Matching in a Static Assignment Model*

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Abstract

This paper adopts deep learning physics-informed neural networks to resolve a static assignment model. Workers and firms match in the labor market, and the firm size is endogenous. A perturbation analysis of the equilibrium is carried out, and it is found that the perturbed matching rule follows a Voterra integral equation.

1 Introduction

Eeckhout and Kircher (2018) claim that the matching between workers and firms is an important determinant of growth. The one-to-one matching models, including those proposed by Kantorovich (1942), Koopmans and Beckmann (1957), Shapley and Shubik (1972), and Becker (1973), cannot analyze endogenous firm size because workers and firms can only match in pairs. Lucas (1978) endogenizes the firm size while splitting the population endogenously into workers and managers. Based on the research by Lucas (1978), Eeckhout and Jovanovic (2011) include worker heterogeneity without considering sorting of managers and workers. Kelso and Crawford (1982) establish a many-to-one

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matching framework in a finite agent environment with heterogenous firms and workers. Gaubert (2018) investigates the matching between heterogeneous firms and cities and found that firm sorting contributes greatly to city's productivity and welfare. By using French matched employer-employee (DADS) data, Orefice and Peri (2020) demonstrate that positive assortative matching exists among immigrant workers.

Eeckhout and Kircher (2018) link the firm size and sorting in a unified framework to analyze how firms with different productivity achieve the tradeoff between the quantity and quality of workers. By investigating the impact of firm size on the demand for workers, Eeckhout and Kircher (2018) find a mechanism under which the matching rule and equilibrium wage can be both determined in the labor market based on a system of three differential equations through governing the equilibrium quantity and quality of workers. This theoretical framework addresses many-to-one matches, which is extremely challenging because it is difficult to integrate the local sorting condition to a global condition due to the endogenous firm size.

In this paper, our basic model closely resembles a stripped-down version of benchmark study by Eeckhout and Kircher (2018). By using the perturbation method, a Volterra integral equation is derived to show how the matching rule varies with respect to key parameters through different channels, such as the direct effect and the wage effect. This theoretical contribution shows the impact of key variables on matching explicitly. Meanwhile, by using the latest deep learning techniques, complex numerical exercises are conducted easily. Future studies can apply this numerical method with deep learning techniques to projects with searching and matching which require intensive computations.

The rest of this paper is organized as follows. Section 2 presents the model and the application of deep learning. Section 3 shows the perturbation on the matching. Section 4 concludes this paper.

2 The model

Our basic model highly resembles a stripped-down version of benchmark study by Eeckhout and Kircher (2018).

2.1 Matching

The labor market has no friction. The firm determines the quality and quantity of labor hiring. Following the research by Eeckhout and Kircher (2018) and Grossman et al. (2017), a firm with a boss of human capital x hires L employees of human capital y to produce output according to the following function,

$$F(x, y, L) = A\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi}, \tag{1}$$

where L represents the firm size, $0 < \phi < 1$ represents the impact coefficient of firm size, $A > 0$ represents the firm's productivity, and $0 < \beta < 1$ denotes the weight of bosses in production. $\alpha > 0$ measures the elasticity of substitution between x and y.[1]

The boss hires workers from the labor market. The boss chooses not only the type of worker (i.e., the worker's human capital level) but also the number of workers, L, to maximize the firm's profits,

$$\pi(x) = \max_{y,L} A\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi} - W(y)L. \tag{2}$$

From the firm's problem, the first-order conditions with respect to L and y can be represented as

$$W(y) = A\phi\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi-1}, \tag{3}$$

and

$$W'(y) = A(1-\beta)\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{1}{\alpha-1}} y^{-\frac{1}{\alpha}} L^{\phi-1}. \tag{4}$$

Equation (4) shows $W'(y) > 0$, indicating that workers with higher productivity receive a higher wage.

Following the study by Eeckhout and Kircher (2018), this study assumes $0 < \alpha < 1$ such that there is positive assortative matching in labor market equilibrium. The matching rule follows

$$x = m(y), \tag{5}$$

[1]Our results do not depend on the function form of production. For example, the function $F(x, y, L) = A\left[\eta x^{\rho} + (1-\eta)(yL)^{\rho}\right]^{\frac{\gamma}{\rho}}$ can be adopted, as in the study by Adamopoulos and Restuccia (2014).

for $y \in [\underline{y}, \bar{y}]$, where $\underline{y}$ denotes the minimum human-capital level of employees, and $\bar{y}$ denotes the maximum human-capital level of employees. We have

$$y = m^{-1}(x) \equiv \nu(x), \tag{6}$$

for $x \in [\underline{x}, \bar{x}]$, where $\underline{x}$ denotes the minimum human-capital level of employers, and $\bar{x}$ represents the maximum human-capital level of employers. Therefore, $m(\cdot)$ and $\nu(\cdot)$ are increasing functions. The matching rule is determined by the human capital distribution of bosses and that of workers.

