$$\Pr\left(\alpha \leq \sum_{j=1}^{\infty} \chi(\phi, e_{t+j-1})\beta^j \leq \alpha + \varepsilon_\phi \middle| e_t \right) \leq 1 - \varepsilon_\phi,$$

for any $\alpha > 0$ and all $e_t \in E$, $t \geq 0$.

For $\beta R = 1$, we know from Lemma 1 that $V_1(a_t, e_t)$ has a limit as t approaches infinity. Since $V_1(a_t, e_t) = Ru_1(c_t, h_t)$, $u_1(c_t, h_t)$ is close to some number ϕ for $t \geq \tau$, and τ is large enough. If $\phi > \bar{\lambda}$, a_t is bounded since $V_1(a_t, e_t)$ is close to $R\phi$. Thus, letting $t \to \infty$, we have the left-hand side of Equation (10), $\frac{a_t}{R^{t-\tau}}$, close to zero. The second term of the right-hand side of Equation (10) is close to $\sum_{j=1}^{\infty} \chi(\phi, e_{\tau+j-1})\beta^j$. Lemma 3 shows that $\sum_{j=1}^{\infty} \chi(\phi, e_{t+j-1})\beta^j$ is sufficiently volatile. However, a_τ is known in period τ. Thus we have a contradiction. We know that $u_1(c_t, h_t)$ converges to $\phi \leq \bar{\lambda}$ as t approaches infinity. If $\bar{\lambda} = 0$, then we have $\lim_{t\to\infty} V_1(a_t, e_t) = 0$ almost surely. Thus we have $\lim_{t\to\infty} a_t = \infty$ almost surely. If $\bar{\lambda} > 0$, then we have $\lim_{t\to\infty} h_t = 1$ almost surely.

THEOREM 2. *If $\beta R = 1$ and Assumptions 1–4 hold, then we have either*

$$\lim_{t\to\infty} h_t = 1 \ a.s.$$

or

$$\lim_{t\to\infty} a_t = \infty \ a.s.$$

Theorem 2 provides a general framework to investigate income fluctuation problems with exogenous labor supply and with endogenous labor supply for $\beta R = 1$. It does not need Assumption 5.

Chamberlain and Wilson (2000) study an income fluctuation problem with exogenous labor supply and find that wealth accumulation is unbounded for $\beta R = 1$. Case (ii) of Assumption 2 in this article corresponds to their work. I also extend their Lemma 4 to cases with endogenous labor supply.

Marcet et al. (2007) study an income fluctuation problem with endogenous labor supply. They show that the agent's wealth converges to a finite level almost surely and labor supply approaches zero almost surely for $\beta R = 1$. I extend their results to more general situations.

PROPOSITION 5. *If $\beta R = 1$ and Assumptions 1–4 hold, then $\bar{k} < \infty$ implies that* $\lim_{t\to\infty} h_t = 1$ *almost surely. Furthermore,*

$$h(a, e) = 1,\ c(a, e) = ra,\ \text{and}\ a'(a, e) = a,$$

for $a \geq \bar{k}$ and all $e \in E$. If $a_0 \in [0, \bar{k}]$ for process $\{(a_t, e_t)\}_{t=0}^{\infty}$, then $a_t \in [0, \bar{k}]$ for all $t \geq 0$, and $\lim_{t\to\infty} a_t = \bar{k}$ *almost surely. Moreover, we have $\bar{k} < \infty$ in Case (A) of Assumption 5.*

These policy functions imply that wealth accumulation is bounded. Theorem 2 implies that the agent's labor supply $1 - h_t$ approaches zero almost surely. It is also known that the agent's wealth converges to a finite level almost surely in this situation. However, the limit depends on the initial state. From Proposition 5, $\bar{k} < \infty$ implies that $a_t = a_0$ for all $t \geq 0$, if $a_0 \geq \bar{k}$. Households with starting asset $a_0 \geq \bar{k}$, keep their starting asset a_0 for $t \geq 0$, and do not supply labor.

PROPOSITION 6. *If $\beta R = 1$ and Assumptions 1–4 hold, then $\bar{k} = \infty$ if and only if*

$$\lim_{t\to\infty} a_t = \infty\ a.s.$$

Proposition 6 investigates situations in which agents always face income shocks before their wealth converges to infinity. Since labor supply is exogenous in Chamberlain and Wilson (2000), we know that $\bar{k} = \infty$ and wealth accumulation accumulation approaches infinity almost surely.

Proposition 6 implies that we have either

$$\Pr(\lim_{t\to\infty} a_t = \infty) = 1,$$

or

$$\Pr\left(\{(a_t, e_t)\}_{t=0}^{\infty} \text{ is bounded}\right) = 1.$$

From Theorem 2, we know that $\lim_{t\to\infty} h_t = 1$ almost surely in the second case.

For $\beta R = 1$, $\lim_{t\to\infty} a_t = \bar{k}$ almost surely, if $a_0 \leq \bar{k}$. The agent accumulates the asset until it reaches $\bar{k}$ if $a_0 \leq \bar{k}$. If $a_0 \geq \bar{k}$, the agent holds the starting asset for all $t \geq 0$. The agents, whose assets are higher than or equal to $\bar{k}$, do not supply labor. They do not suffer from the labor efficiency shock. They hold constant asset levels and have perfectly smooth consumption sequences. Agents whose starting assets are lower than $\bar{k}$ first accumulate assets and reach the target level $\bar{k}$. Then they stop working and reach a perfect self-insurance state. The endogenous labor supply opens a door for agents to enter a perfect insurance situation. This gives agents strong incentives for asset accumulation if their assets are lower than the target level $\bar{k}$. Thus the agent's wealth either approaches infinity almost surely or converges to a finite level almost surely for $\beta R = 1$.

2.3. *The Case of $\beta R < 1$.* The household's policy functions for $\beta R < 1$ are characterized first. The properties of these policy functions determine the existence, uniqueness, and stability of the stationary distribution of state variables. Some properties of the stationary distribution for $\beta R < 1$ are then investigated as done by Aiyagari (1994) and Huggett (1993).

