Online Payment Habit Distribution: A Continuous-Time Approach

By SHENGHAO ZHU AND ZIYUAN WANG*

We set up a heterogeneous agent model to describe users' use of online payments. We obtain the coupled HJB and KFE equations, and use the upwind algorithm (a finite difference method) to calculate online payment times and the distribution of users' online payment habits in different market environments. The optimal charging policy which can maximize the social welfare is also obtained.

I. Introduction

With the progress of science and technology, e-commerce, live streaming and other online consumption forms have become a part of people's lives. Even for offline consumption, most consumers no longer carry cash, but complete transactions through online payment methods such as Alipay and Wechat Pay. These fintech companies charge users a fee to balance the cost of doing business or to cover bank fees. This paper constructs a continuous time version of the heterogeneous agent model with two idiosyncratic risks in payment habit and market environment to solve the optimal online payment frequency and the optimal charging method to maximize the social welfare.

In recent decades, the inclusion of explicit heterogeneity in macroeconomic models has been a key development in macroeconomics research. With the increasing availability of high-quality micro data and the emergence of more powerful computing methods, these heterogeneous agent models have proliferated in number and are now ubiquitous.

Heterogeneity model is highly regarded by economists for many reasons. First, it opens the door to introduce micro data to the table in order to empirically discipline macro theories with micro data. Second, macroeconomists usually want to analyze the impact of a particular shock or policy on welfare, and who is the loser and who is the beneficiary, so distribution considerations are usually not negligible. Third, heterogeneous models often provide significantly different aggregate implications than do representative agent models.

Despite the continuously increasing popularity of macroeconomic models with rich heterogeneity, there is a lack of theoretical and analytical results on heterogeneity models in the literature. Instead, most studies explain the implications of these theories through purely numerical analysis. But even such computational approaches are extremely difficult to solve the models with transition dynamics or with non-differential or non-convex properties.

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When recasting in continuous time, heterogeneous agent models boil down to systems of two coupled partial differential equations. The first equation is a Hamilton-Jacobi-Bellman (HJB) equation for the optimal choices of a invidual who takes the distribution. And the second is a Kolmogorov Forward equation (KFE or Fokker-Planck equation) characterizing the evolution of the distribution, given optimal choices of individuals.

In this paper, we use the method in Achdou et al. (2022), combined with users' online payment habits and market environment, and propose a problem to solve the selection of online payment times in continuous time based on heterogeneous agents model. The HJB equation represents an individual's optimal online payment usage when given a random process of payment habits and market environment. The KFE equation represents the evolution of the joint distribution of payment habits and duration environments. And we use them to solve the optimal online payment frequency and the charging method to maximize the social welfare.

The paper is organized as follows. In Section 2 contains the literature review. In Section 3, we establish a heterogeneous agent model to describe the use of online payment with idiosyncratic risks in payment habit and market environment. Section 4 demonstrates the general stationary equilibrium. Section 5 show the welfare of social and the fintech company and the optimal charging method. Section 6 illustrate our computational algorithm. Section 7 concludes the paper. The derivation of formulas and the algorithm is in appendix.

II. Literature Review

After the studies of Bewley (1986), Hopenhayn (1992), Huggett (1993) and Aiyagari (1994), many economists use the incomplete market model with heterogeneous agents for research. These incomplete market models are often called Bewley models. Achdou (2022) recast the Aiyagari-Bewley-Huggett model of income and wealth distribution in continuous time and develop a simple, efficient, and portable numerical algorithm for computing a wide class of heterogeneous agent models. The algorithm is based on a finite difference method and applies to the computation of both stationary and time-varying equilibria.

III. Model Set

A. Household

Time is continuous, indexed by $t \in [0,\infty)$. There is a continuum of agents with measure 1 in the economy. Online payment habits h indicate users' willingness to use online payment methods. The more willing users are to use online payment methods, the greater their online payment habits will be. When market environment x are bad, such as when the reputation of the fintech company is damaged, users may be less likely to choose online payment methods. Agents have standard preferences over utility flows from online payments habit h_t, usage times of online payment n_t and market environment x_t

discounted at rate $\rho \geq 0$:

$$E_0 \int_0^\infty e^{-\rho t} u(h_t, x_t, n_t) dt \tag{1}$$

$$u(h_t, x_t, n_t) = x(t) \frac{h(t)^{1-\gamma}}{1-\gamma} - C \frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \tag{2}$$

where C is the constant using cost by users. And the online payment habit evolves accroding to:

$$dh(t) = (n(t) - T(n(t)))dt \tag{3}$$

where $n(t)$ is the online payment times per unit time, $T(n(t)) = kn(t) + ph(t)$ is the fee charged by the fintech company according to agent's online panyment times and habit. k and p is determined by fintech company according to agents' payment habits. We let $x(t)$ represent the market environment. Because the market environment is constantly changing and cannot be predicted accurately, we let $x(t)$ evolve stochastically over time on a bounded interval $[\underline{x}, \overline{x}]$ with $\overline{x} > \underline{x} > 0$, according to the stationary diffusion process:

$$dx(t) = (a - \eta x(t))dt + \sigma dB(t) \tag{4}$$

where $B(t)$ is a standard Brownian motion.Equation 4 is a continuous-time analogue of a Markov process. It is also an Ornstein-Uhlenbeck (OU) process and its mean is $\frac{a}{\eta}$, its rate of mean reversion θ is η, its volatility is σ, so its variance is $\frac{\sigma^2}{2\theta}$.

