# The Endogeneity of Skill Distribution, Incidence of Tax Reform, and Optimal Taxation 

Minchung Hsu<br>National Graduate Institute for Policy Studies<br>C.C. Yang<br>Academia Sinica<br>National Chengchi University<br>Feng Chia University

Shenghao Zhu
University of International Business and Economics
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#### Abstract

This paper extends the seminal work of Mirrlees (1971) to the setting where the distribution of people's skills is determined by the accumulation of human capital as well as innate ability via the luck of the draw. We utilize the Kolmogorov forward equation to analytically derive a closed-form solution for the stationary skill distribution and address (i) how the imposition of an income tax schedule shapes the distribution of skills, (ii) how the local perturbation of the income tax schedule (tax reform) alters skill distribution and government revenue, (iii) how the endogeneity of skill distribution modifies the optimal tax formula derived by Diamond (1998) and Saez (2001) and the asymptotic optimal marginal tax rate.


## 1 Introduction

Given a distribution of people's skills (earnings abilities), optimal taxation studies how people's earnings should be taxed via the design of an income tax schedule. This approach is standard in the optimal taxation literature since the seminal work of Mirrlees (1971); see, for example, Diamond (1998), Saez (2000), Scheuer and Werning (2017), and Sachs, Tsyvinsky and Werquin (2020). Our paper turns the approach upside down: given an income tax schedule, we study how the distribution of people's skills is determined by the accumulation of human capital as well as innate ability via the luck of the draw. We utilize the Kolmogorov forward equation (also known as Fokker-Planck equation) to analytically derive a closed-form solution for the stationary skill distribution and address (i) how the imposition of an income tax schedule shapes the distribution of skills, (ii) how the local perturbation of the income tax schedule (tax reform) alters skill distribution and government revenue, (iii) how the endogeneity of skill distribution modifies the optimal tax formula derived by Diamond (1998) and Saez (2001) and the asymptotic optimal marginal tax rate.

Our study qualitatively differs from the work of Stiglitz (1982) and subsequent papers such as Rothschild and Scheuer (2013), Ales, Kurnaz and Sleet (2015), and Sachs, Tsyvinsky and Werquin (2020). These papers address optimal taxation in the tradition of Mirrlees (1971) by assuming an exogenous distribution of people's skills but extend it to allow for endogenous wages, in that different skills are not perfectly substitutable so that they earn different wage rates in general equilibrium (just like labor and capital are different factors of production so as to earn different rates of return in the neoclassical production economy). Our paper sticks to the original setup of Mirrlees (1971), in which different skills are perfectly substitutable in terms of effective units, namely, an individual of skill $n$ having a marginal product equal to $n$.

Since optimal conditions derived yield few clear-cut analytical results, considerable efforts of the optimal taxation literature starting with Mirrlees (1971) have gone into simulations to
quantitatively explore the shape of optimal tax schedules. ${ }^{1}$ Nevertheless, consensus on this important issue seems to remain elusive. Working with quasi-linear preferences in consumption, Diamond (1998) showed that if people's skills are Pareto-distributed and the government has a redistributive taste, optimal marginal tax rates rise with income above the modal skill level. However, in the same paper, he also provided a condition on the skill distribution for yielding an opposite result that optimal marginal tax rates decrease with income above some critical skill level. Utilizing the same framework as Diamond (1998), Li et al. (2013) demonstrated that, depending on the skill distribution assumed, the schedule of optimal marginal tax rates can be almost anything: strictly increasing, strictly decreasing, U-shaped, inverse U-shaped, W-shaped, or M-shaped. Working with quai-linear preferences in labor rather than consumption, Myles (2000) reached an analogous result: except for the zero rate on the highest skilled agent, any qualitative structure of optimal marginal tax rates can be supported by some skill distribution. The findings of these papers strongly suggest the critical role of the skill distribution in shaping the optimal income tax schedue. A question naturally arises: what is the "right" skill distribution? One route to answering the question is to discipline the unobservable skill distribution by observable empirical data. An important contribution of Saez (2001) is to back out the unobservable distribution of skills such that, given the actual taxes imposed, the resulting earnings distribution from the model economy replicates the empirical earnings distribution observed in the real world. Subsequent works elaborating on Saez (2001) include Mankiw, Weinzierl and Yagan (2009), Heathcote, Storesletten and Violante (2017), Rothschild and Scheuer (2013), and Chang and Park (2020). Our paper is complementary to this line of the literature, in that we analytically expose how the distribution of skills is shaped by economic forces and taxation.

Rothschild and Scheuer (2013) considered a model in which agents have a two-dimensional exogenous skill type. Although the same skill type is perfectly substitutable in terms of

[^0]effective units as in Mirrlees (1971), different skill types are not (just like labor (resp. capital) supplied by different agents is perfectly substitutable, but labor and capital are not).

For the first how, we focus on two popular income tax schemes in the literature: (i) the linear income tax, and (ii) the constant rate of progressivity (CRP) tax.

For the second how, we generalize the results in Saez (2001), in which the distribution of people's skills is exogenously rather than endogenously specified.

For the third how, we address how the celebrated formula with regard to optimal marginal tax rates derived by Diamond (1998) and Saez (2001) should be modified. This formula is exposed in detail in Salanie (2011) and Brewer, Saez and Shephard (2010). We extend the formula from the setting of exogenous skill distributions to that of endogenous skill distributions.

## Related literature

The literature on optimal taxation is vast. Here we focus on a limited subset of the studies in the tradition of the Mirrlees (1971) approach that are most relevant to our paper.

Our paper is closely related to the work of Piketty (1997), Saez (2001), Golosov, Tsyvinsky and Werquin (2014), Sachs, Tsyvinsky and Werquin (2020), and Chang and Park (2020). These papers all use the variational approach to study the tax incidence of arbitrary local perturbations of an initial, potentially suboptimal, tax schedule and obtain the optimal tax schedule by imposing the condition that there exist no perturbations that can increase social welfare. In particular, Sachs, Tsyvinsky and Werquin (2020) considered endogenous wages in general equilibrium, and Chang and Park (2020) incorporated the presence of private insurance. We adopt the same approach. Our contribution to the literature is to employ the Fokker-Planck equation (the Kolmogorov forward equation) to obtain an endogenously determined stationary skill distribution that depends on the tax schedule and then utilize the solution to gain new insights into the issues of tax incidence and optimal taxation.

Farhi and Werning (2013) and Golosov, Troshkin and Tsyvinski (2016) studied optimal labor taxation over the life cycle within the framework of New Dynamic Public Finance (NDPF). ${ }^{2}$ As in our paper, people's skills evolve stochastically over time in their papers. However, their skill evolution is exogenously specified and independent of economic forces. Stantcheva (2017) and Kapička and Neira (2019) studied the evolution of skills over the life cycle through the accumulation of risky human capital in the framework of NDPF. Their focus is on the joint determination of optimal tax and human capital policies over the life cycle. By contrast, our focus is not on dynamics but on stationary state. We utilize the Kolmogorov forward equation to derive endogenously-determined stationary skill distributions. Our approach qualitatively differs from the one adopted in NDPF.

Abbott, Gallipoli, Meghir and Violante (2019) studied the equilibrium effects of college financial aid policies using an overlapping-generations life cycle model. Holter, Krueger and Stepanchuk (2019) stuided how tax progressivity and household heterogeneity affect Laffer curves by developing a large scale overlapping generations with endogenous accumulation of human capital through labor market experience.

Heathcote, Storesletten and Violante (2017) considered a model in which individuals differ in their cost of acquiring skill. This way of determining people's skills fundamentally differs from ours. Importantly, they departs from the Mirrlees approach, in that the tax scheme in their model is exogenously specified a priori. Heathcote and Tsujiyama (2019)

Badel, Huggett and Luo (2020) showed that the revenue maximizing top tax rate is approximately $49 \%$ in their quantitative human capital model. This finding is significantly lower than an established view that the revenue maximising top tax rate for the US is approximately $73 \%$. The endogenous response of top earners' human capital to changes in the top tax rate is key to their result.

Following the seminal work of Stiglitz (1982), several papers inlcuding Rothschild and

[^1]Scheuer (2013), Ales, Kurnaz and Sleet (2015), and Sachs, Tsyvinsky and Werquin (2020) have considered imperfect substitution between different types of labor in production and allowed the endogenous determination of wages. To focus on the distribution of people's skill through human capital accumulation, we abstract from this line of extension.

Some papers have introduced additional elements to extend the classical tax formula derived by Diamond (1998) and Saez (2001). Sachs, Tsyvinsky and Werquin (2020) incorporated the impact of the general equilibrium effects with endogenous wages, which are deteremined by the assignment of skills to tasks. Chang and Park (2020) incorporated private insurance and highlighted the interaction between private and public insurance. This paper complements these studies by endogenizing the distribution of people's skill through human capital accumulation.

## 2 Basic Model

We consider the simplest possible model to extend the framework of Mirrlees (1971) to incorporate the accumulation of human capital.

There are a continuum of agents in the economy. Each agent lives for one period; at the end of the period, agents give birth to their children and die. Each agent is assumed to give birth to one child and thus the populaiton size of the economy remains unchanged over time. Agents in any time period are heterogeneous, in that they are endowed with different stocks of human capital.

Given $n_{s}$ (human capital), the agent of generation $s$ faces the following problem:

$$
\max _{c_{s}, e_{s}, l_{s}} U\left(c_{s}, e_{s}, l_{s}\right)=u\left(c_{s}, e_{s}\right)-v\left(l_{s}\right)
$$

subject to

$$
\begin{equation*}
c_{s}+e_{s}=y_{s} \equiv n_{s} l_{s}-T\left(n_{s} l_{s}\right), \tag{1-1}
\end{equation*}
$$

$$
\begin{equation*}
\ln n_{s+1}=e_{s}+\ln n_{s}+\theta_{s+1} \tag{1-2}
\end{equation*}
$$

where $c_{s}$ and $e_{s}$ denote, respectively, consumption and education expenditure; $l_{s}$ is the agent's labor supply, $y_{s}$ disposable income, $T($.$) an income tax schedule imposed on earnings n_{s} l_{s}$, and $\theta_{s+1}$ the innate ability of the agent's child. It is assumed that $v: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is twice continuously differentiable, strictly increasing, and strictly convex; $T: \mathbb{R}_{+} \rightarrow \mathbb{R}$ is twice continuously differentiable. We specify the functional form for $u\left(c_{s}, e_{s}\right)$ below.

The education expenditure $e_{s}$ directly enters into the utility function $u(\cdot)$ because it represents a kind of bequest as in Glomm and Ravikumar (1992). The budget constraint (1-1) shows that the sum of consumption and education expenditure, $c_{s}+e_{s}$, have to come out of disposable income, $y_{s}$, implying the absence of a credit market to finance human capital investment. Benabou (2002) justified this absence by means of that children cannot be held responsible for the debts incurred by their parents.

The motion (1-2) characterizes the process of human capital accumulation across consecutive generations. Similar to Benabou (2002), a child's human capital $n_{s+1}$ is determined by three factors: her own innate ability $\theta_{s+1}$, parental education expenditure $e_{s}$, and the quality of the home or neighborhood environment as captured by parental human capital $n_{s}$. Following Benabou (2002), we assume that the child's innate ability $\theta_{t+1}$ is independent and identically distributed (i.i.d.) with $\theta_{s+1} \sim N\left(0, \sigma^{2}\right)$. This risk represents the luck of the draw in the accumulation of human capital.

Timing for the motion (1-2) is as follows:

1. $\theta_{s+1}$ is realized and the parent observes $\ln n_{s}+\theta_{s+1}$.
2. After observing $\ln n_{s}+\theta_{s+1}$, the parent spends education expenditure $e_{s}$ and consumes $c_{s}$ with $c_{s}+e_{s}=y_{s}$.
3. With the education expenditure $e_{s}$, the child's $\log$ skill $\ln n_{s+1}$ is determined according to (1-2).

In the absence of the agent's decision on $e_{s}$, the motion (1-2) is simply a random walk in $\ln n_{s}$ with no drift. In the presence of the agent's decision on $e_{s}$ (after $\theta_{s+1}$ is realized), the motion (1-2) is a random walk in $\ln n_{s}$ with drift $e_{s}$.

It is assumed that $u($.$) takes the CES form:$

$$
u\left(c_{s}, e_{s}\right)=\left[(1-\beta)\left(\frac{c_{s}}{1-\beta}\right)^{\frac{\phi-1}{\phi}}+\beta\left(\frac{e_{s}}{\beta}\right)^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}}
$$

where $\beta$ is a parameter capturing the agent's tastes for bequest in the form of eduction expenditure. Sovling for the agent's problem with respect to $c_{s}$ and $e_{s}$ gives

$$
\begin{gathered}
c_{s}=(1-\beta) y_{s}, \\
e_{s}=\beta y_{s} .
\end{gathered}
$$

Substituting the derived $e_{s}$ into the motion (1-2) gives the human captial accumulation process:

$$
\begin{equation*}
\ln n_{s+1}=\beta y_{s}+\ln n_{s}+\theta_{s+1} . \tag{1}
\end{equation*}
$$

We also obtain

$$
u\left(c_{s}, e_{s}\right)=y_{s}
$$

which implies that the agent's labor supply $l_{s}$ solves

$$
\max _{l_{s}} y_{s}-v\left(l_{s}\right),
$$

which yields the first-order condition

$$
\begin{equation*}
v^{\prime}\left(l\left(n_{s}\right)\right)=n_{s}\left[1-T^{\prime}\left(n_{s} l\left(n_{s}\right)\right)\right] . \tag{2}
\end{equation*}
$$

This completes the description of the basic model.