LEMMA 4. *If $\beta R < 1$ and Assumptions 1–4 hold, then there exists $e_a \in E$ for each $a > 0$ such that $a'(a, e_a) < a$.*

Lemma 4 implies that, for any asset level, there is a realization of labor efficiency shock such that the household dissaves. Thus, the process $\{a_t\}_{t=0}^{\infty}$ with any initial asset a_0 has a positive probability to be lower than a given level $a_l < a_0$ in finite steps. Let

$$\check{a}(a) = \min_{e \in E}\{a'(a, e)\}.$$

Thus, $\check{a}(a)$ is continuous in a since $a'(a, e)$ is continuous in a by part 3) of Proposition 2. Through Lemma 4, $\check{a}(a) < a$ for all $a > 0$. Let $d = \min\{a - \check{a}(a) : a \in [a_l, a_0]\}$. Thus, $d > 0$. We can pick the realization sequence of labor efficiency shocks $e's$ such that (a, e) moves along $(\check{a}(a), e)$ so that $a_t \in [a_l, a_0]$ decreases by at least d in one step. Thus, starting from a_0, process $\{a_t\}_{t=0}^{\infty}$ reaches levels lower than a_l in at most $[\frac{a_0 - a_l}{d}] + 1$ steps.[15]

Define

$$\hat{a}(a) = \max_{e \in E}\{a'(a, e)\}.$$

Thus, $\hat{a}(a)$ is continuous in a since $a'(a, e)$ is continuous in a by part (3) of Proposition 2.

PROPOSITION 7. *If $\beta R < 1$ and Assumptions 1–4 hold, then $\bar{k} < \infty$ implies that $a'(a, e) < a$ for $a \geq \bar{k}$ and all $e \in E$. Additionally, we have $\bar{k} < \infty$ in Case (A) of Assumption 5.*

From Proposition 7, we know that $\hat{a}(a) < a$ for $a \geq \bar{k}$. If consumption and leisure are complementary and there exists $c > 0$ such that $u_2(c, 1) > 0$, we have $\bar{k} < \infty$ from Proposition 7. Agents with sufficiently high levels of assets do not supply labor supply and reduce assets due to impatience.

EXAMPLE 1. Suppose that

$$u(c, h) = \frac{c^{1-\gamma}}{1-\gamma} e^{-\theta c} + v(h),$$

where $\gamma > 1$ and $\theta > 0$. Furthermore, it is assumed that $v(h)$ is twice continuously differentiable on [0,1], $v'(h) > 0$, $v''(h) < 0$, and $\lim_{h \to 0} v'(h) = \infty$. Apparently, Example 1 satisfies Assumptions 1 and 2. This utility function also satisfies Assumption 3 since it has an upper bound and, by the argument in Footnote 6 of Appendix B in the Supporting Information, we know that optimal consumption has a lower bound that is strictly positive. Example 1 satisfies Case (A) of Assumption 5. However, it does not satisfy Case (B) of Assumption 5 since $\limsup_{c \to \infty} \Psi(c, \Delta) = e^{\theta\Delta} > 1$, for all $\Delta > 0$.

PROPOSITION 8. *If $\beta R < 1$, Assumptions 1–4 and Case (B) of Assumption 5 hold, and then we have*

$$c(a, e) \geq ra, \quad \forall e \in E,$$

for sufficiently large a. Furthermore, there exists $k^b > 0$ such that

$$a'(a, e) < a, \quad \forall e \in E,$$

for $a \geq k^b$.

Rabault (2002), Benhabib et al. (2015), Acikgöz (2018), and Stachurski and Toda (2019) find a similar lower bound of consumption policy functions in models with exogenous labor supply.

[15] Here $[x]$ denotes the largest integer less than or equal to x.

Here I extend this result to a model with endogenous labor supply. The proof of Proposition 8 is in Appendix B of the Supporting Information.

Propostion 8 provides us with a lower bound of the consumption policy function for sufficiently large a. If $r \leq 0$, this lower bound is trivial since $c(a, e) > 0$ for all $a \geq 0$ and $e \in E$. If $r > 0$, this lower bound of the consumption policy function implies that consumption must be more than capital income for agents with high levels of wealth.

Both Cases (A) and (B) of Assumption 5 imply that the agent with sufficiently large wealth dissaves. Thus, process $\{(a_t, e_t)\}_{t=0}^{\infty}$ is contractionary. Although Case (B) is an extension of Assumption 3 in Acikgöz (2018), which concentrates on situations of exogenous labor supply, Case (A) is independent of it. Example 1 satisfies Case (A), but does not satisfy Case (B).

Proposition 8 applies to situations with exogenous labor supply such as works by Aiyagari (1994), Huggett (1993), and Acikgöz (2018). The following example illustrates the connection between income fluctuation problems with endogenous labor supply and those with exogenous labor supply.

EXAMPLE 2. Suppose that

$$u(c, h) = \frac{c^{1-\gamma}}{1-\gamma} + \chi \frac{[c + \mathcal{J}(h)]^{1-\sigma}}{1-\sigma},$$

where $\chi \geq 0$ and $\gamma > \sigma > 1$. Furthermore, it is assumed that $\mathcal{J}(h)$ is twice continuously differentiable on [0,1], $\mathcal{J}(0) = 0$, $\mathcal{J}'(h) > 0$, $\mathcal{J}''(h) < 0$, and $\lim_{h \to 0} \mathcal{J}'(h) = \infty$. This utility function satisfies Assumptions 1–3 as in Example 1. Example 2 satisfies Case (B) of Assumption 5. If $\chi = 0$, we have $u_2(c, h) = 0$ for all $c \geq 0$ and $h \in [0, 1]$. If $\chi > 0$ we have $u_{12}(c, h) = -\sigma\chi[c + \mathcal{J}(h)]^{-\sigma-1}\mathcal{J}'(h) < 0$. Thus, Example 2 does not satisfy Case (A) of Assumption 5. However, we know from part 1) of Proposition 3 that $\lim_{a \to \infty} c(a, e) = \infty$ for all $e \in E$. Furthermore,

$$\lim_{c \to \infty} \frac{u_2(c, h)}{u_1(c, h)} = \lim_{c \to \infty} \frac{\chi}{\left[1 + \frac{\mathcal{J}(h)}{c}\right]^{\sigma} c^{\sigma-\gamma} + \chi} \mathcal{J}'(h) = \mathcal{J}'(h).$$