The agent's problem:

$$\max_{\{n(t)\}_{t=0}^\infty} E_0 \int_0^\infty e^{-\rho t} \left(x(t) \frac{h(t)^{1-\gamma}}{1-\gamma} - C \frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) dt \tag{5}$$

where $\rho > 0$ is the time dicount factor, E_0 is the expectation operator conditional on the information set at time 0.
Let $V(h(t), x(t))$ be the value function of the household's problem:

$$\begin{aligned} V(x(t), h(t)) &= \max_{\{n(t)\}_{t=0}^\infty} E_0 \int_0^\infty e^{-\rho t} \left(x(t) \frac{h(t)^{1-\gamma}}{1-\gamma} - C \frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) dt \\ dh(t) &= [n(t) - T(n(t))]\, dt \\ dx(t) &= (a - \eta x(t))dt + \sigma dB(t) \end{aligned} \tag{6}$$

where E_t is the expectation operator conditional on the information set at time t, C is the using cost by household, and $x(t)$ represent the market environment. We will estimate the haibt parameters of C, γ, ε and market environment parameters of η and σ. From equation 6, we can derive the HJB equation:

$$\rho V(h(t),x(t)) = \max_{\{n(t)\}_{t=0}^{\infty}} \begin{pmatrix} x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \\ +(a-\eta x)V_x(h(t),x(t)) \\ +\frac{1}{2}\sigma^2 V_{xx}(h(t),x(t)) \\ +(n(t)-T(n(t)))V_h(h(t),x(t)) \end{pmatrix} \tag{7}$$

We use the finite difference (FD) method to solve this dynamic problem, and we have the frequency of using online payments in figure 1. It will increase as habits rise. But the environment have little influence on frequency of online payments.

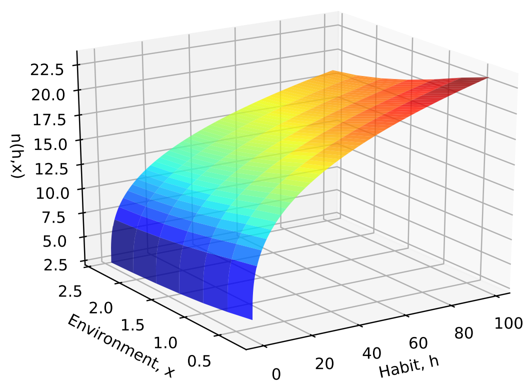

FIGURE 1. THE POLICY FUNCTION OF ONLINE PAYMENT USAGE

B. Household's decision

From Equation 6, we can also get the evolution of the houshold's payment habit $\{h(t)\}_{t=0}^{\infty}$ through the FD method. The result is shown in Figure 2 and the value function is shown in Figure 3. From Figure 2, we find that when individuals are not used to online payment or rarely use it, their habits will increase. However, when individuals enjoying using online payment, their online payment habits will decrease, and the greater their habits are, the rapidly they decrease. This may be caused by the fees charged by fintech company according to individuals' online payment habits. Meanwhile, from Figure 3, we find that the value function has been increased when individuals start to use online payment. And with the increase of online payment habits, the increase of value function is not significant. Different colored lines in the both figures represent different market

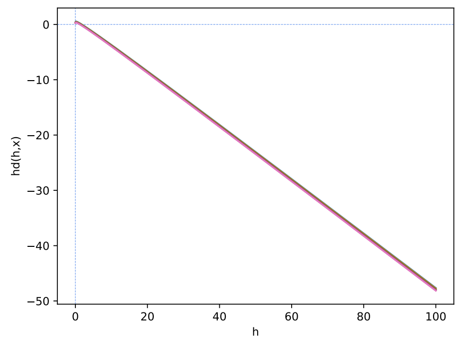

FIGURE 2. DERIVATIVE OF PAYMENT HABIT

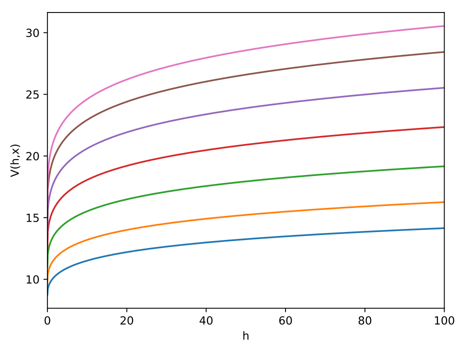

FIGURE 3. VALUE FUNCTION

environment.

IV. Stationary distribution of online payment

From Equation 6, we can derive the Kolmogorov foward (Fokker-Planck) equation (KFE) in Equation 8:

$$0 = -\frac{\partial}{\partial h}\mu_1 f(h,x,t) - \frac{\partial}{\partial x}\mu_2 f(h,x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 f(h,x,t)) \tag{8}$$

where $\mu_1 = n(t) - T(n(t))$ and $\mu_2 = a - \eta x(t)$. The partial differential Equation 8 characterizes the evolution of the payment habits and market environment, we get the distribution of payments in the economy from it, which is shown in Figure 4

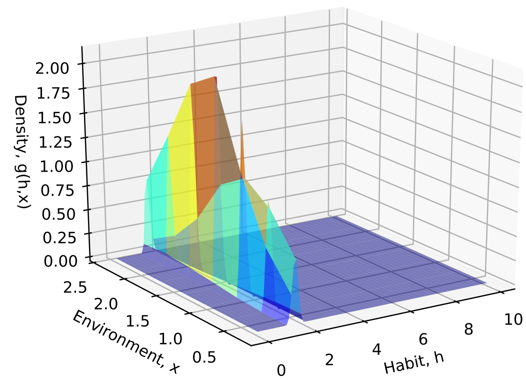

FIGURE 4. ONLINE PAYMENT HABIT DISTRIBUTION

V. Optimal Charge Method

A. *Social Welfare Maximization*

According to the value function and distribution function obtained before, we can calculate social welfare.