## 3 Endogeneity of skill distributions

On the basis of the human capital accumulation process (1), this section is to derive an endogeneously-determined skill distribution. We are particularly interested in how the imposition of the income tax schedule $T($.$) shapes the distribution of skills.$

As $\Delta s \rightarrow 0$, the motion (1) gives ${ }^{3}$

$$
\begin{equation*}
d \ln n(s)=\beta y(n(s)) d s+\sigma d B(s) \tag{3}
\end{equation*}
$$

where the process $\{B(s): s \geq 0\}$ is a standard Brownian motion. Compared to the Brownian motion specified by Farhi and Werning (2013, Eq. (16)) for people's skills, a key difference is that while their evolution is exogenously specified, ours is endogenously determined through the effect of $y(n(s))$ on $d \ln n(s)$.

Let the cross-sectional pdf of $n(s)$ at time $s$ be denoted by $f(n, s)$. Given the motion (3), applying the Kolmogorov forward (KF) equation yields ${ }^{4}$

$$
\frac{\partial f(n, s)}{\partial s}=\frac{1}{2} \frac{\partial^{2}}{\partial n^{2}}\left[\sigma^{2} n^{2} f(n, s)\right]-\frac{\partial}{\partial n}[\beta y(n) n f(n, s)] .
$$

[^2]In stationary state,

$$
\frac{\partial f(n, s)}{\partial s}=0
$$

Thus, a stationary $f(n, s)$, denoted by $f(n)$, satisfies the following differential equation:

$$
\frac{1}{2} \frac{d^{2}}{d n^{2}}\left[\sigma^{2} n^{2} f(n)\right]-\frac{d}{d n}[\beta y(n) n f(n)]=0
$$

Integrating the above equation gives

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d n}\left[\sigma^{2} n^{2} f(n)\right]=[\beta y(n) n f(n)]+C \tag{4}
\end{equation*}
$$

where $C$ denotes an arbitrary constant. Following Achdou, Han, Lasry, Lions, and Moll (2022, proof of Proposition 10), we choose $C=0$ as an implicit boundary condition and later we will verify that the solution for $f(n)$ does satisfy this condition.

With $C=0$, we have from (4)

$$
\begin{equation*}
\frac{f^{\prime}(n)}{f(n)}=\frac{2}{\sigma^{2}} \frac{\beta y(n)}{n}-\frac{2}{n}, \tag{5}
\end{equation*}
$$

which leads to

$$
\int_{1}^{n} \frac{f^{\prime}(\tilde{n})}{f(\tilde{n})} d \tilde{n}=\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}-2 \int_{1}^{n} \frac{1}{\tilde{n}} d \tilde{n} .
$$

Solving for $f(n)$ yields (see Appendix 9.1):

Theorem 1 Under human capital accumulation process (1), the stationary pdf of skills, $f(n)$, satisfies

$$
f(n)=f(1) n^{-2} \exp \left(\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}\right), n \geq 0,{ }^{5}
$$

where the constant $f(1)$ is determined by $\int_{0}^{\infty} f(n) d n=1$.

$$
\begin{aligned}
& { }^{5} \text { If } n \in[0,1], \\
& \qquad \begin{aligned}
f(n) & =f(1) n^{-2} \exp \left\{\frac{2}{\sigma^{2}} \int_{1}^{n} \frac{\beta y(\tilde{n})}{\tilde{n}} d \tilde{n}\right\} \\
& =f(1) n^{-2} \exp \left\{\frac{-2}{\sigma^{2}} \int_{n}^{1} \frac{\beta y(\tilde{n})}{\tilde{n}} d \tilde{n}\right\}
\end{aligned}
\end{aligned}
$$

Remark 1 Given $y(n)=n l(n)-T(n l(n))$, it is clear that $f(n)$ in Theorem 1 depends upon the income tax scheme $T($.$) .$

Remark 2 We have $\lim _{n \rightarrow 0} f(n)=0$ ?? Thus, the size of people with $n=0$ is a zero measure in stationay state. Those with $n=0$ in the real world are not explicable with our model of human capital accumulation.

If $\beta=0$, then $e_{s}=0$. The human capital accumulation process (1) would reduce to

$$
\begin{equation*}
\ln n_{s+1}=\ln n_{s}+\theta_{s+1}, \tag{6}
\end{equation*}
$$

and hence $f(n)=f(1) n^{-2}$, which is a Pareto distribution. The pdf $f(n)$ in Theroem 1 is a modified Pareto distribution, in that it is embedded in human efforts via education expenditures to modify the random force represented by (6).

Tax equilibrium. Let $E \in \mathbb{R}_{+}$be a fixed public spending and $z(n) \equiv n l(n)$ (pre-tax income). Given $E$, a tax equilibrium is a tax schedule $T($.$) , an allocation \{c(n), e(n), l(n)\}$, and a skill distribution $f(n)$ such that (i) $c(n)=(1-\beta) y(n), e(n)=\beta y(n)$ and equation (2) holds, (ii) $E=\int_{0}^{\infty} T(n l(n)) f(n) d n$ (the government budget balanced), and (iii) $\int_{0}^{\infty} z(n) f(n) d n=$ $E+\int_{0}^{\infty}[c(n)+e(n)] f(n) d n$ (the good market clearing condition). Given $y(n)=c(n)+e(n)$, we have that (iii) implies (ii), and that (ii) implies (iii).

### 3.1 Two parametric tax schemes

The stationary pdf $f(n)$ in Theorem 1 depends upon the income tax scheme $T($.$) . To be more$ concrete about the dependence, we consider two popular parametric tax schemes for $T($.$) in$ the literature: affine and constant rate of progressivity.

### 3.1.1 Affine tax scheme

Recall $z(n)=n l(n)$. The affine tax scheme is given by $T(z)=\tau z-m$, where $\tau$ is the constant marginal tax rate and $m$ is the lump-sum transfer. Browning and Johnson (1984)
argued that only the net effect of taxes and transfers is crucial for redistribution, and they provided evidence in support of the hypothesis that an affine income tax can have distributional implications similar to those resulting from the actual tax plus transfer system. By adapting a figure in Heathcote, Storesletten and Violante (2017), Bhandari, Evans, Golosov and Sargent (2017) showed that an affine tax scheme can approximate actual tax and transfer programs of the U.S. economy rather good.

With skill inequality across agents and the availability of the lump-sum transfer $m$, Werning (2007) noted that distributional concerns play a key role in determining the marginal tax rate $\tau$ of the affine tax scheme, since (p. 927) "a positive tax rate ensures that more productive, richer workers bear a heavier tax burden and alleviate that of less productive, poorer workers."

With the imposition of the affine tax, we have

$$
y(n)=z(n)-T(z(n))=(1-\tau) z(n)+m
$$

which leads to

$$
\frac{\beta y(n)}{n}=\frac{\beta[(1-\tau) z(n)+m]}{n} .
$$

### 3.1.2 CRP (constant rate of progressivity) tax scheme

As in STW, we define the local rate of progressivity of the tax schedule $T$ at income level $z$ as (minus) the elasticity of the retention rate $1-T^{\prime}(z)$ with respect to income $z$ as

$$
p(z) \equiv-\frac{\partial \ln \left(1-T^{\prime}(z)\right)}{\partial \ln z}=\frac{z T^{\prime \prime}(z)}{1-T^{\prime}(z)}
$$

The CRP tax scheme is defined by $p(z)=p$ for all $z$ with $T(z)=z-\frac{\lambda}{1-p} z^{1-p}, p<1$; see Benabou (2002), Heathcote, Storesletten and Violante (2017), and Sachs, Tsyvinsky and Werquin (2020). This tax scheme is proportional, progessive, and regressive, respectively, if $p=0, p>0$, and $p<0$. Heathcote, Storesletten and Violante (2017) showed that the CRP
tax scheme approximates the actual tax and transfer system of the U.S. economy pretty well. According to their estimation, $p=0.181$ for the U.S. economy.

Note that while $z(n)=0$ implies $T(z(n))=0$ in the CRP tax scheme, it implies $T(z(n))=$ $-m$ in the affine tax scheme. Thus, the CRP tax scheme is not a generalization of the affine tax scheme.

With the imposition of the CRP tax, we have

$$
y(n)=z(n)-T(z(n))=\frac{\lambda}{1-p} z(n)^{1-p},
$$

which leads to

$$
\frac{\beta y(n)}{n}=\frac{\beta \lambda}{1-p} \frac{z(n)^{1-p}}{n}
$$

### 3.1.3 Numerical results

Given that $f(n)$ in Theorem 1 depends upon the income tax scheme $T($.$) , we would like to$ know how the imposition of a specific tax scheme shapes $f(n)$. We conduct the numerical analysis to address it.

Define

$$
\kappa(n) \equiv \frac{v^{\prime}(l(n))}{l(n) v^{\prime \prime}(l(n))},
$$

which is the elasticity of labor supply $l(n)$ with respect to $r(n) \equiv 1-T^{\prime}(z(n))$ (i.e., the retention rate of agent $n$ ) as $T^{\prime}(z(n))=T^{\prime}$ for all $z(n) .{ }^{6}$ To conduct the numerical analysis, we assume a commonly-used functional form for $v(l)$ :

## Assumption 1

$$
v(l)=\frac{l^{1+\frac{1}{\kappa}}}{1+\frac{1}{\kappa}}, 1 / \kappa>0
$$

[^3]Under Assumption 1, $\kappa(n)=\kappa$ for all $n$. This is the case where Diamond (1998, Propositions 1-3) focused on when addressing the shape of the optimal tax schedule.

If $T(z(n))=\tau z(n)-m$ (affine), the FOC (2) under Assumption 1 gives

$$
\begin{equation*}
l(n)=[n(1-\tau)]^{\kappa} \tag{7}
\end{equation*}
$$

which leads to

$$
\begin{align*}
\frac{\beta y(n)}{n} & =\frac{\beta\left[(n(1-\tau))^{\kappa+1}+m\right]}{n} \\
& =\frac{\beta\left[n^{\kappa+1}\left((1-\tau)^{\kappa+1}+\frac{m}{n^{\kappa+1}}\right)\right]}{n} . \\
& =\frac{\beta n^{(\kappa+1)}\left((1-\tau)^{\kappa+1}+\frac{m}{n^{\kappa+1}}\right)}{n} . \\
\frac{\beta y(n)}{n} & =\frac{\beta n^{(\kappa+1)}\left((1-\tau)^{\kappa+1}+\frac{m}{n^{\kappa+1}}\right)}{n} . \tag{8}
\end{align*}
$$

If $T(z(n))=z(n)-\lambda z(n)^{1-p}$ (CRP), the FOC (2) under Assumption 1 gives

$$
\begin{equation*}
l(n)=\lambda^{\frac{\kappa}{1+p \kappa}} n^{\frac{\kappa(1-p)}{1+p \kappa}}, \tag{9}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
\frac{\eta(n)}{n}=\frac{\beta}{1-p} \lambda^{\left[1+\frac{\kappa(1-p)}{1+p \kappa}\right]} n^{(1-p)\left[1+\frac{\kappa(1-p)}{1+p \hbar}\right]-1} . \tag{10}
\end{equation*}
$$

We numerically calculate the following endogenous $f(n): \tau=0$ (no taxes), low $\tau$ and high $\tau$ in the case of affine; $p=0$ (proportional), $p>0$ (progressive), and $p<0$ (regressive) in the csase of CRP. In the calculation of both affine and CRP, we impose the balanced government budget constraint:

$$
\begin{gathered}
m+E=\tau \int_{0}^{\infty} z(n) f(n) d n \text { (affine), } \\
E=\int_{0}^{\infty}\left[z(n)-\lambda z(n)^{1-p}\right] f(n) d n(\mathrm{CRP}) .
\end{gathered}
$$

### 3.2 Earnings distributions

Recall $z(n)=n l(n)$. Let $f_{Z}(z)$ denote the pdf of $z$. While $z$ (earnings) is empirically observable, $n$ (earnings ability) is not. As such, it is often useful to express results in terms of $f_{Z}(z)$. In fact, the elementary tax refrom we focus on later is to raise the marginal tax rate at some specific earnings level $z^{*}$. Below we derive $f_{Z}(z)$ from $f(n)$.

It is known that $f_{Z}(z)$ and $f(n)$ are related through $f_{Z}(z) z^{\prime}=f(n)$; see Saez (2001, p. 215). Thus, we have

$$
f(n) d n=f_{Z}(z) z^{\prime}(n) d n=f_{Z}(z) d z
$$

We impose the following assmumption to regulate the relationship between $n$ and $z$ :
Assumption 2 The (strong) Spence-Mirrlees single crossing property is satisifed so that $z(n)$ is strictly increasing in $n$, i.e., $z^{\prime}(n)>0$.