If $\mathcal{J}'(1) > e^n w$, then we have $\mathcal{J}'(1) > ew$ for all $e \in E$. Therefore, there exists $a < \infty$ such that $h(a, e) = 1$ for all $e \in E$. We know that $\bar{k} < \infty$. Consequently, both Propositions 7 and 8 apply to these situations. If $\mathcal{J}'(1) \leq e^n w$, then we have

$$\frac{u_2(c, 1)}{u_1(c, 1)} = \frac{\chi}{\left[1 + \frac{\mathcal{J}(1)}{c}\right]^{\sigma} c^{\sigma-\gamma} + \chi} \mathcal{J}'(1) < \mathcal{J}'(1) \leq e^n w, \quad \forall c > 0.$$

Thus, we know that $h(a, e^n) < 1$ for all $a > 0$. Therefore, we have $\bar{k} = \infty$. Proposition 7 does not apply to these situations, but Proposition 8 does.

THEOREM 3. *If $\beta R < 1$ and Assumptions 1–5 hold, then there exists $k^b > 0$ such that $a_t \leq \max\{k^b, a_0\}$ for all $t \geq 0$. Define integer*

$$I = \begin{cases} 0, & \text{if } a_0 \leq k^b \\ \left[\frac{a_0 - k^b}{\theta}\right] + 1, & \text{if } a_0 > k^b \end{cases},$$

where $\theta = \min\{a - \hat{a}(a) : a \in [k^b, a_0]\} > 0$. We have

$$\Pr\left(a_t \leq k^b, \forall t \geq I\right) = 1.$$

Theorem 3 shows that asset accumulation has an upper bound in our model for $\beta R < 1$. In the theorem, two cases corresponding to Cases (A) and (B) of Assumption 5, respectively, are discussed. For $\beta R < 1$, I find that these are two separate sufficient conditions that guarantee the existence of an upper bound for asset accumulation.

If Case (A) of Assumption 5 holds, an agent with a sufficiently high level of assets does not supply labor and then acts as in the deterministic situation. In the deterministic situation, $\beta R < 1$ implies that the agent dissaves. Thus I do not need the restriction on the coefficient of relative risk aversion. The complementarity between consumption and leisure can guarantee the existence of the upper bound. In this case, we can pick $k^b = \bar{k}$. Marcet et al. (2007) find this upper bound for $\beta R < 1$ if the utility function is separable and homogeneous. I extend the result to models with more general utility functions.

For the income fluctuation problem with exogenous labor supply, researchers such as Schechtman and Escudero (1977), Clarida (1987), and Aiyagari (1994) use the bounded coefficient of relative risk aversion to guarantee the upper bound for asset accumulation.[16] Acikgöz (2018) uses the coefficient of absolute risk aversion to extend this condition. The difference between my model and these early works is that labor supply is endogenous in my model. Assumption 5 provides sufficient conditions guaranteeing the existence of the upper bound for the asset accumulation if labor supply is endogenous. Specially, I confine the ratio between marginal utility functions of consumption in different shock states in Case (B) of Assumption 5 and, thus, extend conditions in Schechtman and Escudero (1977) and Acikgöz (2018) to models with endogenous labor supply. Case (B) of my Assumption 5 reveals that the ratio between marginal utility functions of consumption in different shock states plays an important role in determining precautionary savings. This intuition can apply to general situations in which marginal utility functions of consumption suffer from shocks.

Theorem 3 also shows that a_t becomes lower than k^b after I periods if the initial asset level $a_0 > k^b$. Since $\hat{a}(a)$ represents the highest asset accumulation for asset level a and $\hat{a}(a) < a$ if $a \geq k^b$, a_t decreases monotonically whenever it is higher than k^b. As a function of a, $\hat{a}(a)$, has a fixed point in the interval $[0, k^b)$, since $\hat{a}(0) \geq 0$ and $\hat{a}(k^b) < k^b$. Thus, the set $\{a \in [0, k^b) : \hat{a}(a) = a\}$ is not empty. Define

$$\bar{a} = \inf\left\{a \in [0, k^b) : \hat{a}(a) = a\right\}. \tag{11}$$

Thus, we have[17]

$$\hat{a}(a) > a, \ \text{if } a \in [0, \bar{a}),$$

and

$$\hat{a}(\bar{a}) = \bar{a}.$$

Both $\hat{a}(a)$ and $\bar{a}$ depend on the wage rate w and the rate of return on assets r since $a'(a, e)$ depends on w and r. Let $S = [0, \bar{a}] \times E$.

PROPOSITION 9. *If $\beta R < 1$ and Assumptions 1–5 hold, then we have*

$$\Pr\left((a_t, e_t) \in S, \quad \forall t \geq 0 \middle| (a_0, e_0) \in S\right) = 1, \tag{12}$$

[16] Schecktman and Escudero (1977) also investigate a counterexample in which the coefficient of relative risk aversion is unbounded as consumption goes to infinity. In this example, they find that the wealth accumulation approaches infinity almost surely even under $\beta R < 1$.

[17] If $\bar{a} = 0$, $[0, \bar{a})$ is empty.

and

$$\Pr\left(\exists T \geq 1,\ \textit{such that}\ (a_T, e_T) \in S \middle| (a_0, e_0) \notin S\right) = 1. \tag{13}$$

Proposition 9 implies that, starting from $s_0 = (a_0, e_0)$ in S, the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ stays in S. If the process starts outside S, it almost surely arrives in S. Using the Markov property of the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ and combining Equations (12) and (13) in Proposition 9, we know that

$$\Pr\left(\exists T \geq 0,\ \text{such that}\ (a_t, e_t) \in S, \forall t \geq T\right) = 1.$$

LEMMA 5. *If $\beta R < 1$ and Assumptions 1–4 hold, then there exists $\tilde{a} > 0$ and $\tilde{e} \in E$ such that $a'(a, \tilde{e}) = 0$ for $a \in [0, \tilde{a}]$.*

From Assumption 2, we know that $c(0, e) > 0$ and $h(0, e) < 1$ for all $e \in E$. We have

$$V_1(a, e) \leq V_1(0, e) = Ru_1(c(0, e), h(0, e)).$$

Thus, $V_1(a, e)$ is bounded. Iterating the Euler equation (4) forward, it is found that it cannot always hold with equality if $\beta R < 1$. Thus, for any (a_0, e_0), the borrowing constraint (2) is binding at some $t \geq 0$. Using the Markov property of process $\{(a_t, e_t)\}_{t=0}^{\infty}$ we know that the borrowing constraint (2) is binding infinitely often. Lemmas 4 and 5 imply that the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ with any initial asset a_0 has a positive probability to hit the lower bound of assets $a = 0$ in finite steps. State $s^* = (0, \tilde{e})$ plays a crucial role in showing the existence, uniqueness, and stability of the stationary distribution of process $\{(a_t, e_t)\}_{t=0}^{\infty}$.