$$\begin{aligned} &\max \int_{\underline{x}}^{\overline{x}} \int_0^{\infty} V(h) f(h) dh dx \\ &s.t.\ D = \int_0^{\infty} T(n) f(n) dn \end{aligned} \tag{9}$$

In order to meet the constraint, we assume that after fintech company collects fees from users, he will return the fees to society, so the market environment will continue to improve, and the 4 becomes:

$$dx(t) = (a - \eta x)dt + \sigma dB(t) + Ddt$$

we use a bisection algorithm on the stationary fee D charged by the fintech company. We begin an iteration with an initial guess D^0, then keep iterating it until D^{n+1} is close enough to D^n. The results are shown in Figure 5. We can observe that when $p = 0.6$ and $k = 0$, social welfare reaches its maximum value. This means that fintech company should not charge users based on the number of times they use online payment, but only based on their usage habits of online payment. When $k > 0$, social welfare decreases continuously with the increase of k and p.

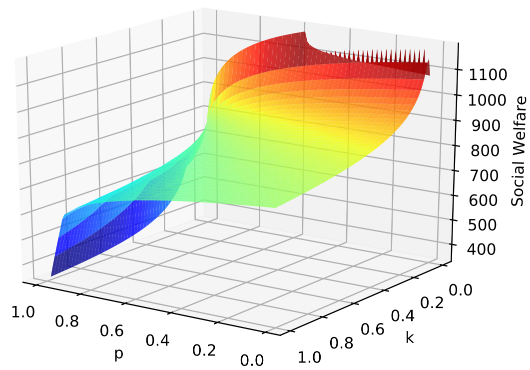

FIGURE 5. SOCIAL WELFARE

B. The Fintech Company Welfare Maximization

We know that the fintech company will charge a fee based to users' online payment times and habits, $T(n) = kn + ph$. So using the distribution function of users' online payment habits and the fee charged by fintech company, we can derive the fintech company's profits.

$$\max_{T(n)} \int_0^\infty T(n) f(n) dn \tag{10}$$

To solve Equation 10, we use the same method as solving the problem of social welfare maximization. The results are shown in Figure 6. We find that when $p = 1$ and $k = 0$, the profit of fintech company reaches the maximum. In addition, the company's welfare decreases with the increase of k, which may be because charging users according to the usage times of them discourages the enthusiasm of users and destroys the habit of using online payment, thus reducing the company's profits.

However, policies that maximize social welfare and firm welfare do not exactly coincide. In this respect, we will further calibrate parameters and update results after we get payment data from Ant Open Research.

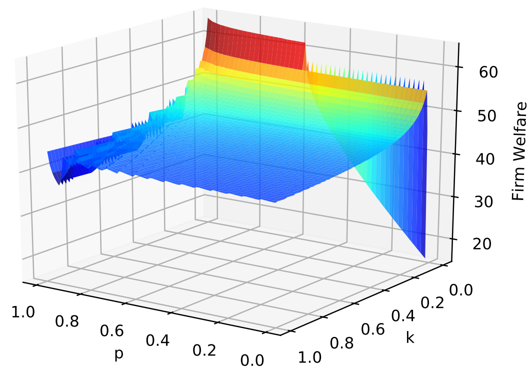

FIGURE 6. FINTECH COMPANY WELFARE

VI. Computation

The economy can be represented by the following system of equations which we aim to solve numerically:

$$\rho V(h,x) = \max_{\{n(t)\}_{t=0}^{\infty}} u(n) + (n - T(n))V_h + (a - \eta x)V_x + \frac{1}{2}\sigma^2 V_{xx} \tag{11}$$

$$0 = -\frac{\partial}{\partial h}(\mu_1(h)f(h,x,t)) - \frac{\partial}{\partial x}(\mu_2(x)f(h,x,t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 f(h,x,t)) \tag{12}$$

$$1 = \int_{\underline{x}}^{\bar{x}} \int_{\underline{h}}^{\infty} f(h,x)dhdx \tag{13}$$

on $(0,\infty)\times(\underline{x},\bar{x})$.

A. HJB Equation

We use a finite difference method to approximate the HJB function $V(h(t),x(t))$ at $I \times J$ discrete points in the space dimension (h_i,x_j), where $i = 1,...,I$, $j = 1,...,J$. We use equispaced grids, denote by Δh the distanse between grid points along h dimension and Δx along x dimension, and use the short-hand notation $V_{i,j} \equiv V(h_i,x_j)$ where i indexes habit and j indexes environment. The derivative in the h dimension is approximated using an upwind method, i.e. using either a forward or a backward difference approximation:

$$\begin{aligned}\partial_{h,B}V_{i,j} &= \frac{V_{i,j}-V_{i-1,j}}{\Delta h}\\ \partial_{h,F}V_{i,j} &= \frac{V_{i+1,j}-V_{i,j}}{\Delta h}\end{aligned} \tag{14}$$

Similarly, we also use an upwind method in the x-direction. For the second-order derivative, we use a central difference approximation. Hence:

$$\begin{aligned}\partial_{x,B}V_{i,j} &= \frac{V_{i,j}-V_{i-1,j}}{\Delta x}\\ \partial_{x,F}V_{i,j} &= \frac{V_{i+1,j}-V_{i,j}}{\Delta x}\\ \partial_{xx}V_{i,j} &= \frac{V_{i,j+1}-2V_{i,j}+V_{i,j-1}}{(\Delta x)^2}\end{aligned} \tag{15}$$