The Spence-Mirrlees single crossing property, a standard assumption imposed in the optimal taxation literature, ensures that $z(n)$ is increasing in $n$. We impose a strong form, in that $z(n)$ is required to be strictly increasing in $n$. It is known that $z^{\prime}(n) \geq 0$ constitutes the second-order condition for agents' incentive compatibility in the optimal taxation literature; see Salanie (2011, chapter 4) for details. However, to abstract from bunching associated with $z^{\prime}(n)=0$, the assumption that $z^{\prime}(n)>0$ is often imposed explicitly or implicitly in theoretic analyses; see, for example, Diamond (1998), Ales, Kurnaz and Sleet (2015) and Scheuer and Werning (2017). Although the strong form of the Spence-Mirrlees single crossing property is not necessary for all of our results, it greatly facilitates our analysis. In our numerical simulations, we check if Assumption 2 does hold.

In the proof of Lemma 1 in the Appendix, we derive (see Eq. (17)):

$$
z^{\prime}(n)=l(n) \frac{1+\kappa(n)}{1+p(z(n)) \kappa(n)} .
$$

Given $\kappa(n)=\frac{v^{\prime}(l(n))}{l(n) v^{\prime \prime}(l(n))}$ so that $\kappa(n) \geq 0$ for all $n$ in our setting, a sufficient condition to ensure $z^{\prime}(n)>0$ with $l(n)>0$ is that $p(z(n)) \kappa(n)>-1$ for all $n$. Sachs, Tsyvinsky and Werquin
(2020) imposed this assumption, noting that it ensures that the second-order condition of the individual problem is satisfied. ${ }^{7}$ From now on, we impose Assumption 2 implicitly whenever it involves the function $n=n(z)$.

Let $n=n_{1}$ solve

$$
n\left[1-T^{\prime}(z(n))\right]=v^{\prime}(1)
$$

Thus, we have $l\left(n_{1}\right)=1$, and $z_{1}=z\left(n_{1}\right)=n_{1} l\left(n_{1}\right)=n_{1} .{ }^{8}$ We obtain:

## Lemma 1

$$
f_{Z}(z)=f(n(z)) \frac{1+p(z) \kappa(z)}{1+\kappa(z)} \frac{1}{l(z)}, z \geq 0
$$

where

$$
l(z)=\exp \left\{\int_{z_{1}}^{z} \frac{[1-p(\tilde{z})] \kappa(\tilde{z})}{\tilde{z}[1+\kappa(\tilde{z})]} d \tilde{z}\right\} .
$$

Lemma 1 implies

$$
n(z)=\frac{z}{l(z)}=z \exp \left\{-\int_{z_{1}}^{z} \frac{[1-p(\tilde{z})] \kappa(\tilde{z})}{\tilde{z}[1+\kappa(\tilde{z})]} d \tilde{z}\right\}, z \geq 0
$$

If $\kappa(z)=\kappa$ and $p(z)=p$ for all $z$, then

$$
n(z)=\frac{z}{l(z)}=z_{1}^{\frac{(1-p) \kappa}{1+\kappa}} z^{\frac{1+p \kappa}{1+\kappa}}, z \geq 0 .
$$

Thus, we have $n^{\prime}(z)>0$ as long as $p k>-1$.
Using the above lemma and imposing Assumption 1, in what follows we derive $f_{Z}(z)$ for the affine and the CRP tax scheme, respectively.

[^4]
### 3.2.1 Affine tax scheme

If the tax scheme is affine, we have $p(z)=0$ for all $z$. Under Assumption 1, we have $\kappa(z)=\kappa$ for all $z$. According to (7), we obtain

$$
n(z)=z^{\frac{1}{\kappa+1}}(1-\tau)^{\frac{-\kappa}{\kappa+1}} ; z_{1}=(1-\tau)^{-1}
$$

Thus, Lemma 1 leads to

$$
\begin{aligned}
f_{Z}(z) & =f(n(z)) \frac{1}{1+\kappa} \exp \left\{-\frac{\kappa}{1+\kappa} \int_{z_{1}}^{z} \frac{1}{\tilde{z}} d \tilde{z}\right\} \\
& =f(n(z)) \frac{1}{1+\kappa}\left(\frac{z}{z_{1}}\right)^{-\frac{\kappa}{1+\kappa}} \\
& =\frac{f(1)}{1+\kappa}(1-\tau)^{1+\frac{2 \kappa}{\kappa+1}} z^{\frac{-2-\kappa}{\kappa+1}} \exp \left(\frac{2}{\sigma^{2}} \int_{z(1)}^{z} \frac{\beta y(n(\tilde{z}))}{n(\tilde{z}) \tilde{z}^{\prime}} d \tilde{z}\right) \quad \text { (using Theorem) } \\
& =\frac{f(1)}{1+\kappa}(1-\tau)^{1+\frac{2 \kappa}{\kappa+1}} z^{-1-\frac{1}{\kappa+1}} \exp \left(\frac{2}{\sigma^{2}} \int_{z(1)}^{z} \frac{\beta y(n(\tilde{z}))}{\tilde{z}(1+\kappa)} d \tilde{z}\right), \quad \text { (using Eq.(17)). }
\end{aligned}
$$

### 3.2.2 CRP tax scheme

If the tax scheme is CRP, we have $p(z)=p$ for all $z$. Under Assumption 1, we have $\kappa(z)=\kappa$ for all $z$. According to (9), we obtain

$$
n(z)=[\lambda(1-p)]^{\frac{-\kappa}{1+\kappa}} z^{\frac{1+p \kappa}{1+\kappa}} ; z_{1}=[\lambda(1-p)]^{\frac{-\kappa}{1-p) \kappa}} .
$$

Thus, Lemma 1 leads to

$$
\begin{aligned}
f_{Z}(z) & =f(n(z)) \frac{1+p \kappa}{1+\kappa} \exp \left\{-\frac{(1+p) \kappa}{1+\kappa} \int_{z_{1}}^{z} \frac{1}{\tilde{z}} d \tilde{z}\right\} \\
& =f(n(z)) \frac{1+p \kappa}{1+\kappa}\left(\frac{z}{z_{1}}\right)^{-\frac{(1+p) \kappa}{1+\kappa}} \\
& =f(1) \frac{1+p \kappa}{1+\kappa}[\lambda(1-p)]^{\frac{2 \kappa}{1+\kappa}-\frac{\kappa}{(1-p) \kappa} \frac{(1+p) \kappa}{1+\kappa}} z^{-2 \frac{1+p \kappa}{1+\kappa}-\frac{(1+p) \kappa}{1+\kappa}} \exp \left(\frac{2}{\sigma^{2}} \int_{z(1)}^{z} \frac{\beta y(n(\tilde{z}))}{n(\tilde{z}) \tilde{z}^{\prime}} d \tilde{z}\right) \quad \text { (using Theorem) } \\
& =f(1) \frac{1+p \kappa}{1+\kappa}[\lambda(1-p)]^{\frac{\kappa}{1+\kappa}\left[2-\frac{1+p}{1-p}\right]} z^{-\frac{2+k+3 p k}{1+\kappa}} \exp \left(\frac{2}{\sigma^{2}} \int_{z(1)}^{z} \frac{\beta y(n(\tilde{z}))}{\tilde{z} \frac{1+\kappa}{1+p \kappa}} d \tilde{z}\right), \quad \text { (using Eq.(17)). }
\end{aligned}
$$

## 4 Incidence of tax reforms

Adopting the variational approach as in the work of Piketty (1997), Saez (2001), Golosov, Tsyvinsky and Werquin (2014), Sachs, Tsyvinsky and Werquin (2020), and Chang and Park (2020), this section considers an initial, potentially suboptimal, tax schedule $T($.$) and de-$ rives the tax incidence of arbitrary local perturbations of this tax schedule ("tax reforms"). Although tax incidence belongs to a positive analysis, its derivation will pave the way for delivering the characterization of the optimal tax schedule. Our analysis follows Sachs, Tsyvinsky and Werquin (2020) (STW hereafter) closely. As noted in the Introduction, a major difference between their work and our work is that while they address endogenous wages, we address endogenous skill distributions.

Consider an arbitrary reform of the initial tax schedule $T($.$) , which can be represented by$ $T()+.b \hat{T}($.$) , where \hat{T}($.$) is a continuously differentiable function on \mathbb{R}_{+}$and $b \in \mathbb{R}$ parameterizes the size of the reform. We derive the first-order effect of this perturbation on individual utility, labor supply, and skill distribution in this section and on government revenue in the next section. As exposed in STW, this first-order effect can be formally represented by the Gateaux derivative in the direction of $\hat{T}($.$) :$

$$
\hat{x}(n) \equiv \lim _{b \rightarrow 0} \frac{1}{b}[x(n ; T+b \hat{T})-x(n ; T)]
$$

where $x$ represents a variable and $\hat{x}(n)$ denotes the change in $x(n ; T)$ in response to the tax reform $\hat{T}$. The Gateaux derivative implies approximately (in the sense of the first-order effect with a small $b$ ):

$$
\begin{equation*}
x(n ; T+b \hat{T})=x(n ; T)+b \hat{x}(n) . \tag{11}
\end{equation*}
$$

For convenience, we often express $x(n ; T)$ simply as $x(n)$.

### 4.1 Effects on individual utility and labor supply

Let $U(n)$ denote the utility attained by type- $n$ agents. We have:

Proposition 1 The effect of a tax reform $\hat{T}$ of the initial tax schedule $T$ on individual utility, $\hat{U}(n)$, and individual labor supply, $\hat{l}(n)$, satisfies

$$
\begin{gathered}
\hat{U}(n)=-\hat{T}(z(n)) \\
\frac{\hat{l}(n)}{l(n)}=-\varepsilon(n) \frac{\hat{T}^{\prime}(z(n))}{1-T^{\prime}(z(n))}
\end{gathered}
$$

where

$$
\varepsilon(n) \equiv \frac{\partial \ln l(n)}{\partial \ln \left(1-T^{\prime}(z(n))\right)}=\frac{\kappa(n)}{1+p(z(n)) \kappa(n)}
$$

The first result of the proposition is the same as Eq. (14) derived by STW if we ignore the part that is related to the endogenous wages in their derivation. It shows that the tax reform $\hat{T}$ reduces $U(n)$ exactly by $\hat{T}(z(n))$. Kelven (2020, p. 6) succinctly summarized this well-known result: "the utility effect of any arbitrary, small reform equals the mechanical revenue effect." (italics original)

The second result of the proposition is the same as Eq. (9) derived by STW if we ignore the part that is related to the endogenous wages in their derivation. As explained in STW, the tax reform causes a percentage change in the retention rate $\frac{\hat{r}(z)}{r(z)}=-\frac{\hat{T}^{\prime}(z(n))}{1-T^{\prime}(z(n))}$, which induces a percentage change in labor supply $\frac{\hat{l}(n)}{l(n)}$ equal to $\varepsilon(n) \frac{\hat{r}(z)}{r(z)}$.

### 4.2 Effects on skill distribution

Applying (11) gives

$$
f(n ; T+b \hat{T})=f(n)+b \hat{f}(n)
$$

We obtain:
Proposition 2 The effect of a tax reform $\hat{T}$ of the initial tax schedule $T$ on skill distribution, $\hat{f}(n)$, satisfies

$$
\frac{\hat{f}(n)}{f(n)}=\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}}\left[\int _ { 1 } ^ { n } \frac { 1 } { \tilde { n } } \left(\left[1-T^{\prime}(z(\tilde{n})] z\left(\tilde{n} \frac{\hat{l}(\tilde{n})}{l(\tilde{n})}-\hat{T}(z(\tilde{n}))\right) d \tilde{n}\right], n \geq 0\right.\right.
$$

where $\frac{\hat{f}(1)}{f(1)}$ is determined by $\int_{0}^{\infty} \frac{\hat{f}(n)}{f(n)} f(n) d n=0$.

Theorem 1 tells us that the stationary pdf of skills, $f(n)$, depends upon the after-tax income $y(n)=z(n)-T(z(n))$ and hence on the tax scheme $T($.$) . The above result shows$ that the effect of the tax reform on $\frac{\hat{f}(n)}{f(n)}$ is via two channels: (i) the tax reform itself, $\hat{T}($.$) ,$ and (ii) the induced effect through $\frac{\hat{l}(n)}{l(n)}$. This result is absent in STW, since the distribution of their agent types is exogenously given and thereby $\frac{\hat{f}(n)}{f(n)}=0$.