Let $\mathbf{B}(S)$ be the Borel $\sigma-$algebra on S. I define the transition function $P(\cdot, \cdot)$ of the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ on S as

$$P((a, e), A \times \{e'\}) = \begin{cases} \pi(e'|e) & \text{if } a'(a, e) \in A \\ 0 & \text{otherwise} \end{cases},$$

for all $(a, e) \in S$, $A \times \{e'\} \in \mathbf{B}(S)$.

THEOREM 4. *If $\beta R < 1$ and Assumptions 1–5 hold, then the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ is uniformly ergodic. Precisely, $\{(a_t, e_t)\}_{t=0}^{\infty}$ has a unique stationary distribution μ on S. Moreover, there exists $\rho \in (0, 1)$ such that*[18]

$$||P^n(s, \cdot) - \mu|| \leq 2\rho^n,\ \forall s \in S, \tag{14}$$

where $|| \cdot ||$ is the total variation norm.

Theorem 4 implies that process $\{(a_t, e_t)\}_{t=0}^{\infty}$ has a unique stationary distribution. The borrowing constraint and precautionary savings cause the lower bound of asset $a = 0$ to work as a reflecting barrier for the process. The balance of the lower reflecting barrier and the contraction property of $\{(a_t, e_t)\}_{t=0}^{\infty}$ helps us to obtain the stationary distribution of the wealth accumulation process.

In the literature, researchers usually use the monotone-Markov-process method to prove the existence and uniqueness of the stationary distribution. Aiyagari (1993) proves the result for i.i.d. shocks. Huggett (1993) proves the result for two-state Markov chain shocks. For multiple-state Markov chain shocks, the monotonicity condition is very restrictive. Furthermore, it is

[18] For all $s \in S$ and $B \in \mathbf{B}(S)$, $P^n(s, B)$ denotes the probability that the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ at state s enters set $B \in \mathbf{B}(S)$ in n steps.

difficult to verify the monotonicity condition for the joint Markov chain of the state variables, including exogenous shocks and endogenous variables. The method used here does not require the monotonicity of the Markov chain of the state variables. The crucial observation is that the lower bound of the state space is an accessible atom. Starting from any asset level, the state variables have positive probability to hit the lower bound in finite periods. This follows from the boundedness of the marginal utility of asset, $V_1(a, e)$, and $\beta R < 1$. Assumption 5 guarantees that asset accumulation has upper bounds for $\beta R < 1$. However, Assumption 5 only provides two sufficient conditions. There may be other conditions guaranteeing that the state space is compact.

For any problem with a compact state space and bounded $V_1(a, e)$, Theorem 4 should go through for finite-state Markov shocks. The proof of Theorem 4 provides a new method to show the existence and uniqueness of the stationary distribution for the Markov dynamic systems. This method can apply to more general situations.

Furthermore, Equation (14) implies that, starting from any initial distribution, the stochastic process converges to the unique stationary distribution. It is also known that it converges to the stationary distribution geometrically fast under the total variation norm. The rate of convergence is ρ.

After I weaken the monotonicity of the Markov chain shocks, results are more applicable in simulation exercises. In simulations, researchers use the Tauchen method, first introduced by Tauchen (1986), to approximate autoregressive processes estimated from real economic data. The Tauchen method chooses values for the state variables and the transition probabilities so that the resulting finite-state Markov chain closely mimics an underlying autoregression. A multiple-state Markov chain surely fits the autoregressive process better than a two-state Markov chain.

PROPOSITION 10. *If $\beta R < 1$ and Assumptions 1–5 hold, then the Law of Large Numbers holds for any $\mathbf{B}(S)$−measurable function f satisfying $\int_S |f| d\mu < \infty$, that is*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} f(a_i, e_i) = \int_S f d\mu, \ \mathbf{P}_\mu - a.s.$$

Since S is compact, any continuous function is integrable with respect to the probability measure. Proposition 10 implies that the Law of Large Numbers holds for all of the moments of the asset distribution. Thus, in order to compute the mean wealth in the stationary economy, we do not have to simulate the asset accumulation processes for many households to find the approximate cross-section stationary distribution and then to compute the mean wealth. We can simulate an asset accumulation process for a long enough period and then compute the sample path mean of the asset to approximate the cross-section mean in the stationary distribution.[19] Proposition 10 shows that the path mean converges to the cross-section mean almost surely.

PROPOSITION 11. *If $\beta R < 1$ and Assumptions 1–5 hold, then we have $\mu(\{(a, e) : a = 0\}) > 0$, that is, the lower bound of the asset space 0 is a mass point in the stationary distribution.*

State $s^* = (0, \tilde{e})$ is an accessible atom. The borrowing constraint (2) is binding infinitely often. Whenever the borrowing constraint (2) is binding, there is a positive probability such that the process reaches s^* in the next period. For any (a_0, e_0), we have

$$\Pr((a_t, e_t) \text{ visits } (0, \tilde{e}) \text{ infinitely often}) = 1.$$

[19] This may raise the question of how long is long enough? In practice, we can solve this problem by the following procedure: (a) set an arbitrary large number, say 3000, for the simulation period, and after simulating for these periods, (b) set an arbitrary small convergence criterion to test whether the path average of the simulated data for two consecutive periods is smaller than the convergence criterion. If the test result is true, the simulation can stop.