The finite difference approximation to equation (11) is:

$$\rho V_{i,j} = u(n_{i,j}) + (n_{i,j} - T(n_{i,j}))\partial_h V_{i,j} + (a-\eta x_j)\partial_x V_{i,j} + \frac{1}{2}\sigma^2 \partial_{xx} V_{i,j} \tag{16}$$

where $\partial_h V_{i,j}$ and $\partial_x V_{i,j}$ are either the forward or the backward difference approximation. There are two complications. The first question is when to use a forward and when a backward difference approximation. It turns out that this is actually quite important for the stability properties of the scheme. The second is that the HJB equations are highly non-linear, and therefore so is the system of equations (16). It therefore has to be solved using an iterative scheme (rather than simply inverting a matrix).
The approach we choose to update the value function $V^m_{i,j}, m = 1, \ldots$ is the implicit method because it is both more efficient and more reliable:

$$\begin{aligned}\frac{V^{m+1}_{i,j}-V^m_{i,j}}{\Delta} + \rho V^{m+1}_{i,j} =& u(n^m_{i,j}) + (n^m_{i,j} - T(n^m_{i,j}))\partial_h V^{m+1}_{i,j}\\ &+ (a-\eta x_j)\partial_x V^{m+1}_{i,j} + \frac{1}{2}\sigma^2 \partial_{xx} V^{m+1}_{i,j}\end{aligned} \tag{17}$$

where $\dot{h}^m_{i,j} = n^m_{ij} - T(n^m_{i,j}) = (1-k)n^m_{i,j} - ph^m_i$ and $n^m_{i,j} = (u')^{-1}((k-1)\partial_h V^m_{i,j})$. The parameter Δ is the step size of the implicit method. The main advantage of implicit scheme is that the step size Δ can ge arbitrarily large.

UPWIND SCHEME

As already mentioned, it is important whether and when a forward or a backward difference approximation is used. The correct way of doing this is to use a so-called "upwind scheme." The idea is to use a forward difference approximation whenever the drift of the state variable is positive and to use a backwards difference whenever it is negative. In practice, this is done as follows: first compute the derivative of payment habit h, $\dot{h}$, according to both the backwards and forward difference approximations $\partial_{h,B}V_{i,j}$ and $\partial_{h,F}V_{i,j}$

$$\begin{aligned}\dot{h}_{i,j,F} &= (1-k)(u')^{-1}((k-1)\partial_{h,F}V_{i,j}) - ph_i\\ \dot{h}_{i,j,B} &= (1-k)(u')^{-1}((k-1)\partial_{h,B}V_{i,j}) - ph_i\end{aligned}$$

where we suppress m suerscripts for notational simplicity. And we use the following finite difference approximation to equation (11):

$$\begin{aligned}\frac{V_{i,j}^{m+1}-V_{i,j}^{m}}{\Delta}+\rho V_{i,j}^{m+1} =&u(n_{i,j}^{m})+\partial_{h,F}V_{i,j}^{m+1}(\dot{h}_{i,j,F}^{m})^{+}+\partial_{h,B}V_{i,j}^{m+1}(\dot{h}_{i,j,B}^{m})^{-}\\ &+\partial_{x,F}V_{i,j}^{m+1}(a-\eta x_j)^{+}+\partial_{x,B}V_{i,j}^{m+1}(a-\eta x_j)^{-}\\ &+\frac{\sigma^2}{2}\partial_x^2 V_{i,j}^{m+1}\end{aligned} \tag{18}$$

For any number x, the notation x^+ means "the positive part of x", i.e. $x^+ = \max\{x,0\}$ and analogously $x^- = \min\{x,0\}$.

Solution

Equation 18 constitutes a system of $I \times J$ linear equations, and it can be written in matrix notation using the following steps. Substituting the definition of the derivatives Equations 14 and 15, Equation 18 is:

$$\begin{aligned}\frac{V_{i,j}^{m+1}-V_{i,j}^{m}}{\Delta}+\rho V_{i,j}^{m+1} =&u(n_{i,j}^{m})+\frac{V_{i+1,j}^{m+1}-V_{i,j}^{m+1}}{\Delta h}(\dot{h}_{i,j,F}^{m})^{+}+\frac{V_{i,j}^{m+1}-V_{i-1,j}^{m+1}}{\Delta h}(\dot{h}_{i,j,B}^{m})^{-}\\ &+\frac{V_{i,j+1}^{m+1}-V_{i,j}^{m+1}}{\Delta x}(a-\eta x_j)^{+}+\frac{V_{i,j}^{m+1}-V_{i,j-1}^{m+1}}{\Delta h}(a-\eta x_j)^{-}\\ &+\frac{\sigma^2}{2}\frac{V_{i,j+1}^{m+1}-2V_{i,j}^{m+1}+V_{i,j-1}^{m+1}}{(\Delta x)^2}\end{aligned}$$