## 5 Effects on government revenue

Government revenue is given by

$$
\Re(T)=\int_{0}^{\infty} T(z(n)) f(n) d n
$$

and hence

$$
\Re(T+b \hat{T})=\int_{0}^{\infty}[T(z(n ; T+b \hat{T}))+b \hat{T}(z(n ; T+b \hat{T}))] f(n ; T+b \hat{T}) d n
$$

We show in the proof of Proposition 3:

$$
\begin{aligned}
\hat{\Re}(\hat{T}) & \equiv \lim _{b \rightarrow 0} \frac{1}{b}[\Re(T+b \hat{T})-\Re(T)] \\
& =\int_{0}^{\infty} \hat{T}(z(n)) f(n) d n+\int_{0}^{\infty} T^{\prime}(z(n)) \frac{\hat{l}(n)}{l(n)} z(n) f(n) d n+\int_{0}^{\infty} T(z(n)) \frac{\hat{f}(n)}{f(n)} f(n) d n,
\end{aligned}
$$

where the first term of $\hat{\Re}(\hat{T})$ represents the mechanical effect of the tax reform, the second term the behavioral response due to changes in $l(n)$, and the third term the impact due to changes in $f(n)$. This result is the same as Eq. (15) derived by STW if (i) letting $\hat{f}(n) \equiv 0$ in our derived $\hat{\Re}(\hat{T})$ and (ii) ignoring the part that is related to the endogenous wages in their derivation.

### 5.1 Elementary tax reform

Let $F_{Z}(z)$ denote the cdf of $z$. As in STW, we focus on the elementary tax reform:

$$
\hat{T}^{\prime}\left(z ; z^{*}\right)=\frac{1}{1-F_{Z}\left(z^{*}\right)} \delta\left(z-z^{*}\right)
$$

where the marginal tax rates are perturbed by the Dirac delta function $\delta($.$) at z^{*}$. The elementary tax reform consists of raising the marginal tax rate by $\frac{1}{1-F_{Z}\left(z^{*}\right)}$ at only one earnings level $z^{*}$, implying that agents with earnings no less than $z^{*}$ all increase their tax payment by a constant amount $\frac{1}{1-F_{Z}\left(z^{*}\right)}$, that is,

$$
\hat{T}\left(z ; z^{*}\right)=\frac{1}{1-F_{Z}\left(z^{*}\right)} I_{\left\{z \geq z^{*}\right\}}, \text { where } I_{\left\{z \geq z^{*}\right\}}=\left\{\begin{array}{ll}
1 & \text { if } z \geq z^{*} \\
0 & \text { if } z<z^{*}
\end{array} .\right.
$$

The focus on the elementary tax reform is without loss of generality, since any other tax reforms can be expressed as a weighted sum of elementary tax reforms. ${ }^{9}$ We use notation $\hat{\Re}\left(z^{*}\right)$ to replace $\hat{\Re}(\hat{T})$ when tax reform is elementary.

We have the mechanical effect,

$$
\int_{0}^{\infty} \hat{T}\left(z(n) ; z^{*}\right) f(n) d n=\frac{1}{1-F_{Z}\left(z^{*}\right)} \int_{z^{*}}^{\infty} f_{Z}(z) d z=1, \quad\left(\left[\begin{array}{c}
\text { first term of } \hat{\Re}(\hat{T}) \text { and } \\
\text { by definition of } \hat{T}\left(z ; z^{*}\right)
\end{array}\right]\right)
$$

and the behavioral response,

$$
\begin{aligned}
& \left.\left.\int_{0}^{\infty} T^{\prime}(z(n)) \frac{\hat{l}(n)}{l(n)} z(n) f(n) d n\right|_{\hat{T}=\hat{T}\left(z(n) ; z^{*}\right)} \quad \text { (second term of } \hat{\Re}(\hat{T})\right) \\
& =-\int_{0}^{\infty} T^{\prime}(z) \varepsilon(z) \frac{\hat{T}^{\prime}\left(z ; z^{*}\right)}{1-T^{\prime}(z)} z f_{Z}(z) d z \text { (using Proposition 1) } \\
& =-\varepsilon\left(z^{*}\right) \frac{T^{\prime}\left(z^{*}\right)}{1-T^{\prime}\left(z^{*}\right)} \frac{z^{*} f_{Z}\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)} .\left(\text { by definition of } \hat{T}^{\prime}\left(z ; z^{*}\right)\right) .
\end{aligned}
$$

The elementary tax reform implies that the mechanical effect of the tax reform on government revenue is exactly equal to $\$ 1$. If the skill distribution were exogenous, $\hat{f}(n) \equiv 0$ and the third

[^5]term of $\hat{\Re}(\hat{T})$ would vanish. Then the incidence of the elementary tax reform at income $z^{*}$ on tax revenue would equal the sum of the first and second terms of $\hat{\Re}\left(z^{*}\right)$ (i.e., the mechanical effect plus the behavioral response):
$$
\hat{\Re}_{e x}\left(z^{*}\right) \equiv 1-\varepsilon\left(z^{*}\right) \frac{T^{\prime}\left(z^{*}\right)}{1-T^{\prime}\left(z^{*}\right)} \frac{z^{*} f_{Z}\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)},
$$
which is the same as Eq. (16) in STW and that derived by Diamond (1998) and Saez (2001). As noted in STW, the second term of $\hat{\Re}_{e x}\left(z^{*}\right)$ represents the marginal excess burden of tax reform, which captures the loss in government revenue due to the behavior response of agents.

Evaluating the third term of $\hat{\Re}(\hat{T})$ with the elementary tax reform gives rise to:

Proposition 3 The effect of the elementary tax reform at income $z^{*}$ on government revenue, $\hat{\Re}\left(z^{*}\right)$, is given by

$$
\hat{\Re}\left(z^{*}\right)=\hat{\Re}_{e x}\left(z^{*}\right)+\frac{2 \beta}{\sigma^{2}} \frac{\chi\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}
$$

where $\chi\left(z^{*}\right)=\left\{\begin{array}{c}{\left[\begin{array}{c}-\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)}\left[T\left(z^{*}\right)-E\right] f_{Z}\left(z^{*}\right) \\ -\int_{z^{*}}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right)[T(z)-E] f_{Z}(z) d z\end{array}\right] \quad \text { if } z^{*} \geq z(1) ;} \\ {\left[\begin{array}{c}\kappa\left(z^{*}\right) \\ 1+\kappa\left(z^{*}\right)\end{array} f_{Z}\left(z^{*}\right)\left[T\left(z^{*}\right)-E\right]\right.} \\ +\int_{z^{*}}^{z(1)}\left(\int_{z(1)}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right)[T(z)-E] f_{Z}(z) d z \\ -\int_{z(1)}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right)[T(z)-E] f_{Z}(z) d z\end{array}\right] \quad$ if $z^{*}<z(1)$.
Note that $E=\int_{0}^{\infty} T(z) f_{Z}(z) d z$ (the balanced government budget). Proposition 2 shows that the effect of the tax reform on $\frac{\hat{f}(n)}{f(n)}$ consists of (i) the tax reform itself, $\hat{T}$ (.), and (ii) the induced effect through $\frac{\hat{l}(n)}{l(n)}$. The term $\chi\left(z^{*}\right)$ in Proposition 3 displays this effect on government revenue in the case of the elementary tax reform. If $\beta=0$ (no education expenditures in (1-2)), we would have $\hat{\Re}\left(z^{*}\right)=\hat{\Re}_{e x}\left(z^{*}\right)$.

Suppose that $p(z) \geq 0$ and $\kappa(z)=\kappa>0$ for all $z$. Proposition 3 tells us that whether $\chi\left(z^{*}\right)$ is positive or negative critically depends on $T(z)$ relative to $E$ at $z^{*}$ or above. Let us consider $z^{*} \geq z(1)$ and $z^{*}<z(1)$, separately.

In the case where $z^{*} \geq z(1), T(z)>E$ tends to hold at high $z$ and the opposite tends to hold at low $z$. Therefore, according to Proposition 3, the presence of the term $\chi\left(z^{*}\right)$ provides a tendency to uphold the result that $\hat{\Re}\left(z^{*}\right)<\hat{\Re}_{e x}\left(z^{*}\right)$ at high $z^{*}$ but $\hat{\Re}\left(z^{*}\right)>\hat{\Re}_{e x}\left(z^{*}\right)$ at low $z^{*}$.

In the case where $z^{*}<z(1)$, the first two terms of $\chi\left(z^{*}\right)$ tend to be negative because $T(z)<E$ tends to hold. As to the third term of $\chi\left(z^{*}\right)$, its sign tends to be negative as well, given that those $z$ below $z(1)$ are excluded.

Putting together, the presence of $\chi\left(z^{*}\right)$ tends to uphold $\hat{\Re}\left(z^{*}\right)<\hat{\Re}_{e x}\left(z^{*}\right)$ at high or low $z^{*}$, whereas it tends to uphold $\hat{\Re}\left(z^{*}\right)>\hat{\Re}_{e x}\left(z^{*}\right)$ for middle $z^{*}$ which are neither high nor low.

### 5.2 Numerical simulations

We calibrate our model to the U.S. economy and evaluate quantitatively the effects of elementary tax reforms on government revenue using the formula in Proposition 3.

We impose Assumption 1 and choose $\kappa=0.33$ as in STW.
Using PSID data for years 2000-2006, Heathcote, Storesletten and Violante (2017) showed that the CRP tax scheme approximates the actual tax and transfer system of the U.S. economy pretty well. According to their estimation, $p=0.181$ for the U.S. economy. However, this estimation may need be modified if including the very rich seriously. First, as noted by Heathcote, Storesletten and Violante (2017), the PSID undersamples the very rich. Second, note that the local rate of progressivity of the tax schedule $T$ at income level $z$ is given by $p(z)=\frac{z T^{\prime \prime}(z)}{1-T^{\prime}(z)}$. The CRP tax scheme is defined by $p(z)=p$ for all $z$, but the very rich typically face $p(z)=0$ rather than $p(z)>0$ for $z$ above some threshold $\bar{z}$. For example, the top marginal tax rate of the U.S. federal personal income tax in 2021 remains at $37 \%$ for those individuals whose taxable incomes are $\$ 523,600$ and higher. Put differently, the current U.S. personal income tax code features $p=0$ rather than $p>0$ for $z$ above $\bar{z}=\$ 523,600$. In view of
these two points, we let the CRP tax scheme be the one estimated by Heathcote, Storesletten and Violante (2017) up to a threshold earnings, above which we append a different CRP tax scheme with $p=0$. This different CRP tax scheme de fact imposes a flat rate on income above the threshold.

With the tax scheme described, both equations (9) and (10) remain unchanged, except that $p=0.181$ if $z \leq \bar{z}$ and $p=0$ if $z>\bar{z}$ (recall that $n=\lambda^{\frac{-\kappa}{1+\kappa}} z^{\frac{1+p \kappa}{1+\kappa}}$ ). As a result, the corresponding $f_{Z}(z)$ reamins the same as that reported in Section 3.2.2, except that $p=0.181$ if $z \leq \bar{z}$ and $p=0$ if $z>\bar{z}$.

We calibrate the parameters, $\rho$ and $\sigma$, by matching with the earnings distribution of the U.S. economy documented by Diaz-Gíménez, Glover and Ríos-Rull (2011). Table 1 reports the matching result. Figure 1 reports the resulting hazard ratio $\frac{1-F_{Z}(z)}{z f_{Z}(z)}$.

We have

$$
\hat{\Re}_{e x}\left(z^{*}\right)=1-\varepsilon\left(z^{*}\right) \frac{T^{\prime}\left(z^{*}\right)}{1-T^{\prime}\left(z^{*}\right)} \frac{z^{*} f_{Z}\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}
$$

where $\varepsilon\left(z^{*}\right)=\frac{\kappa}{1+p \kappa}$ and $T^{\prime}\left(z^{*}\right)=1-\lambda\left(z^{*}\right)^{-p}$ with $p=0.181$ if $z^{*} \leq \bar{z}$ and $p=0$ if $z^{*}>\bar{z}$.
*We let the US tax schedule be represented by CRP with parameters $p=0.181$ as in STW. As such,

$$
\begin{aligned}
& \varepsilon\left(z^{*}\right)=\frac{\kappa\left(z^{*}\right)}{1+p\left(z^{*}\right) \kappa\left(z^{*}\right)}=\frac{0.33}{1+0.151 \times 0.33}= \\
& T(z)=z-\frac{\lambda}{1-p} z^{1-p}=z-\frac{4}{0.849} z^{0.849} \\
& T^{\prime}\left(z^{*}\right)=1-\lambda\left(z^{*}\right)^{-p}=1-4\left(z^{*}\right)^{-0.151}
\end{aligned}
$$

From Lemma 1,

$$
\begin{aligned}
& l(z)=\exp \left\{\int_{z_{1}}^{z} \frac{[1-p(\tilde{z}]) \kappa(\tilde{z})}{\tilde{z}[1+\kappa \tilde{z})]} d \tilde{z}\right\}=\exp \left\{\int_{z_{1}}^{z} \frac{0.894 \times 0.33}{\tilde{z}[1+0.33]} d \tilde{z}\right\}=\exp \left\{\frac{0.894 \times 0.33}{1.33} \int_{z_{1}}^{z} \frac{1}{\tilde{z}} d \tilde{z}\right\}=\frac{z}{z_{1}} \exp \left\{\frac{0.894 \times 0.33}{1.33}\right\} \\
& z_{1}=\lambda^{\frac{-\kappa}{(1-p) \kappa}}=4^{0.849 \times 0.33}
\end{aligned}
$$