3. THE GENERAL EQUILIBRIUM

There is a continuum of households with measure 1 in the economy. Each household faces an income fluctuation problem with endogenous labor supply, as in Section 2. There is uncertainty at the individual household level but there is no aggregate uncertainty in the economy.[20] There is a single firm in the economy.

3.1. *The Firm's Problem.* The single firm rents capital K and hires labor L from competitive markets. It has an aggregate production function $F(K, L)$ satisfying

ASSUMPTION 6. *F displays constant returns to scale, with $F_1, F_2 > 0$, and $F_{11}, F_{22} < 0$. F satisfies Inada conditions* $\lim_{K\to\infty} F_1(K, 1) = 0$ *and* $\lim_{K\to 0} F_1(K, 1) = \infty$.

The firm maximizes its profits in each period. Its profit-maximization problem is

$$\max_{K,L} \{F(K, L) - (r + \delta)K - wL\},$$

where $\delta \in (0, 1)$ is the depreciation rate of capital, $r = R - 1$ is the net rate of return on capital, and w is the wage rate of the labor efficiency unit. The first-order conditions of the firm's profit-maximization problem are

$$F_1(K, L) = r + \delta, \tag{15}$$

and

$$F_2(K, L) = w. \tag{16}$$

Through the property of constant returns to scale, Equation (15) implies that

$$F_1\left(\frac{K}{L}, 1\right) = r + \delta, \tag{17}$$

and Equation (16) implies that

$$F_2\left(1, \frac{L}{K}\right) = w. \tag{18}$$

From Equation (17), "demand" for the capital–labor ratio is derived,

$$D(r) = \frac{K}{L}(r).$$

Following Assumption 6, $D(r)$ represents a negative relationship between capital–labor ratio $\frac{K}{L}$ and interest rate r. It is also known that

$$\lim_{r\downarrow -\delta} D(r) = \infty,$$

and

$$\lim_{\frac{K}{L}\downarrow 0} r = \infty.$$

[20] To facilitate the Law of Large Numbers in the economy with a continuum of households with measure 1, I use the construction proposed by Sun (2006).

Furthermore, Equations (17) and (18) determine a function between wage rate w and interest rate r, which is denoted as $w(r)$. By Assumption 6, $w(r)$ is continuous and strictly decreasing in r.

3.2. *Definition of the Stationary Equilibrium.* Let $X = [0, \infty) \times E$. Let φ be a probability measure defined on measurable space $(X, \mathbf{B}(X))$, where $\mathbf{B}(X)$ is the Borel σ−algebra on X. From the household problem in Section 2, we can find policy functions $c(s)$, $h(s)$, and $a'(s)$ for all $s \in X$. Thus it is easy to extend the definition of the transition function $P(\cdot, \cdot)$ to all $s \in X$, and $B \in \mathbf{B}(X)$.[22] Let $K(\varphi)$ and $L(\varphi)$ denote the aggregate capital and labor as functions of the distribution φ. I then introduce the definition of the stationary recursive competitive equilibrium for the economy.

DEFINITION 1. A stationary recursive competitive equilibrium with incomplete markets is a list of functions $(c(s), h(s), a'(s), \varphi, K, L)$ and a pair of prices (r, w) such that:
(1) $c(s), h(s), a'(s)$ are optimal decision rules given (r, w).
(2) (r, w) satisfy the firm's profit-maximization conditions.
(3) Market clearing conditions are satisfied:
(i) $\int_X a'(s)d\varphi = K(\varphi)$,
(ii) $\int_X e[1 - h(s)]d\varphi = L(\varphi)$.
(4) φ is a stationary distribution under the transition function $P(\cdot, \cdot)$ implied by the household's decision rules. Formally, φ satisfies

$$\varphi(B) = \int_X P(s, B)d\varphi, \forall B \in \mathbf{B}(X).$$

For $\beta R > 1$, the household's assets converge to infinity by Theorem 1. Therefore, there is no general equilibrium. For $\beta R = 1$, the household's labor supply converges to zero by Theorem 2. Therefore, there is no general equilibrium. Thus, we must have $\beta R < 1$ in a general equilibrium. We obtain Theorem 5, which extends Proposition 3 of Marcet et al. (2007) to more general situations.

THEOREM 5. *In a stationary equilibrium under incomplete markets,* $\beta R < 1$.

The interest rate has to be smaller than the time discount rate in the stationary general equilibrium under incomplete markets. In the general equilibrium under complete markets, $\beta R = 1$. Thus, the capital–labor ratio is higher under incomplete markets than under complete markets. As in the work by Marcet et al. (2007), precautionary savings cause a high capital–labor ratio in the stationary general equilibrium under incomplete markets.

By Theorem 4, for $\beta R < 1$, the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ has a unique stationary distribution μ defined on $(S, \mathbf{B}(S))$, where $S = [0, \bar{a}] \times E$ and $\mathbf{B}(S)$ is the Borel σ−algebra on S. From Proposition 9, we know that, starting from $s_0 = (a_0, e_0)$ outside S, the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ eventually arrives at S almost surely, and stays there. Thus, the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ has a unique stationary distribution on X by extending measure μ on S. The unique stationary distribution on X is constructed by combining the stationary distribution μ on S and zero measure on $(\bar{a}, \infty) \times E$. Thus, the extension of the measure does not influence any integral with respect to the stationary distribution μ.