Collecting terms with the same subscriprs on the right-hand side:

$$
\begin{aligned}
\frac{V_{i,j}^{m+1}-V_{i,j}^{m}}{\Delta}+\rho V_{i,j}^{m+1} =& u(n_{i,j}^{m})+V_{i-1,j}^{m+1}b_{i,j}+V_{i,j}^{m+1}(c_{i,j}+\upsilon_j)+V_{i+1,j}^{m+1}d_{i,j} \\
&+V_{i,j-1}^{m+1}\chi_j+V_{i,j+1}^{m+1}\zeta_j \\
b_{i,j} =& -\frac{(\dot{h}_{i,j,B})^-}{\Delta h} \\
c_{i,j} =& -\frac{(\dot{h}_{i,j,F})^+}{\Delta h}+\frac{(\dot{h}_{i,j,B})^-}{\Delta h} \\
d_{i,j} =& \frac{(\dot{h}_{i,j,F})^+}{\Delta h} \\
\chi_j =& -\frac{(a-\eta x)^-}{\Delta x}+\frac{\sigma^2}{2(\Delta x)^2} \\
\upsilon_j =& -\frac{(a-\eta x)^+}{\Delta x}+\frac{(a-\eta x)^-}{\Delta x}-\frac{\sigma^2}{(\Delta x)^2} \\
\zeta_j =& \frac{(a-\eta x)^+}{\Delta x}+\frac{\sigma^2}{2(\Delta x)^2}
\end{aligned}
\tag{19}
$$

At the boundaries in the i dimension, the equations become:

$$
\begin{aligned}
\frac{V_{1,j}^{m+1}-V_{1,j}^{m}}{\Delta}+\rho V_{1,j}^{m+1} =& u(n_{1,j}^{m})+V_{1,j}^{m+1}(2b_{1,j}+c_{1,j}+\upsilon_j)+V_{2,j}^{m+1}(d_{1,j}-b_{1,j}) \\
&+V_{1,j-1}^{m+1}\chi_j+V_{1,j+1}^{m+1}\zeta_j \\
\frac{V_{I,j}^{m+1}-V_{I,j}^{m}}{\Delta}+\rho V_{I,j}^{m+1} =& u(n_{I,j}^{m})+V_{I-1,j}^{m+1}(b_{I,j}-d_{I,j})+V_{I,j}^{m+1}(c_{I,j}+\upsilon_j+2d_{I,j}) \\
&+V_{I,j-1}^{m+1}\chi_j+V_{I,j+1}^{m+1}\zeta_j
\end{aligned}
$$

where in the first equation, we have used that $\partial_{h,B}V_{1,j}=\frac{V_{1,j}-V_{0,j}}{\Delta h}=\frac{V_{2,j}-V_{1,j}}{\Delta h}$ and hence $V_{0,j}=2V_{1,j}-V_{2,j}$. Similarly, in the second equation, $\partial_{h,F}V_{I,j}=\frac{V_{I+1,j}-V_{I,j}}{\Delta h}=\frac{V_{I,j}-V_{I-1,j}}{\Delta h}$ and hence $V_{I+1,j}=2V_{I,j}-V_{I-1,j}$.

We use the same method at the boundaries in the j dimension, and the equations become:

$$
\begin{aligned}
\frac{V_{i,1}^{m+1}-V_{i,1}^{m}}{\Delta}+\rho V_{i,1}^{m+1} =& u(n_{i,1}^{m})+V_{i-1,1}^{m+1}b_{i,1}+V_{i,1}^{m+1}(c_{i,1}+\upsilon_1+2\chi_j)+V_{i+1,1}^{m+1}d_{i,1} \\
&+V_{i,j+1}^{m+1}\zeta_j \\
\frac{V_{i,J}^{m+1}-V_{i,J}^{m}}{\Delta}+\rho V_{i,J}^{m+1} =& u(n_{i,J}^{m})+V_{i-1,J}^{m+1}b_{i,J}+V_{i,J}^{m+1}(c_{i,J}+\upsilon_J+2\zeta_J)+V_{i+1,J}^{m+1}d_{i,J} \\
&+V_{i,J-1}^{m+1}(\chi_J-\zeta_J)
\end{aligned}
$$

where in the first equation, we have used that $\partial_{x,B}V_{i,1} = \frac{V_{i,1}-V_{i,0}}{\Delta x} = \frac{V_{i,2}-V_{i,1}}{\Delta x}$ and hence $V_{i,0} = 2V_{i,1} - V_{i,2}$. Similarly, in the second equation, $\partial_{x,F}V_{i,J} = \frac{V_{i,J+1}-V_{i,J}}{\Delta x} = \frac{V_{i,J}-V_{i,J-1}}{\Delta x}$ and hence $V_{i,J+1} = 2V_{i,J} - V_{i,J-1}$.
Equation 19 is a system of $I \times J$ linear equations which can be written in matrix notation as:

$$\frac{1}{\Delta}(V^{m+1} - V^m) + \rho V^{m+1} = u^m + \mathbf{A}^m V^{m+1} \tag{20}$$

where V^n is a vector of length $I \times J$ with entries $(V_{1,1}, ..., V_{I,1}, V_{1,2}, ..., V_{I,2}, ..., V_{I,J})$ and $\mathbf{A}^m = \widetilde{A}^m + \mathbf{C}$ where the $(I \times J) \times (I \times J)$ matrices $\widetilde{A}^m$ and $\mathbf{C}$ are:

$$\widetilde{A}^m = \begin{bmatrix}
c_{1,1} & d_{1,1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
b_{2,1} & c_{2,1} & d_{2,1} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & b_{I,1} & c_{I,1} & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & 0 & c_{1,2} & d_{1,2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & b_{2,2} & c_{2,2} & d_{2,2} & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & c_{1,J} & d_{1,J} & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & b_{2,J} & c_{2,J} & d_{2,J} & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_{I,J} & c_{I,J}
\end{bmatrix}$$

$$
C=\begin{bmatrix}
\upsilon_1+\chi_1 & 0 & \cdots & \cdots & 0 & \zeta_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \upsilon_1+\chi_1 & 0 & \ddots & \ddots & 0 & \zeta_1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & 0 & \upsilon_1+\chi_1 & 0 & \ddots & \ddots & 0 & \zeta_1 & 0 & \ddots & \ddots & \vdots \\
\chi_2 & 0 & \ddots & 0 & \upsilon_2 & 0 & \ddots & \ddots & 0 & \zeta_2 & 0 & \ddots & \vdots \\
0 & \chi_2 & 0 & \ddots & 0 & \upsilon_2 & 0 & \ddots & \ddots & 0 & \zeta_2 & 0 & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \vdots \\
\vdots & \ddots & 0 & \chi_2 & 0 & \ddots & 0 & \upsilon_2 & 0 & \ddots & \ddots & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \chi_J & 0 & \ddots & \ddots & \upsilon_J+\zeta_J & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \chi_J & 0 & \ddots & \upsilon_J+\zeta_J & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \chi_J & 0 & \cdots & 0 & \upsilon_J+\zeta_J
\end{bmatrix}
$$