From Lemma 1,

$$
f_{Z}(z)=f(n(z)) \frac{1+p(z) \kappa(z)}{1+\kappa(z)} \frac{1}{l(z)}=f(n(z)) \frac{1+0.151 \times 0.33}{1+0.33} \frac{z_{1}}{z} \exp \left\{-\frac{0.894 \times 0.33}{1.33}\right\}
$$

where $n(z)=\lambda^{\frac{-\kappa}{1+\kappa}} z^{\frac{1+p \kappa}{1+\kappa}}=4^{\frac{-0.33}{1.33}} z^{\frac{1+0.151 \times 0.33}{1.33}}$

## 6 Optimal taxation

As is standard in the literature, the government is assumed to maximize

$$
W \equiv \int_{0}^{\infty} \frac{1}{q} G(U(n)) f(n) d n
$$

where $G(U(n))$ is a social welfare function with $G($.$) being increasing and concave in U(n)$ (the concavity is to reflect the redistributive motive of the government), and $q$ denotes the marginal value of public funds. ${ }^{10}$ The government needs to keep its budget balanced:

$$
E=\int_{0}^{\infty} T(z) f_{Z}(z) d z
$$

From the work of Saez (2001) and STW, we know that a necessary condition for optimal taxation is

$$
\hat{\Re}\left(z^{*}\right)+\left.\int_{0}^{\infty} \hat{U}(n(z))\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} g(n(z)) f_{Z}(z) d z=0
$$

where $g(n(z))=G^{\prime}(U(n(z))) / q$ represents the social marginal welfare weight on agent $n$, expressed in terms of the marginal value of public funds. That is, optimal taxation satisfies the condition that there exists no tax reform that has a positive effect on social welfare. However, due to the property that $f(n)$ is endogenous rather than exogenously specified in our setting, this necessary condition need be modified to
$\hat{\Re}\left(z^{*}\right)+\left.\int_{0}^{\infty} \hat{U}(n(z))\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} g(n(z)) f_{Z}(z) d z+\left.\int_{0}^{\infty} \frac{1}{q} G(U(n(z))) \frac{\hat{f}(n(z))}{f(n(z))}\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} f_{Z}(z) d z=0$.
Let $W(z) \equiv \frac{1}{q} G(U(n(z)))$. Following the same logic of proving Proposition 3, we have

$$
\left.\int_{0}^{\infty} \frac{1}{q} G(U(n(z))) \frac{\hat{f}(n(z))}{f(n(z))}\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} f_{Z}(z) d z=\frac{2 \beta}{\sigma^{2}} \frac{\psi\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}
$$

[^6]where $\psi\left(z^{*}\right)=\left\{\begin{array}{c}{\left[\begin{array}{c}-\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)}\left[W\left(z^{*}\right)-W\right] f_{Z}\left(z^{*}\right) \\ -\int_{z^{*}}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right)[W(z)-W] f_{Z}(z) d z\end{array}\right] \quad \text { if } z^{*} \geq z(1) ;} \\ {\left[\begin{array}{c}\frac{\kappa\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} f_{Z}\left(z^{*}\right)\left[W\left(z^{*}\right)-W\right] \\ +\int_{z^{*}}^{z(1)}\left(\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa \kappa \tilde{z}))} d \tilde{z}\right)[W(z)-W] f_{Z}(z) d z \\ -\int_{z(1)}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right)[W(z)-W] f_{Z}(z) d z\end{array}\right] \quad \text { if } z^{*}<z(1) .}\end{array}\right.$
Note that $\psi\left(z^{*}\right)$ has the same form as $\chi\left(z^{*}\right)$ of Proposition 3, in that $W(z)$ is in place of $T(z)$ while $W$ is in place of $E$. This result is not surprising in view of that $\left.\int_{0}^{\infty} W(z) \frac{\hat{f}(n(z))}{f(n(z))}\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} f_{Z}(z) d z$ is in place of $\left.\int_{0}^{\infty} T(z) \frac{\hat{f}(n(z))}{f(n(z))}\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} f_{Z}(z) d z$ in the derivation.

Using Propositions 1 and 3, the necessary condition for optimality yields

$$
\begin{aligned}
& \underbrace{-\int_{0}^{\infty} \hat{T}\left(z(n) ; z^{*}\right) g(z(n)) f_{Z}(z) d z}_{\hat{U}(n(z))} \\
& +\underbrace{\int_{0}^{\infty} \hat{T}\left(z(n) ; z^{*}\right) f_{Z}(z) d z}_{\text {mechanical effect }}-\underbrace{\varepsilon\left(z^{*}\right) \frac{T^{\prime}\left(z^{*}\right)}{1-T^{\prime}\left(z^{*}\right)} \frac{z^{*} f_{Z}\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}}_{\text {behavioral response }}+\underbrace{\frac{2 \beta}{\sigma^{2}} \frac{\chi\left(z^{*}\right)+\psi\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}}_{\frac{f(n)}{f(n)}} \\
& =\underbrace{-\frac{\int_{z^{*}}^{\infty} g(z) f_{Z}(z) d z}{1-F_{Z}\left(z^{*}\right)}}_{\hat{U}(n(z))}+\underbrace{1}_{\text {mechanical effect }}-\underbrace{-\varepsilon\left(z^{*}\right) \frac{T^{\prime}\left(z^{*}\right)}{1-T^{\prime}\left(z^{*}\right)} \frac{z^{*} f_{Z}\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}}_{\hat{U}(n(z))}+\underbrace{\frac{2 \beta}{\sigma^{2}} \frac{\chi\left(z^{*}\right)+\psi\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}}_{\text {behavioral response }} \\
& =0 .
\end{aligned}
$$

We then obtain

Proposition 4 The optimal marginal tax rate at income $z^{*}$ is given by

$$
\frac{T^{\prime}\left(z^{*}\right)}{1-T^{\prime}\left(z^{*}\right)}=\frac{1}{\varepsilon\left(z^{*}\right)} \frac{1-F_{Z}\left(z^{*}\right)}{z^{*} f_{Z}\left(z^{*}\right)}\left\{\begin{array}{c}
{\left[1-\frac{\int_{z^{*}}^{\infty} g(z) f_{Z}(z) d z}{1-F_{Z}\left(z^{*}\right)}\right]}  \tag{12}\\
+\frac{2 \beta}{\sigma^{2}} \frac{\chi\left(z^{*}\right)+\psi\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}
\end{array}\right\} .
$$

The first term on the right-hand side of (12) is derived by Diamond (1998) and Saez (2001) under the setting where the distribution of people's skills is exogenous. ${ }^{11}$ It shows

[^7]that the optimal marginal tax rate at income $z^{*}$ is decreasing in $\varepsilon\left(z^{*}\right)$ (the labor supply elasticity w.r.t. the retention rate), $\frac{z^{*} f_{Z}\left(z^{*}\right)}{1-F_{Z}\left(z^{*}\right)}$ (the hazard rate of the earnings distribution), and $\int_{z^{*}}^{\infty} g(z) f_{Z}(z) d z /\left[1-F_{Z}\left(z^{*}\right)\right]$ (the average social marginal welfare weight of earnings above $\left.z^{*}\right)$. The second term on the right-hand side of (12) is new and it represents our modification to the conventional formula by letting the distribution of people's skills become endogenous according to the human capital accumulation process (1). When the formula (12) is evaluated at $\beta=0, f(n)$ becomes exogenously given and, therefore, the formula (12) collapses to the result derived by Diamond (1998) and Saez (2001).

After presenting Proposition 3, we explain that the presence of $\chi\left(z^{*}\right)$ tends to uphold $\hat{\Re}\left(z^{*}\right)<\hat{\Re}_{e x}\left(z^{*}\right)$ at high or low $z^{*}$, whereas it tends to uphold $\hat{\Re}\left(z^{*}\right)>\hat{\Re}_{e x}\left(z^{*}\right)$ for middle $z^{*}$ which are neither high nor low. We see from (12) that this tendency provides a force to cut down $T^{\prime}\left(z^{*}\right)$ for high or low $z^{*}$ but raise up $T^{\prime}\left(z^{*}\right)$ for middle $z^{*}$. Given that $\psi\left(z^{*}\right)$ takes the same form as $\chi\left(z^{*}\right)$ and that $G(U(z))$ is increasing in $U(z)$, the presence of $\psi\left(z^{*}\right)$ in (12) reinforces the force provided by $\chi\left(z^{*}\right)$.

From now on, we impose Assumption 1 so that $\kappa(z)=\kappa$. As noted earlier, this is the case where Diamond (1998, Propositions 1-3) focused on. Under Assumption 1, we have

$$
\begin{equation*}
l(n)=\left[n\left(1-T^{\prime}(z(n))\right)\right]^{\kappa} . \tag{13}
\end{equation*}
$$

To evaluate the right-hand side of (12), we need three pieces of inputs regarding the shape of the tax schedule: $T(z), T^{\prime}(z)$ and $T^{\prime \prime}(z)$. Given a set $\left\{T^{\prime}\left(z_{i}\right)\right\}$ in which $T^{\prime}\left(z_{i}\right)$ denotes the marginal tax rate imposed on income $z \in\left[z_{i-1}, z_{i}\right)$, we calculate $\left\{T^{\prime \prime}\left(z_{i}\right)\right\}$ and $\left\{T(z) \mid z \in\left[z_{i-1}, z_{i}\right)\right\}$ as follows:

$$
\begin{gathered}
\left.T^{\prime \prime}\left(z_{i}\right) \approx \frac{T^{\prime}\left(z_{i+1}\right)-T^{\prime}\left(z_{i}\right)}{z_{i+1}-z_{i}} \text { (a finite difference approximation to } T^{\prime \prime}(.)\right), \\
T(z)=T(0)+\sum_{j=1}^{i-1} T^{\prime}\left(z_{j}\right)\left(z_{j}-z_{j-1}\right)+T^{\prime}\left(z_{i}\right)\left(z-z_{i-1}\right), \quad z_{0}=0, z \in\left[z_{i-1}, z_{i}\right)
\end{gathered}
$$

where $T(0)$ is a lump-sum transfer that can be determined through the balanced government budget constraint.

Starting with an initial set $\left\{T^{\prime}\left(z_{i}\right)\right\}$, we use the formula (12) to compute a new set $\left\{T^{\prime}\left(z_{i}\right)\right\}^{\prime}$. This loop is repeated until a fixed-point $\left\{T^{\prime}\left(z_{i}\right)\right\}^{*}$ (i.e. $\left.\left\{T^{\prime}\left(z_{i}\right)\right\}=\left\{T^{\prime}\left(z_{i}\right)\right\}^{\prime}\right)$ is found. We use the optimal marginal tax rates obtained at $\beta=0$ as the initial $\left\{T^{\prime}\left(z_{i}\right)\right\}$ and investigate how $\left\{T^{\prime}\left(z_{i}\right)\right\}^{*}$ differs from the initial $\left\{T^{\prime}\left(z_{i}\right)\right\}$.

## 7 Asymptotic skill distributions and optimal marginal tax rates

To addres asymptotic optimal marginal tax rates, we need to first address asymptotic skill distributions. We show that $f(n)$ in Theorem 1 has a Pareto tail asymptotically.

### 7.1 Asymptotic skill distributions

We follow a constructuve proof as in Achdou, Han, Lasry, Lions, and Moll (2022, proof of Proposition 10).

Integrating both sides of 5 from $n_{1}$ to $n_{2}$ with $n_{1}<n_{2}$, where both $n_{1}$ and $n_{2}$ are large enough, we have

$$
\int_{n_{1}}^{n_{2}} \frac{f^{\prime}(\tilde{n})}{f(\tilde{n})} d \tilde{n}=\frac{2 \beta}{\sigma^{2}} \int_{n_{1}}^{n_{2}} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}-\int_{n_{1}}^{n_{2}} \frac{2}{\tilde{n}} d \tilde{n}
$$

Note that, for $n \in\left[n_{1}, n_{2}\right]$, there exists a positive constant $\bar{C}<\infty$ such that $n y(n) \leq \bar{C}$ and hence $y(n) / n \leq \bar{C} / n^{2}$. We then have

$$
\int_{n_{1}}^{n_{2}} \frac{f^{\prime}(\tilde{n})}{f(\tilde{n})} d \tilde{n}+\int_{n_{1}}^{n_{2}} \frac{2}{\tilde{n}} d \tilde{n}=\frac{2 \beta}{\sigma^{2}} \int_{n_{1}}^{n_{2}} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n} \leq \frac{2 \beta}{\sigma^{2}} \int_{n_{1}}^{n_{2}} \frac{\bar{C}}{\tilde{n}^{2}} d \tilde{n}=\frac{2 \beta}{\sigma^{2}} \bar{C}\left(\frac{1}{n_{1}}-\frac{1}{n_{2}}\right)
$$

which leads to

$$
\ln \left[f\left(n_{2}\right) n_{2}^{2}\right]-\ln \left[f\left(n_{1}\right) n_{1}^{2}\right] \leq \frac{2 \beta}{\sigma^{2}} \bar{C}\left(\frac{1}{n_{1}}-\frac{1}{n_{2}}\right)
$$

Therefore, there exists $\bar{\xi}<\infty$ such that

$$
\lim _{n \rightarrow \infty} \ln \left[f(n) n^{2}\right]=\bar{\xi}
$$

Equivalently, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} f(n)=\exp (\bar{\xi}) n^{-2} \tag{14}
\end{equation*}
$$

that is, $f(n)$ converges to a Pareto distribution in the limit.
To sum up, we state:

Proposition 5 The stationary $p d f f(n)$ in Theorem 1 follows an asymptotic power law with $\lim _{n \rightarrow \infty} f(n)=\exp (\bar{\xi}) n^{-2}$.