From the firm's profit-maximization conditions in Section 3.1, wage rate w is a function of interest rate r, and is denoted as $w(r)$. Thus, to search for a stationary equilibrium, we only

[22] The transition function $P(\cdot, \cdot)$ of the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ on X is defined as

$$P((a, e), A \times \{e'\}) = \begin{cases} \pi(e'|e) & \text{if } a(a, e) \in A \\ 0 & \text{otherwise} \end{cases},$$

for all $(a, e) \in X, A \times \{e'\} \in \mathbf{B}(X)$.

need to find the equilibrium interest rate r^*. Let $\bar{r} = \frac{1}{\beta} - 1$. From Theorem 4, we know that the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ in the household's problem has a unique stationary distribution for each $r \in (-1, \bar{r})$. The stationary distribution can be expressed as $\mu(r)$ to emphasize its dependence on the interest rate r. We define the aggregate capital supply with respect to $\mu(r)$ as

$$A(r) = \int_S a d\mu(r).$$

Similarly, we express the policy function $h(a, e)$ as $h(s; r)$. Then we can define the aggregate labor supply,

$$L(r) = \int_S e[1 - h(s; r)] d\mu(r).$$

For each $r \in (-1, \bar{r})$, $a = 0$ is a mass point in the stationary distribution $\mu(r)$, by Proposition 11. From Assumption 2, we know that $c(0, e) > 0$ and $h(0, e) < 1$ for all $e \in E$. Therefore, we have $L(r) > 0$ for all $r \in (-1, \bar{r})$. [22] Thus "supply" of the capital–labor ratio is

$$\zeta(r) = \frac{A(r)}{L(r)}. \tag{19}$$

3.3. *The Continuity of $A(r)$ and $L(r)$.* Following Laitner (1992), Aiyagari (1994), and Acikgöz (2018), I will show that $\zeta(r)$ is a continuous function of $r \in (-1, \bar{r})$. I will prove that both $A(r)$ and $L(r)$ are continuous functions of $r \in (-1, \bar{r})$. To show that, I will find a common bounded support for sequence $\{\mu(r_m)\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} r_m = r_0 \in (-1, \bar{r})$.

Lemma 6 is a general result for real functions. I thank Prof. Jushan Bai for providing the proof of this lemma to me.

LEMMA 6. *$\{f_n\}_{n=1}^{\infty}$ is a sequence of functions on $[b, d] \subseteq$. Assume that $f_n(x)$ is weakly increasing in x,*

$$\lim_{n\to\infty} f_n(x) = f(x), \ \forall x \in [b, d],$$

and $f(x)$ is a continuous function of $x \in [b, d]$. Then f_n converges uniformly to f.

There is also parametric continuity of optimal policy functions.

PROPOSITION 12. *If Assumptions 1–4 and 6 hold, then $c(s; r)$, $h(s; r)$, and $a(s; r)$ are continuous in s and r.*

We describe an important property of collection $\{\mu(r) : r \in (-1, \bar{r})\}$.

LEMMA 7. *If Assumptions 1–6 hold, then we can find $\varepsilon > 0$ for each $r_0 \in (-1, \bar{r})$, such that $\{\mu(r) : r \in (r_0 - \varepsilon, r_0 + \varepsilon)\}$ has a common bounded support.*

Although I cannot show that there exists a uniform upper bound of $\bar{a}(r)$ for all $r \in (-1, \bar{r})$, I can always find a local uniform upper bound $k^M(r_0)$ for asset accumulation within a neighborhood of $r_0 \in (-1, \bar{r})$. A common bounded support of $\{\mu(r) : r \in (r_0 - \varepsilon, r_0 + \varepsilon)\}$ helps show that $A(r)$ and $L(r)$ are continuous at $r_0 \in (-1, \bar{r})$. The $\mu(r)$ is extended from $[0, \bar{a}(r)] \times E$ to $[0, k^M(r_0)] \times E$.

[22] By Assumption 4, we have $0 < e^1 < e^2 < \cdots < e^n$.

However, the extension of the measure does not influence any integral with respect to the stationary distribution because $\mu((\bar{a}(r), k^M(r_0)] \times E) = 0$ for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$.

THEOREM 6. *If Assumptions 1–6 hold, then we have* $\lim_{m\to\infty} A(r_m) = A(r_0)$ *and* $\lim_{m\to\infty} L(r_m) = L(r_0)$ *for sequence* $\{r_m\}_{m=1}^{\infty}$ *such that* $\lim_{m\to\infty} r_m = r_0 \in (-1, \bar{r})$.

Theorem 6 implies that aggregate labor supply $L(r)$ moves continuously with respect to r. At the same time, aggregate capital supply $A(r)$ is a continuous function of r by the definition of the weak convergence of measures on a common bounded support. Theorem 6 also reveals the connection between parametric continuity of stationary distributions and that of optimal policy functions.

3.4. *Existence of the Stationary Equilibrium.* We also want to investigate how aggregate labor supply $L(r)$ and aggregate capital supply $A(r)$ move when r approaches $\bar{r} = \frac{1}{\beta} - 1$ from below. To achieve this, we need more tools of probability limit theories.

Let X be any subset of $\mathbb{R}^l$. $\mathbf{B}(X)$ is the Borel $\sigma-$algebra on X. Let $\Lambda(X, \mathbf{B}(X))$ be the collection of probability measures on measurable space $(X, \mathbf{B}(X))$. A subset Ω of $\Lambda(X, \mathbf{B}(X))$ is called tight if, for any $\varepsilon > 0$, there exists a compact set $C \subset X$ such that $\sup_{\lambda\in\Omega} \lambda(X\backslash C) \le \varepsilon$.[23] Let $\Theta \subset \mathbb{R}^m$ be the space of parameters. For each $\theta \in \Theta$, let $P_\theta(\cdot,\cdot)$ be a transition function on $(X, \mathbf{B}(X))$.

THEOREM 7. *Assume that*

a) $X \subset \mathbb{R}^l$ *and* $\Theta \subset \mathbb{R}^m$*;*

b) if $\{(x_n, \theta_n)\}_{n=1}^{\infty}$ *is a sequence in* $X \times \Theta$ *converging to* (x_0, θ_0)*, then the sequence* $\{P_{\theta_n}(x_n, \cdot)\}_{n=1}^{\infty}$ *in* $\Lambda(X, \mathbf{B}(X))$ *converges weakly to* $P_{\theta_0}(x_0, \cdot)$*;*

c) for each $n \ge 1$, $\mu_n \in \Lambda(X, \mathbf{B}(X))$ *is a fixed point of* $P_{\theta_n}(\cdot,\cdot)$*; and*

d) $\{\mu_n\}_{n=1}^{\infty}$ *is tight.*

If $\{\theta_n\}$ *is a sequence in* Θ *converging to* θ_0*, then there exists a subsequence* $\{\theta_{n_i}\}_{i=1}^{\infty}$ *such that* $\{\mu_{n_i}\}_{i=1}^{\infty}$ *converges weakly to* $\hat{\mu} \in \Lambda(X, \mathbf{B}(X))$*, and* $\hat{\mu}$ *is a fixed point of* $P_{\theta_0}(\cdot,\cdot)$*.*

Theorem 7 mainly extends Theorem 12.13 posited by Stokey and Lucas (1989). Its proof is in Appendix C of the Supporting Information.