B. Kolmogorov Forward Equation and Equilibrium

We now turn to the solution of Equation 12, which also have to satisfy Equation 13. The rough idea is to discretize these as

$$0=-\frac{\partial}{\partial h}(\mu_1(h_i)f_{i,j})-\frac{\partial}{\partial x}(\mu_2(x_j)f_{i,j})+\frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma^2 f_{i,j} \tag{21}$$

$$1=\sum_{i=1}^{I}\sum_{j=1}^{J} f_{i,j}\Delta h\Delta x \tag{22}$$

We can use the same method in section VI.A to solve HJB.

$$\begin{aligned}\frac{V_{i,j}^{m+1}-V_{i,j}^{m}}{\Delta}+\rho V_{i,j}^{m+1}=&u(n_{i,j}^{m})+V_{i-1,j}^{m+1}b_{i,j}+V_{i,j}^{m+1}(c_{i,j}+\upsilon_j)+V_{i+1,j}^{m+1}d_{i,j}\\&+V_{i,j-1}^{m+1}\chi_j+V_{i,j+1}^{m+1}\zeta_j\end{aligned}$$

UPWIND SCHEME

There is again a question when to use a forward or backward approximation for the derivatives. It turns out that the most convenient approximation of equation 21 is as follows:

$$
\begin{aligned}
0 = &- \frac{f_{i,j}(\dot{h}^n_{i,j,F})^+ - f_{i-1,j}(\dot{h}^n_{i-1,j,F})^+}{\Delta h} - \frac{f_{i+1,j}(\dot{h}^n_{i+1,j,B})^- - f_{i,j}(\dot{h}^n_{i,j,B})^-}{\Delta h} \\
&- \frac{f_{i,j}\mu_2(x_j)^+ - f_{i,j-1}\mu_2(x_{j-1})^+}{\Delta x} - \frac{f_{i,j+1}\mu_2(x_{j+1})^- - f_{i,j}\mu_2(x_j)^-}{\Delta x} \\
&+ \frac{\sigma^2}{2}\frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta x)^2}
\end{aligned}
$$

Collecting terms, we have:

$$
\begin{aligned}
&0 = f_{i+1,j}b_{i+1,j} + f_{i,j}(c_{i,j} + \upsilon_j) + f_{i-1,j}d_{i-1,j} + f_{i,j-1}\chi_{j-1} + f_{i,j+1}\zeta_{j+1} \\
&b_{i,j} = -\frac{(\dot{h}_{i,j,B})^-}{\Delta h} \\
&c_{i,j} = -\frac{(\dot{h}_{i,j,F})^+}{\Delta h} + \frac{(\dot{h}_{i,j,B})^-}{\Delta h} \\
&d_{i,j} = \frac{(\dot{h}_{i,j,F})^+}{\Delta h} \\
&\chi_j = -\frac{\mu_2(x_j)^-}{\Delta x} + \frac{\sigma^2}{2(\Delta x)^2} \\
&\upsilon_j = -\frac{\mu_2(x_j)^+}{\Delta x} + \frac{\mu_2(x_j)^-}{\Delta x} - \frac{\sigma^2}{(\Delta x)^2} \\
&\zeta_j = \frac{\mu_2(x_j)^+}{\Delta x} + \frac{\sigma^2}{2(\Delta x)^2}
\end{aligned}
$$

VII. Conclusion

We set up a heterogeneous agent model to describe users' use of online payments. We obtain the coupled HJB and KFE equations, and use the upwind algorithm (a finite difference method) to calculate online payment times and the distribution of users' online payment habits in different market environments. The optimal charging policy which can maximize the social welfare is also obtained.

However, there are also some problems that we need to improve. After obtaining online payment data from Ant Open Research, we will calibrate the parameters and get a more accurate charging method to maximize social welfare.

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SOLUTION

The reason this is the preferred approximation is that it can be written in matrix form in a way that is closely related to the approximation used for the HJB equation

$$\mathbf{A}^T f = 0 \tag{1}$$

where $\mathbf{A}^T$ is the transpose os the intensity matrix $\mathbf{A}$ from the HJB equation with two diffusion processes($\mathbf{A}^n$ from the final HJB iteration). This makes sense: Besides making sense, this approximation is also convenient: once one hae constructed the matrix $\mathbf{A}$ for solving the HJB equation using an implicit method, almost no extra work is needed.

To solve the eigenvalue problem 1 while imposing 22, the simplest procedure is as follows. Fix $f_{i,j} = 0.1$ (any other number will do as well) for an arbitrary (i, j), to then solve the system for some $\widetilde{f}$ and then to remormalize $f_{i,j} = \widetilde{f}_{i,j}/(\sum_{i=1}^{I}\sum_{j=1}^{J} f_{i,j}\Delta h\Delta x)$. Fixing $f_{i,j} = 0.1$ is achieved by replacing the corresponding entry of the zero vector in equation 1 by 0.1, and the corresponding row of $\mathbf{A}^T$ by a row of zeros everywhere except for one on the diagonal. Without this "dirty fix," the matrix $\mathbf{A}^T$ is singular and so cannot be inverted.