This result shows that while $f(n)$ in Theroem 1 is a modified Pareto distribution, $f(n)$ as $n \rightarrow \infty$ is a Pareto distribution.

Now we verify that there must be $C=0$ for equation (4). Integrating (4) gives

$$
\begin{equation*}
\frac{1}{2}\left[\sigma^{2} n^{2} f(n)\right]=\int_{0}^{n}\left[(\beta y(\tilde{n}))^{\rho} \tilde{n} f(\tilde{n})\right] d \tilde{n}+C n+c, \tag{15}
\end{equation*}
$$

where $c$ is some constant. Given $d n(t)=\beta y(n(t)) n(t) d t+\sigma n(t) d B(t)$, a stationary $f(n)$ implies: $0=\int_{i} d n(t)=\int_{i} \beta y(n(t)) n(t) d t$, where notation $\int_{i}$ denotes a summation across all agents. Therefore,

$$
0=\int_{0}^{\infty} \beta y(n) n f(n) d n
$$

Given the result shown in (14), the left side of (15) converges to $\frac{1}{2} \sigma^{2} \exp (\bar{\xi})$, a constant, as $n \rightarrow \infty$. With $\int_{0}^{n} \beta y(\tilde{n}) \tilde{n} f(\tilde{n}) d \tilde{n} \rightarrow 0$ as $n \rightarrow \infty$, it implies that $C n$ in the right side of (15) must go to zero as $n \rightarrow \infty$. Thus, the solution must satisfy $C=0$.

### 7.2 Asymptotic optimal marginal tax rates

One can rewrite (12) as

$$
\frac{T^{\prime}\left(z^{*}\right)}{1-T^{\prime}\left(z^{*}\right)}=\frac{1}{\varepsilon\left(z^{*}\right)}\left\{\begin{array}{c}
{\left[\frac{1-F_{Z}\left(z^{*}\right)}{z^{*} f_{Z}\left(z^{*}\right)}-\frac{\int_{z^{*}}^{\infty} g(z) f_{Z}(z) d z}{z^{*} f_{Z}\left(z^{*}\right)}\right]} \\
+\frac{2 \beta}{\sigma^{2}} \frac{\chi\left(z^{*}\right)+\psi\left(z^{*}\right)}{z^{*} f_{Z}\left(z^{*}\right)}
\end{array}\right\}
$$

Using $f_{Z}(z) z^{\prime}=f(n)$, we have

## 8 Conclusion

## 9 Appendix

### 9.1 Proof of Theorem 1

$$
\begin{aligned}
\int_{1}^{n} \frac{f^{\prime}(\tilde{n})}{f(\tilde{n})} d \tilde{n} & =\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}-2 \int_{1}^{n} \frac{1}{\tilde{n}} d \tilde{n} \Rightarrow \\
\int_{1}^{n} d \ln f(\tilde{n}) d \tilde{n} & =\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}-2 \int_{1}^{n} d \ln \tilde{n} d \tilde{n} \Rightarrow \\
\left.\ln f(\tilde{n})\right|_{1} ^{n} & =\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}-\left.2 \ln \tilde{n}\right|_{1} ^{n} \Rightarrow \\
\ln \frac{f(n)}{f(1)} & =\frac{2}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}-2 \ln n \Rightarrow \\
\frac{f(n)}{f(1)} & =\exp \left(\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}-2 \ln n\right) \Rightarrow \\
\frac{f(n)}{f(1)} & =\exp \left(\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}\right) \exp (-2 \ln n) \Rightarrow \\
f(n) & =f(1) n^{-2} \exp \left(\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n})}{\tilde{n}} d \tilde{n}\right) .
\end{aligned}
$$

### 9.2 Proof of Lemma 1

From the FOC (2), we have $v^{\prime}(l(n))=n\left[1-T^{\prime}(z(n))\right]$. This then leads to

$$
l^{\prime}(n)=\frac{1-T^{\prime}(z(n))-z(n) T^{\prime \prime}(z(n))}{n^{2} T^{\prime \prime}(z(n))+v^{\prime \prime}(l(n))}
$$

which in turn leads to

$$
\begin{equation*}
\frac{l^{\prime}(n)}{l(n)}=\frac{1}{n} \frac{[1-p(z(n))] \kappa(n)}{1+p(z(n)) \kappa(n)} \tag{16}
\end{equation*}
$$

Using (16), we obtain

$$
z^{\prime}(n)=l(n)+n l^{\prime}(n)=l(n)\left[1+n \frac{l^{\prime}(n)}{l(n)}\right]=\frac{z(n)}{n} \frac{1+\kappa(n)}{1+p(z(n)) \kappa(n)}
$$

which yields

$$
\begin{equation*}
\frac{z^{\prime}(n)}{z(n)}=\frac{1}{n} \frac{1+\kappa(n)}{1+p(z(n)) \kappa(n)} \tag{17}
\end{equation*}
$$

Combining (16) and (17) yields

$$
\frac{l^{\prime}(n)}{l(n) z^{\prime}(n)}=\frac{[1-p(z(n))] \kappa(n)}{z(n)[1+\kappa(n)]}
$$

Therefore, we have

$$
\frac{\frac{d l}{d z}}{l}=\frac{[1-p(z)] \kappa(z)}{z[1+\kappa(z)]}
$$

Using the boundary condition $l\left(z_{1}\right)=l\left(n_{1}\right)=1$, we then have $l(z)$ stated in Lemma 1.
From

$$
f(n)=f_{Z}(z) z^{\prime}(n)
$$

we obtain

$$
\begin{aligned}
f_{Z}(z) & =f(n) \frac{1}{z^{\prime}(n)} \\
& =f(n) \frac{1}{l(n)} \frac{1+p(z(n)) \kappa(n)}{1+\kappa(n)}(\text { using }(17)) \\
& \left.=f(n(z)) \frac{1+p(z) \kappa(z)}{1+\kappa(z)} \exp \left\{-\int_{z_{1}}^{z} \frac{[1-p(\tilde{z})] \kappa(\tilde{z})}{\tilde{z}[1+\kappa(\tilde{z})]} d \tilde{z}\right\}, z \geq z(0) . \text { (using the derived } l(z)\right)
\end{aligned}
$$

### 9.3 Proof of Proposition 1

## Individual utility:

By definition,

$$
U(n ; T+b \hat{T})=y(n ; T+b \hat{T})-v(l(n ; T+b \hat{T}))
$$

Applying the first-order Taylor series expansion, we have

$$
v(l(n ; T+b \hat{T}))=v(l(n ; T))+v^{\prime}(l(n ; T))[l(n ; T+b \hat{T})-l(n ; T)] .
$$

Applying (11), we have

$$
\begin{aligned}
U(n ; T+b \hat{T}) & =U(n ; T)+b \hat{U}(n), \\
y(n ; T+b \hat{T}) & =y(n ; T)+b \hat{y}(n)
\end{aligned}
$$

Substituting the above three equations into the first equation and using $U(n ; T)=y(n ; T)-$ $v(l(n ; T))$ yields

$$
\begin{equation*}
\hat{U}(n)=\hat{y}(n)-v^{\prime}(l(n)) \hat{l}(n) \tag{18}
\end{equation*}
$$

By definition,

$$
\begin{aligned}
y(n ; T+b \hat{T}) & =z(n ; T+b \hat{T})-[T(z(n ; T+b \hat{T}))+b \hat{T}(z(n ; T+b \hat{T}))] \\
y(n ; T) & =z(n ; T)-T(z(n ; T)) .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& y(n ; T+b \hat{T})-y(n ; T) \\
& =n l(n ; T+b \hat{T})-n l(n ; T)-[T(z(n ; T+b \hat{T}))-T(z(n ; T))]-b \hat{T}(z(n ; T+b \hat{T})) \\
& =n l(n ; T+b \hat{T})-n l(n ; T)-T^{\prime}(z(n ; T))[n l(n ; T+b \hat{T})-n l(n ; T)]-b \hat{T}(z(n ; T+b \hat{T})) \\
& =n\left[1-T^{\prime}(z(n ; T))\right][l(n ; T+b \hat{T})-l(n ; T)]-b \hat{T}(z(n ; T+b \hat{T})) \\
& =n\left[1-T^{\prime}(z(n ; T))\right] b \hat{l}(n ; T)-b\left\{\hat{T}(z(n ; T))+\hat{T}^{\prime}(z(n ; T))[z(n ; T+b \hat{T})-z(n ; T)]\right\} \\
& =b n\left[1-T^{\prime}(z(n ; T))\right] \hat{l}(n ; T)-b \hat{T}(z(n ; T))-b \hat{T}^{\prime}(z(n ; T)) b \hat{z}(n ; T)
\end{aligned}
$$

Approximating the first-order effect by leaving out the term that involves $b^{2}$ gives

$$
\begin{equation*}
\hat{y}(n)=n\left[1-T^{\prime}(z(n)] \hat{l}(n)-\hat{T}(z(n)) .\right. \tag{19}
\end{equation*}
$$

Substituting (19) into (18) and invoking the FOC (2) leads to the result stated in the proposition.

## Individual labor supply:

Let $r(n) \equiv 1-T^{\prime}(n l(n))$. First, we show $\varepsilon(n)=\frac{\partial \ln l(n)}{\partial \ln r(n)}=\frac{\kappa(n)}{1+p(z(n)) \kappa(n)}$.
From the FOC (2), we have ${ }^{12}$

$$
v^{\prime}(l(n)+d l(n))=n\left[1-T^{\prime}(n[l(n)+d l(n)])+d r(n)\right] .
$$

Applying the first-order Taylor expansion to the above equation and using the FOC (2) gives ${ }^{13}$

$$
\frac{d l(n)}{d r(n)}=\frac{n}{v^{\prime \prime}(l(n))+n^{2} T^{\prime \prime}(n l(n))}
$$

Therefore, we have (using the FOC (2))

$$
\begin{aligned}
\frac{\partial \ln l(n)}{\partial \ln r(n)} & =\frac{\partial l(n)}{\partial r(n)} \frac{r(n)}{l(n)} \\
& =\frac{n\left[1-T^{\prime}(n l(n))\right]}{\left[v^{\prime \prime}(l(n))+n^{2} T^{\prime \prime}(n l(n))\right] l(n)} \\
& =\frac{1}{\frac{l(n) v^{\prime \prime}(l(n))}{v^{\prime}(l(n))}+\frac{n l(n) T^{\prime \prime}(n l(n))}{1-T^{\prime}(n l(n))}} \\
& =\frac{1}{\frac{1}{\kappa(n)}+p(n l(n))} \\
& =\frac{\kappa(n)}{1+p(z(n)) \kappa(n)}
\end{aligned}
$$

[^8]Next, we show $\frac{\hat{l}(n)}{l(n)}=-\varepsilon(n) \frac{\hat{T}^{\prime}(z(n))}{1-T^{\prime}(z(n))}$.
The FOC (2) gives

$$
\begin{aligned}
v^{\prime}(l(n ; T)) & =n\left[1-T^{\prime}(n l(n ; T))\right] \\
v^{\prime}(l(n ; T+b \hat{T})) & =n\left\{1-\left[T^{\prime}(n l(n ; T+b \hat{T}))+b \hat{T}^{\prime}(n l(n ; T+b \hat{T}))\right]\right\},
\end{aligned}
$$

which leads to

$$
v^{\prime}(l(n ; T+b \hat{T}))-v^{\prime}(l(n ; T))=-n\left[T^{\prime}(n l(n ; T+b \hat{T}))-T^{\prime}(n l(n ; T))\right]-b n \hat{T}^{\prime}(n l(n ; T+b \hat{T})) .
$$

Applying the first-order Taylor series expansion to the above equation gives

$$
\begin{aligned}
v^{\prime \prime}(l(n ; T))[l(n ; T+b \hat{T})-l(n ; T)] & =-n^{2} T^{\prime \prime}(n l(n ; T))[l(n ; T+b \hat{T})-l(n ; T)] \\
& -b n\left\{\hat{T}^{\prime}(n l(n ; T))+\hat{T}^{\prime \prime}(n l(n ; T)) n[l(n ; T+b \hat{T})-l(n ; T)]\right\}
\end{aligned}
$$