The concept of "tightness" of a collection of probability measures plays a crucial role in Theorem 7. Specifically, I use tightness to replace compactness in the famous theorem by Stokey and Lucas (1989). Tightness uniformly confines the tail of a collection of probability measures, but does not require the support of the distribution to be bounded. Therefore, I relax the assumption and extend the application scope of the theorem. Although I obtain a result weaker than that by Stokey and Lucas (1989) after relaxing the assumption, it turns out that this is a convenient tool for us to find whether mass of the probability distribution is "escaping to infinity" as its parameters change. This new result is used to investigate the dependence of stationary distributions on model parameters and highlight the connection between parametric continuity of stationary distributions and that of optimal policy functions.

Using Theorem 7 and parametric continuity of optimal policy functions, the limit of $\zeta(r)$ as $r \uparrow \bar{r}$ is investigated.

PROPOSITION 13. *If Assumptions 1–6 hold, then* $\zeta(r)$ *is a continuous function of* $r \in (-1, \bar{r})$*. Additionally,*

$$\lim_{r\uparrow\bar{r}} \zeta(r) = \infty.$$

[23] $X\backslash C = \{x \in X : x \notin C\}$.

From Proposition 6, we know that the agent's wealth either approaches infinity almost surely or converges to a finite level almost surely as $t \to \infty$ for the case of $\beta R = 1$. In proof of Proposition 13, these two situations are investigated. If the agent's wealth approaches infinity almost surely as $t \to \infty$ for the case of $\beta R = 1$, then aggregate capital supply converges to infinity as $r \uparrow \bar{r}$. This situation includes models with exogenous labor supply, such as those by Laitner (1992), Aiyagri (1994) and Acikgöz (2018). If the agent's wealth converges to a finite level almost surely as $t \to \infty$ for the case of $\beta R = 1$, then aggregate labor supply approaches zero as $r \uparrow \bar{r}$. Marcet et al. (2007) show a special case of this situation. These limit results are due to the continuity of optimal policy functions with respect to parameters including interest rate r. In either case, we know that $\lim_{r \uparrow \bar{r}} \zeta(r) = \infty$.

Proof of Proposition 13 reveals the connection between the case of $\beta R = 1$ and the limits of $A(r)$ and $L(r)$ as $r \uparrow \bar{r}$. If the agent's wealth approaches infinity almost surely as $t \to \infty$ for the case of $\beta R = 1$, suppose that $A(r)$ does not diverge as $r \uparrow \bar{r}$, then there exists a sequence $\{r_m\}_{m=1}^{\infty}$ such that $r_m \uparrow \bar{r}$ and $\{A(r_m)\}_{m=1}^{\infty}$ is bounded. Consequently, we know that $\{\mu(r_m)\}_{m=1}^{\infty}$ is tight. Thus, we can apply Theorem 7 to claim that there exists a stationary distribution for the case of $\beta R = 1$. Then we have a contradiction.

If the agent's wealth converges to a finite level almost surely as $t \to \infty$ for the case of $\beta R = 1$, then we know from Proposition 6 that there exists $\bar{k}(\bar{r}) < \infty$ such that $h(a, e) = 1$ for $a \geq \bar{k}(\bar{r})$ and $e \in E$. From the parametric continuity of optimal policy functions we know that there exists $\varepsilon > 0$ such that collection $\{\mu(r) : r \in (\bar{r} - \varepsilon, \bar{r})\}$ has a common bounded support. Therefore, the collection of stationary distributions is tight. Using Theorem 7 we know that there exists a stationary distribution for the case of $\beta R = 1$, and there exists a sequence $\{r_m\}_{m=1}^{\infty}$ such that $r_m \uparrow \bar{r}$ and $\{\mu(r_m)\}_{m=1}^{\infty}$ converges weakly to this stationary distribution. Since we have the capital–labor ratio equal to ∞ in any stationary distribution for the case of $\beta R = 1$, we know that $\frac{A(r)}{L(r)}$ approaches infinity as $r \uparrow \bar{r}$.

The "demand" curve for the capital–labor ratio $D(r)$, defined in Section 3.1, approaches the horizontal line of $r = -\delta$, as $\frac{K}{L}$ tends to infinity. The "supply" and "demand" curves for the capital–labor ratio are combined to determine the equilibrium interest rate r^* and the equilibrium capital–labor ratio $\frac{K}{L}$:

$$D(r^*) = \zeta(r^*).$$

THEOREM 8. *If Assumptions 1–6 hold, then there exists a stationary equilibrium.*

From proof of Theorem 8 in Appendix A of the Supporting Information, we know that $r < \frac{1}{\beta} - 1$ and $\frac{K}{L} > 0$ in the stationary equilibrium. Thus, we have the equilibrium wage rate $w > 0$.

Assuming that labor supply is exogenous and the earnings shock is i.i.d., Aiyagari (1994) shows the existence of the stationary equilibrium. Following this, I show the existence of the stationary equilibrium by finding the intersection of two continuous curves.[24] The horizontal axis of Figure IIb in the paper by Aiyagari (1994) is K. I extend this idea to models with endogenous labor supply. Figure 1 here is obtained by simply replacing the aforementioned K by $\frac{K}{L}$ on the horizontal axis of Figure IIb. The "supply" and "demand" curves for the capital–labor ratio are displayed in Figure 1.

I do not include Greenwood–Hercowitz–Huffman (GHH) preferences in the present article directly since they could violate $\lim_{c \to 0} u_1(c, h) = \infty$ in Assumption 2 easily. However, using a change-of-variables method (see Section 4.3 of Acikgöz, 2018), we can also investigate GHH preferences as an application of the case with exogenous labor supply.

This article illustrates that the "supply" curve of the capital–labor ratio is continuous. It is still not clear whether this curve is monotone. Thus, the uniqueness of the equilibrium is ambiguous.