DERIVATION OF HJB AND KFE

This appendix shows how to derivve the HJB equation with a diffusion process and the Kolmogorov Forward equation (Fokker-Planck equation) with a diffusion process.

A1. Derivation of Hamilton-Jacobi-Bellman Equation

Consider the Bellman equation:

$$\begin{aligned} V(h(t),x(t)) &= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E\int_t^{\infty} e^{-\rho(s-t)}u(n(s))ds \\ dh(t) &= [n(t)-T(n(t))]\,dt \\ dx(t) &= (a-\eta x)dt+\sigma dB_t \end{aligned}$$

Then we have:

$$
\begin{aligned}
& V(h(t),x(t)) \\
&= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \int_t^{\infty} e^{-\rho(s-t)} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \\
&= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \left(\begin{array}{c} \int_t^{t+\Delta t} e^{-\rho(s-t)} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \\ +e^{-\rho\Delta t} \int_{t+\Delta t}^{\infty} e^{-\rho(s-(t+\Delta t))} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \end{array} \right) \\
&= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \left(\begin{array}{c} \int_t^{t+\Delta t} e^{-\rho(s-t)} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \\ +E_{t+\Delta t} \left(e^{-\rho\Delta t} \int_{t+\Delta t}^{\infty} e^{-\rho(s-(t+\Delta t))} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \right) \end{array} \right) \\
&= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \left(\begin{array}{c} \int_t^{t+\Delta t} e^{-\rho(s-t)} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \\ +e^{-\rho\Delta t} E_{t+\Delta t} \left(\int_{t+\Delta t}^{\infty} e^{-\rho(s-(t+\Delta t))} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \right) \end{array} \right) \\
&= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \left(\begin{array}{c} \int_t^{t+\Delta t} e^{-\rho(s-t)} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \\ +\max \left(e^{-\rho\Delta t} E_{t+\Delta t} \left(\int_{t+\Delta t}^{\infty} e^{-\rho(s-(t+\Delta t))} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \right) \right) \end{array} \right) \\
&= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \left(\begin{array}{c} \int_t^{t+\Delta t} e^{-\rho(s-t)} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \\ +e^{-\rho\Delta t} \max \left(E_{t+\Delta t} \left(\int_{t+\Delta t}^{\infty} e^{-\rho(s-(t+\Delta t))} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \right) \right) \end{array} \right) \\
&= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \left(\begin{array}{c} \int_t^{t+\Delta t} e^{-\rho(s-t)} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) ds \\ +e^{-\rho\Delta t} V(h(t+\Delta t), x(t+\Delta t)) \end{array} \right) \\
&= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \left(\begin{array}{c} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) \Delta t + (1-\rho\Delta t) \\ \left(V(h(t),x(t)) + V_x \Delta x + \frac{1}{2} V_{xx} (\Delta x)^2 + V_h \Delta h + \frac{1}{2} V_{hh} (\Delta h)^2 \right) \end{array} \right)
\end{aligned}
$$

We then use Ito's formula

$$
= \max_{\{n(t),h(t)\}_{t=0}^{\infty}} E_t \left(\begin{array}{c} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \right) \Delta t + (1-\rho\Delta t) \\ (V(h(t),x(t)) + V_x(h(t),x(t)) \Delta x + \frac{1}{2} V_{xx}(h(t),x(t)) \sigma^2 \Delta t \\ +V_h(h(t),x(t)) \Delta h) \end{array} \right)
$$

Note that $E_t\Delta B = 0$. Taking expectation operator, we have

$$\begin{aligned}
&V(h(t),x(t)) \\
=&\max_{\{n(t),h(t)\}_{t=0}^{\infty}} \begin{pmatrix} \left(x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}\right)\Delta t \\ +(1-\rho\Delta t)\begin{pmatrix} V(h(t),x(t)) + V_x(h(t),x(t))(a-\eta x)\Delta t \\ +\frac{1}{2}V_{xx}(h(t),x(t))\sigma^2\Delta t + ((n(t)-T(n(t)))V_h\Delta t) \end{pmatrix} \end{pmatrix}
\end{aligned}$$

Dividing by Δt on both sides and letting $\Delta t \to 0$, we have HJB

$$\rho V(h(t),x(t)) = \max_{\{n(t),h(t)\}_{t=0}^{\infty}} \begin{pmatrix} x(t)\frac{h(t)^{1-\gamma}}{1-\gamma} - C\frac{n(t)^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}} \\ +(a-\eta x)V_x(h(t),x(t)) + \frac{1}{2}\sigma^2 V_{xx}(h(t),x(t)) \\ +(n(t)-T(n(t)))V_h(h(t),x(t)) \end{pmatrix}$$

A2. Derivation of Kolmogorov Forward Equation

We know that

$$\begin{aligned}
dh(t) &= \mu_1 dt \\
dx(t) &= \mu_2 dt + \sigma dB(t)
\end{aligned}$$

where $\mu_1 = n(t) - T(n(t))$ and $\mu_2 = a - \eta x(t)$ The derivation of KFE is as follows. For all functions $\varphi(h,x)$, we have

$$E[\varphi(h(t+\Delta t), x(t+\Delta t))] = \int_0^{\infty}\int_{\underline{h}}^{\infty} \varphi(h,x) f(h,x,t+\Delta t)dhdx$$