By (11), it then leads to

$$
v^{\prime \prime}(l(n ; T)) b \hat{l}(n)=-n^{2} T^{\prime \prime}(n l(n ; T)) b \hat{l}(n)-b n\left[\hat{T}^{\prime}(n l(n ; T))-\hat{T}^{\prime \prime}(n l(n ; T)) b n \hat{l}(n)\right] .
$$

Approximating the first-order effect by leaving out the term that involves $b^{2}$ yields

$$
v^{\prime \prime}(l(n ; T)) \hat{l}(n)=-n^{2} T^{\prime \prime}(n l(n ; T)) \hat{l}(n)-n \hat{T}^{\prime}(n l(n ; T))
$$

which in turn yields

$$
\hat{l}(n)=\frac{-n \hat{T}^{\prime}(n l(n ; T))}{v^{\prime \prime}(l(n ; T))+n^{2} T^{\prime \prime}(n l(n ; T))} .
$$

Thus, we have (using the FOC (2))

$$
\begin{aligned}
\frac{\hat{l}(n)}{l(n)} & =\frac{n\left[1-T^{\prime}(n l(n ; T))\right]}{l(n) v^{\prime \prime}(l(n ; T))+n^{2} l(n) T^{\prime \prime}(n l(n ; T))} \frac{-\hat{T}^{\prime}(n l(n ; T))}{\left[1-T^{\prime}(n l(n ; T))\right]} \\
& =\frac{\kappa(n)}{1+p(n l(n)) \kappa(n)} \frac{-\hat{T}^{\prime}(n l(n ; T))}{1-T^{\prime}(n l(n ; T))} \\
& =-\varepsilon(n) \frac{\hat{T}^{\prime}(n l(n))}{1-T^{\prime}(n l(n))} .
\end{aligned}
$$

### 9.4 Proof of Proposition 2

From Theorem 1, we have

$$
f(n ; T)=f(1) n^{-2} \exp \left\{\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n} ; T)}{\tilde{n}} d \tilde{n}\right\}
$$

which leads to

$$
\begin{aligned}
\frac{f(n ; T+b \hat{T})}{f(n ; T)} & =\frac{f(1 ; T+b \hat{T})}{f(1 ; T)} \exp \left\{\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n} ; T+b \hat{T})-y(\tilde{n} ; T)}{\tilde{n}} d \tilde{n}\right\} \\
& =\frac{f(1 ; T+b \hat{T})}{f(1 ; T)} \exp \left\{\ln \left[1+\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{y(\tilde{n} ; T+b \hat{T})-y(\tilde{n} ; T)}{\tilde{n}} d \tilde{n}\right]\right\} .(\text { using } \ln (1+x) \approx x)
\end{aligned}
$$

Thus,

$$
1+b \frac{\hat{f}(n)}{f(n)}=\left(1+b \frac{\hat{f}(1)}{f(1)}\right)\left(1+b \frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{\hat{y}(\tilde{n})}{\tilde{n}} d \tilde{n}\right)
$$

which gives

$$
\frac{\hat{f}(n)}{f(n)}=\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{\hat{y}(\tilde{n})}{\tilde{n}} d \tilde{n} \text {. (approximation by leaving out the term that involves } b^{2} \text { ) }
$$

From (19), we have

$$
\hat{y}(n)=\left[1-T^{\prime}(z)\right] z(n) \frac{\hat{l}(n)}{l(n)}-\hat{T}(z(n))
$$

Therefore,

$$
\begin{aligned}
\frac{\hat{f}(n)}{f(n)} & =\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}} \int_{1}^{n} \frac{\hat{y}(\tilde{n})}{\tilde{n}} d \tilde{n} \\
& =\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}}\left[\int_{1}^{n} \frac{1}{\tilde{n}}\left(\left[1-T^{\prime}(z(\tilde{n})] z(\tilde{n}) \frac{\hat{l}(\tilde{n})}{l(\tilde{n})}-\hat{T}(z(\tilde{n}))\right) d \tilde{n}\right]\right.
\end{aligned}
$$

With $f(n ; T+b \hat{T})=f(n ; T)+b \hat{f}(n)$, we have $\int_{0}^{\infty} \hat{f}(n) d n=0$. Thus, $\frac{\hat{f}(1)}{f(1)}$ is determined by $\int_{0}^{\infty} \frac{\hat{f}(n)}{f(n)} f(n) d n=0$. It gives

$$
\frac{\hat{f}(1)}{f(1)}=-\frac{2 \beta}{\sigma^{2}} \int_{0}^{\infty}\left[\int_{1}^{n} \frac{1}{\tilde{n}}\left(\left[1-T^{\prime}(z(\tilde{n})] z(\tilde{n}) \frac{\hat{l}(\tilde{n})}{l(\tilde{n})}-\hat{T}(z(\tilde{n}))\right) d \tilde{n}\right] f(n) d n\right.
$$

### 9.5 Proof of Proposition 3

Part 1: We show the components of $\hat{\Re}(\hat{T})$.
From

$$
\Re(T)=\int_{0}^{\infty} T(n l(n) f(n) d n
$$

we have

$$
\begin{aligned}
& \Re(T+b \hat{T})-\Re(T) \\
& =\int_{0}^{\infty}[T(n l(n ; T+b \hat{T}))+b \hat{T}(n l(n ; T+b \hat{T}))] f(n ; T+b \hat{T}) d n \\
& -\int_{0}^{\infty} T(n l(n ; T)) f(n ; T) d n \\
& =\int_{0}^{\infty}\left\{\begin{array}{c}
T(n l(n ; T))+T^{\prime}(n l(n ; T))[n l(n ; T+b \hat{T})-n l(n ; T)] \\
+b\left[\hat{T}(n l(n ; T))+\hat{T}^{\prime}(n l(n ; T))[n l(n ; T+b \hat{T})-n l(n ; T)]\right]
\end{array}\right\}[f(n ; T)+b \hat{f}(n)] d n \\
& -\int_{0}^{\infty} T(n l(n ; T)) f(n ; T) d n \\
& =\int_{0}^{\infty}\left\{\begin{array}{c}
T(n l(n ; T))+T^{\prime}(n l(n ; T)) n b \hat{l}(n) \\
+b\left[\hat{T}(n l(n ; T))+\hat{T}^{\prime}(n l(n ; T)) n b \hat{l}(n)\right]
\end{array}\right\}[f(n ; T)+b \hat{f}(n)] d n \\
& -\int_{0}^{\infty} T(n l(n ; T)) f(n ; T) d n
\end{aligned}
$$

Approximating the first-order effect by leaving out the term that involves $b^{2}$ gives

$$
\begin{aligned}
& \hat{\Re}(\hat{T})=\int_{0}^{\infty}\left[\hat{T}(n l(n)) f(n ; T)+T^{\prime}(n l(n)) n \hat{l}(n) f(n ; T)+T(n l(n)) \hat{f}(n)\right] d n \\
& =\int_{0}^{\infty} \hat{T}(n l(n)) f(n) d n+\int_{0}^{\infty} T^{\prime}(n l(n)) n \hat{l}(n) f(n) d n+\int_{0}^{\infty} T(n l(n)) \hat{f}(n) d n \\
& =\int_{0}^{\infty} \hat{T}(n l(n)) f(n) d n+\int_{0}^{\infty} T^{\prime}(n l(n)) n l(n) \frac{\hat{l}(n)}{l(n)} f(n) d n+\int_{0}^{\infty} T(n l(n)) f(n) \frac{\hat{f}(n)}{f(n)} d n .
\end{aligned}
$$

Part 2: We derive the third term of $\hat{\Re}(\hat{T})$ for the elementary tax reform.
Step 1. Derive $\frac{\hat{f}(n(z))}{f(n(z))}$.
From Proposition 2, we have

$$
\frac{\hat{f}(n)}{f(n)}=\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}}\left[\int_{1}^{n} \frac{1}{\tilde{n}}\left(z(\tilde{n})\left[1-T^{\prime}(z(\tilde{n})] \frac{\hat{l}(\tilde{n})}{l(\tilde{n})}-\hat{T}(z(\tilde{n}))\right) d \tilde{n}\right]\right.
$$

Thus, if $z \geq z(1)$,

$$
\begin{aligned}
\frac{\hat{f}(n(z))}{f(n(z))} & =\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}}\left[\int_{1}^{n} \frac{1}{\tilde{n} z^{\prime}(\tilde{n})}\left(z(\tilde{n})\left[1-T^{\prime}(z(\tilde{n})] \frac{\hat{l}(\tilde{n})}{l(\tilde{n})}-\hat{T}(z(\tilde{n}))\right) z^{\prime}(\tilde{n}) d \tilde{n}\right]\right. \\
& =\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}}\left[\int_{z(1)}^{z} \frac{1}{\tilde{z} \frac{1+\kappa(\tilde{z})}{1+p(\tilde{z}) \kappa(\tilde{z})}}\left(\left.\tilde{z}\left[1-T^{\prime}(\tilde{z})\right] \frac{\hat{l}(n)}{l(n)}\right|_{n=n(\tilde{z})}-\hat{T}(\tilde{z})\right) d \tilde{z}\right] \quad(\text { using }(17)) \\
& =\frac{\hat{f}(1)}{f(1)}-\frac{2 \beta}{\sigma^{2}}\left[\int_{z(1)}^{z} \frac{1}{\left.\tilde{z} \frac{1+\kappa(\tilde{z})}{1+p(\tilde{z} \kappa(\tilde{z})}\left(\tilde{z} \varepsilon(\tilde{z}) \hat{T}^{\prime}(\tilde{z})+\hat{T}(\tilde{z})\right) d \tilde{z}\right] \quad(\text { using Proposition } 1)}\right. \\
& \left.=\frac{\hat{f}(1)}{f(1)}-\frac{2 \beta}{\sigma^{2}}\left[\int_{z(1)}^{z} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}(\tilde{z}) d \tilde{z}+\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}(\tilde{z}) d \tilde{z}\right] \quad \text { (using definition of } \varepsilon(\tilde{z})\right) \\
& \equiv \frac{\hat{f}(1)}{f(1)}-\frac{2 \beta}{\sigma^{2}} \Psi(z),(\text { i) }
\end{aligned}
$$

if $z<z(1)$,

$$
\begin{aligned}
\frac{\hat{f}(n(z))}{f(n(z))} & =\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}}\left[\int_{z}^{z(1)} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}(\tilde{z}) d \tilde{z}+\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}(\tilde{z}) d \tilde{z}\right] \\
& \equiv \frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}} \Phi(z) .(\mathrm{ii})
\end{aligned}
$$

Using $\int_{0}^{\infty} \frac{\hat{f}(n)}{f(n)} f(n) d n=0$ and $f(n) d n=f_{Z}(z) d z$ gives

$$
0=\int_{z(1)}^{\infty}\left[\frac{\hat{f}(1)}{f(1)}-\frac{2 \beta}{\sigma^{2}} \Psi(z)\right] f_{Z}(z) d z+\int_{0}^{z(1)}\left[\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}} \Phi(z)\right] f_{Z}(z) d z
$$

which leads to

$$
\begin{aligned}
\frac{\hat{f}(1)}{f(1)} & =\frac{2 \beta}{\sigma^{2}}\left[\int_{z(1)}^{\infty} \Psi(z) f_{Z}(z) d z-\int_{0}^{z(1)} \Phi(z) f_{Z}(z) d z\right] \\
& =\frac{2 \beta}{\sigma^{2}}\left[\begin{array}{c}
\int_{z(1)}^{\infty}\left[\int_{z(1)}^{z} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z}} \hat{T}^{\prime}(\tilde{z}) d \tilde{z}+\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}(\tilde{z}) d \tilde{z}\right] f_{Z}(z) d z \\
-\int_{0}^{z(1)}\left[\int_{z}^{z(1)} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}(\tilde{z}) d \tilde{z}+\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}(\tilde{z}) d \tilde{z}\right] f_{Z}(z) d z
\end{array}\right] .
\end{aligned}
$$

Let $\frac{\hat{f}(1)}{f(1)}\left(z^{*}\right)$ denote $\frac{\hat{f}(1)}{f(1)}$ with the elementary tax reform at $z^{*}$. If $z^{*} \geq z(1)$,

$$
\begin{aligned}
\frac{\hat{f}(1)}{f(1)}\left(z^{*}\right) & =\frac{2 \beta}{\sigma^{2}}\left[\begin{array}{c}
\int_{z(1)}^{\infty}\left[\int_{z(1)}^{z} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}(\tilde{z}) d \tilde{z}+\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{( }(1+\kappa(\tilde{z}))} \hat{T}(\tilde{z}) d \tilde{z}\right] f_{Z}(z) d z \\
-\int_{0}^{z(1)}\left[\int_{z}^{z(1)} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}(\tilde{z}) d \tilde{z}+\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}(\tilde{z}) d \tilde{z}\right] f_{Z}(z) d z
\end{array}\right] \\
& =\frac{2 \beta}{\sigma^{2}}\left[\int_{z^{*}}^{\infty}\left[\int_{z(1)}^{z} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}\left(\tilde{z} ; z^{*}\right) d \tilde{z}+\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}\left(\tilde{z} ; z^{*}\right) d \tilde{z}\right] f_{Z}(z) d z\right] \text { (the part involvil } \\
& =\frac{2 \beta}{\sigma^{2}} \frac{1}{1-F_{Z}\left(z^{*}\right)}\left[\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} f_{Z}\left(z^{*}\right)+\int_{z^{*}}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) f_{Z}(z) d z\right], \text { (by definition of } \hat{T}^{\prime}(\tilde{z}
\end{aligned}
$$