[24] Following Aiyagari (1994), Acikgöz (2018) shows the existence of the stationary equilibrium for a model in which labor supply is exogenous and the earnings shock follows a multiple-state Markov chain.

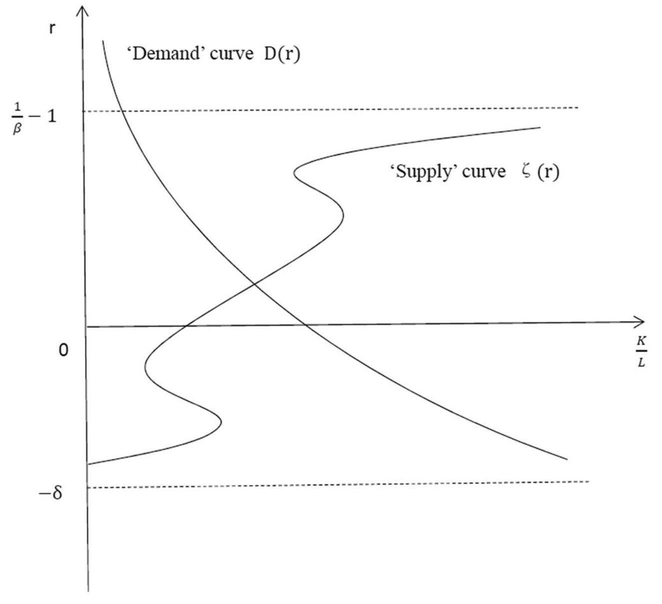

FIGURE 1

"SUPPLY" AND "DEMAND" CURVES FOR THE CAPITAL–LABOR RATIO

The uniqueness is an important starting point to study the movement of equilibrium variables, such as aggregate production and consumption, when the fundamentals of the economy, such as preferences and technology, change. In the future, we could find conditions that determine the uniqueness of the equilibrium.[25]

In this article, I assume that the interest rate is deterministic. It is shown that the asset accumulation is bounded in the general equilibrium. Stachurski and Toda (2019) show an impossibility theorem that concludes that the wealth distribution in Bewley models with the deterministic interest rate inherits the tail behavior of income shocks. Setting a Bewley model with stochastic investment returns, Benhabib et al. (2015) show that the stationary wealth distribution has unbounded support and displays a fat tail. Since Proposition 8 of the present article applies to more general situations in which marginal utility functions suffer from shocks, it seems that it is difficult to generate a fat tail of the wealth distribution as soon as $\beta R < 1$ and the ratio between marginal utility functions of consumption in different shock states is confined. However, stochastic investment returns or stochastic time discount factors could generate large variations of this ratio, and, thus generate a fat tail of the wealth distribution, which is close to the pattern in real data. In the future, we could also investigate an income fluctuation problem with endogenous labor supply and stochastic investment returns.

3.5. *An Algorithm for Finding the Stationary Equilibrium.* My existence proof of the stationary equilibrium has a specific advantage. Aiyagari (1994), using the proof of the existence of the stationary equilibrium, develops a bisection algorithm to find the equilibrium interest rate r^*. Similarly, we can use a bisection algorithm to find a stationary equilibrium.

Step (1): Guess an initial r_1, which should be larger than $-\delta$ and be close to $-\delta$, and an initial r_2, which should be smaller than $\frac{1}{\beta}-1$ and be close to $\frac{1}{\beta}-1$.

[25] Light (2019) finds a group of sufficient conditions guaranteeing the uniqueness of the general equilibrium in a Bewley–Aiyagari model with exogenous labor supply.

Step (2): Set

$$r_3 = \frac{r_1 + r_2}{2}.$$

Step (3): If $D(r_3) - \zeta(r_3) > 0$, let $r_1 = r_3$. Otherwise, let $r_2 = r_3$.

Step (4): If $r_2 - r_1 < \varepsilon$, stop the algorithm and let $r^* = r_3$. Otherwise, go back to step (2).

I provide a general framework to show the existence of the stationary equilibrium in incomplete-market models wtih exogenous labor supply and with endogenous labor supply. Thus, the algorithm here also applies to incomplete-market models with exogenous labor supply and with endogenous labor supply. I extend the algorithm used by Aiyagari (1994) to models with endogenous labor supply. Therefore, my article offers guidance to simulation works on incomplete-market models with endogenous labor supply.

4. CONCLUSION

This article first investigates an income fluctuation problem with endogenous labor supply. For $\beta R = 1$, I show that the agent's wealth either approaches infinity almost surely or converges to a finite level almost surely. If wealth converges to a finite level almost surely, then the agent's labor supply approaches zero almost surely. I provide a general framework to investigate income fluctuation problems with exogenous labor supply and with endogenous labor supply. I also find sufficient conditions guaranteeing that wealth accumulation has upper bounds for cases of $\beta R = 1$ and $\beta R < 1$.

I use a new method to show the existence, uniqueness, and stability of the stationary distribution, and I do not need the monotonicity assumption of the Markov chain. The crucial observation is that the lower bound of the state space for $\beta R < 1$ is an accessible atom. Starting from any asset level, the state variables have positive probability to hit the lower bound in finite periods. That the borrowing constraint is binding infinitely often in an income fluctuation problem implies that the lower bound of the state space for is an accessible atom.

To show the existence of the stationary equilibrium, I find the intersection of the "supply" and "demand" curves for the capital–labor ratio in the economy. The "supply" curve for the capital–labor ratio is the ratio of the aggregate capital supply to the aggregate labor supply. I show that the "supply" curve is a continuous function of the interest rate r and tends to infinity as r approaches $\frac{1}{\beta} - 1$ from below. From the firm's profit-maximization problem the "demand" curve for the ratio is derived, which approaches infinity as r tends to $-\delta$. Following the work by Aiyagari (1994), I show the existence of the stationary equilibrium by finding the intersection of these two continuous curves. My existence proof of the stationary equilibrium also shows that a bisection algorithm can find a general equilibrium. Therefore, my article offers guidance to simulation works on incomplete-market models with endogenous labor supply.

Supporting Information

Additional supporting information may be found online in the Supporting Information section at the end of the article.

Supplimentary Material
Data S1

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