On the one hand, by Itô's lemma, we have

$$\begin{aligned}
d\varphi(h(t),x(t)) =&\varphi_h(h(t),x(t))dh + \varphi_x(h(t),x(t))dx + \frac{1}{2}\varphi_{hh}(h(t),x(t))(dh)^2 \\
&+ \frac{1}{2}\varphi_{xx}(h(t),x(t))(dx)^2 + \varphi_{hx}(h(t),x(t))dhdx \\
=&\varphi_h(h(t),x(t))\mu_1 dt + \varphi_x(h(t),x(t))\mu_2 dt \\
&+ \frac{1}{2}\varphi_{xx}(h(t),x(t))\sigma^2 dt + \varphi_x \sigma dB(t)
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{\int_0^{\infty}\int_{\underline{h}}^{\infty} \varphi(h,x) f(h,x,t+\Delta t)dhdx - \int_0^{\infty}\int_{\underline{h}}^{\infty} \varphi(h,x) f(h,x,t)dhdx}{\Delta t} \\
=&\frac{E_t[\varphi(h(t+\Delta t),x(t+\Delta t))] - E_t[\varphi(h(t),x(t))]}{\Delta t} \\
=&\frac{E_t(E_t[\varphi(h(t+\Delta t),x(t+\Delta t))|h(t),x(t)]) - E_t[\varphi(h(t),x(t))]}{\Delta t}
\end{aligned}$$

$$
\begin{aligned}
=&E_t\left(\frac{(E_t[\varphi(h(t+\Delta t),x(t+\Delta t))-\varphi(h(t),x(t))|h(t),x(t)])}{\Delta t}\right)\\
=&E_t\left(\frac{\Delta\varphi(h(t),x(t))}{\Delta t}\right)\\
=&E_t\left[\varphi_h(h(t),x(t))\mu_1(h(t))\right]+E_t\left[\varphi_x(h(t),x(t))\mu_2(x(t))+\frac{1}{2}\varphi_{xx}(h(t),x(t))\sigma_2^2\right]\\
=&\int_0^\infty\int_{\underline{h}}^\infty\left[\varphi_h(h,x)\mu_1(h(t))\right]f(h,x,t)dhdx\\
&+\int_0^\infty\int_{\underline{h}}^\infty\left[\varphi_x(h,x)\mu_2(x(t))+\frac{1}{2}\varphi_{xx}(h,x)\sigma^2\right]f(h,x,t)dhdx
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{\int_0^\infty\int_{\underline{h}}^\infty\varphi(h,x)f(h,x,t+\Delta t)dhdx-\int_0^\infty\int_{\underline{h}}^\infty\varphi(h,x)f(h,x,t)dhdx}{\Delta t}\\
=&\frac{\int_0^\infty\int_{\underline{h}}^\infty\varphi(h,x)(f(h,x,t+\Delta t)-f(h,x,t))dhdx}{\Delta t}\\
=&\int_0^\infty\int_{\underline{h}}^\infty\varphi(h,x)\frac{f(h,x,t+\Delta t)-f(h,x,t)}{\Delta t}dhdx\\
=&\int_0^\infty\int_{\underline{h}}^\infty\varphi(h,x)\frac{\partial}{\partial t}f(h,x,t)dhdx
\end{aligned}
$$

Therefore, let $\varphi(\underline{h},x)=\varphi(\infty,x)=0$, $\varphi(h,\underline{x})=\varphi(h,\bar{x})=0$ and we have

$$
\begin{aligned}
&\int_0^\infty\int_{\underline{h}}^\infty\varphi(h,x)\frac{\partial}{\partial t}f(h,x,t)dhdx\\
=&\int_0^\infty\int_{\underline{h}}^\infty\left[\varphi_h(h,x)\mu_1(h(t))\right]f(h,x,t)dhdx\\
&+\int_0^\infty\int_{\underline{h}}^\infty\left[\varphi_x(h,x)\mu_2(x(t))+\frac{1}{2}\varphi_{xx}(h,x)\sigma^2\right]f(h,x,t)dhdx\\
=&-\int_0^\infty\int_{\underline{h}}^\infty\left[\varphi(h,x)\frac{\partial}{\partial h}(\mu_1(h)f(h,x,t))\right]dhdx\\
&-\int_0^\infty\int_{\underline{h}}^\infty\left[\varphi(h,x)\frac{\partial}{\partial x}(\mu_2(x)f(h,x,t))\right]dhdx\\
&+\frac{1}{2}\int_0^\infty\int_{\underline{h}}^\infty\left[\varphi(h,x)\frac{\partial^2}{\partial x^2}(\sigma^2f(h,x,t))\right]dhdx
\end{aligned}
$$

Thus, we have

$$\int_0^\infty \int_{\underline{h}}^\infty \varphi(h,x)[\frac{\partial}{\partial t} f(h,x,t) + \frac{\partial}{\partial h}(\mu_1(h) f(h,x,t)) + \frac{\partial}{\partial x}(\mu_2 f(h,x,t)) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 f(h,x,t))] dh dx = 0$$

Since $\varphi(h,x)$ is arbitrary, we have

$$\frac{\partial}{\partial t} f(h,x,t) + \frac{\partial}{\partial h}(\mu_1(h) f(h,x,t)) + \frac{\partial}{\partial x}(\mu_2(x) f(h,x,t)) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 f(h,x,t)) = 0$$

Thus, we have

$$\frac{\partial}{\partial t} f(h,x,t) = -\frac{\partial}{\partial h}(\mu_1(h) f(h,x,t)) - \frac{\partial}{\partial x}(\mu_2(x) f(h,x,t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 f(h,x,t))$$