(iii)

$$
\begin{align*}
& \text { if } z^{*}<z(1), \\
& \begin{aligned}
& \frac{\hat{f}(1)}{f(1)}\left(z^{*}\right)=\frac{2 \beta}{\sigma^{2}}\left[\begin{array}{c}
\int_{z(1)}^{\infty}\left[\int_{z(1)}^{z} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}(\tilde{z}) d \tilde{z}+\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}(\tilde{z}) d \tilde{z}\right] \\
-\int_{0}^{z(1)}\left[\int_{z}^{z(1)} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}(\tilde{z}) d \tilde{z}+\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}(\tilde{z}) d \tilde{z}\right] f_{Z}(z) d z
\end{array}\right] \\
&=\frac{2 \beta}{\sigma^{2}}\left[\begin{array}{c}
\int_{z(1)}^{\infty}\left[\int_{z(1)}^{z} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}\left(\tilde{z} ; z^{*}\right) d \tilde{z}+\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}\left(\tilde{z} ; z^{*}\right) d \tilde{z}\right] f_{Z}(z) d z \\
-\int_{z^{*}}^{z(1)}\left[\int_{z(1)}^{z(1)} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}\left(\tilde{z} ; z^{*}\right) d \tilde{z}+\int_{z}^{z(1)} \frac{1+p(\tilde{z} \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}\left(\tilde{z} ; z^{*}\right) d \tilde{z}\right] f_{Z}(z) d z
\end{array}\right] \text { (he part involving } \\
& \text { is relevant } \\
&=\frac{2 \beta}{\sigma^{2}} \frac{1}{1-F_{Z}\left(z^{*}\right)}\left[\begin{array}{c}
\int_{z(1)}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) f_{Z}(z) d z \\
-\left[\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} f_{Z}\left(z^{*}\right)+\int_{z^{*}}^{z(1)}\left(\int_{z(1)}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) f_{Z}(z) d z\right]
\end{array}\right] . \text { (by definition of } \hat{T}^{\prime}(\tilde{z}
\end{aligned}
\end{align*}
$$

Step 2. Derive $\left.\int_{0}^{\infty} T(n l(n)) f(n) \frac{\hat{f}(n(z))}{f(n(z))}\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} d n$. If $z^{*} \geq z(1)$,

$$
\begin{aligned}
& \left.\int_{0}^{\infty} T(n l(n)) \frac{\hat{f}(n(z))}{f(n(z))}\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} f(n) d n \\
& =\int_{0}^{\infty}\left[\frac{\hat{f}(1)}{f(1)}-\frac{2 \beta}{\sigma^{2}} \Psi(z)\right] T(z) f_{Z}(z) d z(\text { using (i)) } \\
& =\frac{\hat{f}(1)}{f(1)}\left(z^{*}\right) \int_{0}^{\infty} T(z) f_{Z}(z) d z-\frac{2 \beta}{\sigma^{2}} \int_{z^{*}}^{\infty} \Psi(z) T(z) f_{Z}(z) d z \text { (elementary tax reform at } z^{*} \text { ) } \\
& =-\frac{2 \beta}{\sigma^{2}} \int_{z^{*}}^{\infty}\left[\int_{z(1)}^{z} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}\left(\tilde{z} ; z^{*}\right) d \tilde{z}+\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}\left(\tilde{z} ; z^{*}\right) d \tilde{z}\right] T(z) f_{Z}(z) d z \\
& \left.+\frac{\hat{f}(1)}{f(1)}\left(z^{*}\right) \times E \text { (using definition of } \Psi(z) \text { and } E=\int_{0}^{\infty} T(z) f_{Z}(z) d z\right) \\
& =-\frac{2 \beta}{\sigma^{2}} \frac{1}{1-F_{Z}\left(z^{*}\right)}\left[\begin{array}{c}
\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} T\left(z^{*}\right) f_{Z}\left(z^{*}\right)+ \\
\int_{z^{*}}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) T(z) f_{Z}(z) d z
\end{array}\right] \\
& +\frac{\hat{f}(1)}{f(1)}\left(z^{*}\right) \times E\left(\text { by definition of } \hat{T}^{\prime}\left(\tilde{z} ; z^{*}\right) \text { and } \hat{T}\left(\tilde{z} ; z^{*}\right)\right) \\
& =-\frac{2 \beta}{\sigma^{2}} \frac{1}{1-F_{Z}\left(z^{*}\right)}\left\{\begin{array}{c}
{\left[\begin{array}{c}
\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} T\left(z^{*}\right) f_{Z}\left(z^{*}\right)+ \\
{\left[\int_{z^{*}}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) T(z) f_{Z}(z) d z\right.}
\end{array}\right]-} \\
{\left[\begin{array}{c}
\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} f_{Z}\left(z^{*}\right)+ \\
\int_{z^{*}}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) f_{Z}(z) d z
\end{array}\right] \times E}
\end{array}\right\} \text { (using (iii)) } \\
& =\frac{2 \beta}{\sigma^{2}} \frac{1}{1-F_{Z}\left(z^{*}\right)}\left\{\left[\begin{array}{c}
-\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)}\left[T\left(z^{*}\right)-E\right] f_{Z}\left(z^{*}\right)- \\
\int_{z^{*}}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right)[T(z)-E] f_{Z}(z) d z
\end{array}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
& \text { if } z^{*}<z(1) \text {, } \\
& \left.\int_{0}^{\infty} T(n l(n)) \frac{\hat{f}(n(z))}{f(n(z))}\right|_{\hat{T}=\hat{T}\left(z ; z^{*}\right)} f(n) d n \\
& =\int_{0}^{z(1)}\left[\frac{\hat{f}(1)}{f(1)}+\frac{2 \beta}{\sigma^{2}} \Phi(z)\right] T(z) f_{Z}(z) d z+\int_{z(1)}^{\infty}\left[\frac{\hat{f}(1)}{f(1)}-\frac{2 \beta}{\sigma^{2}} \Psi(z)\right] T(z) f_{Z}(z) d z(\text { using (i) and (ii)) } \\
& =\frac{\hat{f}(1)}{f(1)}\left(z^{*}\right) \int_{0}^{\infty} T(z) f_{Z}(z) d z+\frac{2 \beta}{\sigma^{2}} \int_{z^{*}}^{z(1)} \Phi(z) T(z) f_{Z}(z) d z \\
& -\frac{2 \beta}{\sigma^{2}} \int_{z(1)}^{\infty} \Psi(z) T(z) f_{Z}(z) d z\left(\left[\begin{array}{c}
\text { given } z^{*}<z(1), \\
\text { the part involving } \int_{z(1)}^{\infty} \text { is relevant }
\end{array}\right]\right) \\
& =\frac{2 \beta}{\sigma^{2}} \int_{z^{*}}^{z(1)}\left[\int_{z}^{z(1)} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}\left(\tilde{z} ; z^{*}\right) d \tilde{z}+\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}\left(\tilde{z} ; z^{*}\right) d \tilde{z}\right] T(z) f_{Z}(z) d z \\
& -\frac{2 \beta}{\sigma^{2}} \int_{z(1)}^{\infty}\left[\int_{z(1)}^{z} \frac{\kappa(\tilde{z})}{1+\kappa(\tilde{z})} \hat{T}^{\prime}\left(\tilde{z} ; z^{*}\right) d \tilde{z}+\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} \hat{T}\left(\tilde{z} ; z^{*}\right) d \tilde{z}\right] T(z) f_{Z}(z) d z \\
& \left.+\frac{\hat{f}(1)}{f(1)}\left(z^{*}\right) \times E \text { (using definition of } \Phi(z) \text { and } \Psi(z) \text { and } E=\int_{0}^{\infty} T(z) f_{Z}(z) d z\right) \\
& =\frac{2 \beta}{\sigma^{2}} \frac{1}{1-F_{Z}\left(z^{*}\right)}\left[\begin{array}{c}
\frac{\kappa\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} T\left(z^{*}\right) f_{Z}\left(z^{*}\right)+\int_{z^{*}}^{z(1)}\left(\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) T(z) f_{Z}(z) d z \\
-\int_{z(1)}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) T(z) f_{Z}(z) d z
\end{array}\right] \\
& +\frac{\hat{f}(1)}{f(1)}\left(z^{*}\right) \times E \quad\left(\text { by definition of } \hat{T}^{\prime}\left(\tilde{z} ; z^{*}\right) \text { and } \hat{T}\left(\tilde{z} ; z^{*}\right)\right) \\
& =\frac{2 \beta}{\sigma^{2}} \frac{1}{1-F_{Z}\left(z^{*}\right)}\left\{\begin{array}{c}
{\left[\begin{array}{c}
\frac{\kappa\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} T\left(z^{*}\right) f_{Z}\left(z^{*}\right)+\int_{z^{*}}^{z(1)}\left(\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) T(z) f_{Z}(z) d z \\
-\int_{z(1)}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) T(z) f_{Z}(z) d z \\
{\left[\begin{array}{c}
\int_{z(1)}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) f_{Z}(z) d z \\
-\left[\frac{k\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} f_{Z}\left(z^{*}\right)+\int_{z^{*}}^{z(1)}\left(\int_{z}^{z(1)} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right) f_{Z}(z) d z\right]
\end{array}\right]+}
\end{array}\right\} \times E}
\end{array}\right\} \text { (using (iv) } \\
& =\frac{2 \beta}{\sigma^{2}} \frac{1}{1-F_{Z}\left(z^{*}\right)}\left\{\left[\begin{array}{c}
\frac{\kappa\left(z^{*}\right)}{1+\kappa\left(z^{*}\right)} f_{Z}\left(z^{*}\right)\left[T\left(z^{*}\right)-E\right] \\
+\int_{z^{*}}^{z(1)}\left(\int_{z}^{z(1)} \frac{1+p(z) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right)[T(z)-E] f_{Z}(z) d z \\
-\int_{z(1)}^{\infty}\left(\int_{z(1)}^{z} \frac{1+p(\tilde{z}) \kappa(\tilde{z})}{\tilde{z}(1+\kappa(\tilde{z}))} d \tilde{z}\right)[T(z)-E] f_{Z}(z) d z
\end{array}\right]\right\} .
\end{aligned}
$$

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[^0]:    ${ }^{1}$ See Tuomala (1990) for a survey of earlier simulation results.

[^1]:    ${ }^{2}$ For reviews on NDPF, see Golosov, Tsyvinski, and Werning (2006) and Kocherlakota (2010).

[^2]:    ${ }^{3}$ See section 16.4 of Martin, Hurn and Harris (2013).
    ${ }^{4}$ For the applications of the KF equation, see, for example, Benhabib, Bisin and Zhu (2016), Jones and Kim (2018), Nuno and Moll (2018), and Achdou, Han, Lasry, Lions, and Moll (2020).

[^3]:    ${ }^{6}$ Using the FOC $(2), v^{\prime}(l(n))=n\left[1-T^{\prime}(n l(n))\right] \equiv w(n)$, we have

    $$
    \frac{\partial l(n)}{\partial w(n)} \frac{w(n)}{l(n)}=\frac{\partial l(n)}{\partial r(n)} \frac{r(n)}{l(n)}=\frac{v^{\prime}(l(n))}{l(n) v^{\prime \prime}(l(n))}=\kappa(n)
    $$

[^4]:    ${ }^{7}$ See Assumption 1 in the supplement to their paper.
    ${ }^{8}$ We assume that $p(z)<1$ for all $z \geq 0$, which implies $\frac{d}{d n}\left\{n\left[1-T^{\prime}(z(n))\right]\right\}>0$. This guarantees the existence of $n_{1} \geq 0$.

[^5]:    ${ }^{9}$ For more on the elementary tax reform, see Supplement to STW.

[^6]:    ${ }^{10} q=\int_{0}^{\infty} G^{\prime}(U(n(z))) f_{Z}(z) d z$; see Diamond (1998).

[^7]:    ${ }^{11}$ It extends the result derived by Piketty (1997).

[^8]:    ${ }^{12}$ See the first equation in Supplement to STW.
    ${ }^{13} v^{\prime}(l(n)+d l(n))=v^{\prime}(l(n))+v^{\prime}(l(n)) d l(n)$,
    $T^{\prime}(n[l(n)+d l(n)])=T^{\prime}\left(n[l(n)]+n T^{\prime}(n l(n)) d l(n)\right.$.
    $\Rightarrow$
    $v^{\prime}(l(n))+v^{\prime}(l(n)) d l(n)=$
    $n\left(1-T^{\prime}\left(n[l(n)]-n T^{\prime}(n l(n)) d l(n)\right]+d r(n)\right)$.
    $\Rightarrow v^{\prime}(l(n)) d l(n)+n^{2} T^{\prime}(n l(n)) d l(n)=n d r(n)$.