Combining Equations (3) and (4) yields

$$\frac{W'(y)}{W(y)} = \frac{1-\beta}{\phi} \frac{y^{-\frac{1}{\alpha}}}{\beta m(y)^{\frac{\alpha-1}{\alpha}} + (1-\beta) y^{\frac{\alpha-1}{\alpha}}}, \tag{7}$$

for $y \in [\underline{y}, \bar{y}]$. Thus, by Equation (7), the wage function is obtained,

$$W(y) = W(\underline{y}) \exp\left[\frac{1-\beta}{\phi} \int_{\underline{y}}^{y} \frac{z^{-\frac{1}{\alpha}}}{\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}} dz\right], \tag{8}$$

for $y \in [\underline{y}, \bar{y}]$.

From Equation (3), the labor demand is derived as follows:

$$L(x) = \left\{ \frac{A\phi \left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)\nu(x)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}}{W(\nu(x))} \right\}^{\frac{1}{1-\phi}}, \tag{9}$$

which is referred to as the size of the firm with the boss' human capital x. Firm size connects the matching rule and equilibrium wage in the labor market, and it is the crucial force of labor market equilibrium in the study by Eeckhout and Kircher (2018).

2.2 Equilibrium

The equilibrium can be defined as follows.

Definition 1 *Equilibrium consists of*

(i) the solutions to production: wage $W(y)$ and firm size $L(x)$ determined by Equations (8) and (9);

(ii) the matching rule: $m(x)$ determined by the labor market clearing condition,

$$\int_{\underline{x}}^{m(y)} L(z) f_X(z) dz = \int_{\underline{y}}^{y} f_Y(z) dz, \tag{10}$$

for $y \in [\underline{y}, \bar{y}]$ and $m(y) \in [\underline{x}, \bar{x}]$, where $f_X(z)$ and $f_Y(z)$, i.e., the density functions of X and Y, are exogenously given.

Equation (10) indicates that

$$m'(y) = \frac{f_Y(y)}{L\,(m(y))\, f_X\,(m(y))} > 0, \tag{11}$$

and

$$m(y) = \int_{\underline{y}}^{y} \frac{f_Y(z)}{L\,(m(z))\, f_X\,(m(z))} dz + \underline{x}, \tag{12}$$

for $y \in [\underline{y}, \bar{y}]$.

The equilibrium is determined by two endogenous functions: the wage function $W(y)$ and the matching rule $m(y)$. These two functions are determined by two differential equations (7) and (11). Since these two equations are nonlinear, explicit solutions are not permitted.

2.3 Application of the Deep Learning Technique

Traditionally, the shooting method is commonly used to solve matching models. For instance, in the model presented in this paper, the two integral equations, i.e., Equations (8) and (12), can be solved in the following steps. Firstly, the bounds of $W(y)$ needs to be set, and then the bisection method is adopted to obtain $W(\underline{y})$. Meanwhile, a vector $m(y)$ need to be set with an initial value in $[\underline{x}, \bar{x}]$. Secondly, given the initial values of $W(\underline{y})$ and $m(y)$, $W(y)$ can be obtained by using Equation (8). Thirdly, $L(x)$ can be obtained based on Equation (9). Fourthly, Equation (12) is used to calculate the new $m(y)$. Finally, it is checked whether the bounds of the new $m(y)$ are the same as those of x. If it is not, the bisection method is used to obtain a new $W(\underline{y})$, and steps 1 to 4 are repeated. Another way to use the shooting method is to follow the approach proposed by Eeckhout and Kircher (2018). They assume the bounds of the firm size $L(x)$ first and then solve Equation (11) to obtain $m(y)$. The other steps are similar to the former ones. Both methods require setting the initial bounds. If the bounds are not reasonable, e.g., the upper bound is too small (the initial value of $W(\underline{y})$ or $L(x)$ is smaller than the equilibrium solution), the model cannot be solved. Then, the bounds should be reset manually. Additionally, if the value of the bound deviates greatly from the equilibrium solution, it will take much time to find the equilibrium solution. Moreover, to ensure the accuracy of the integral equation and differential equation solutions, a grid of very fine points needs to be used, which also increases the computational time.

Fortunately, when the deep learning Physics-Informed Neural Networks (PINN) is adopted to solve the model, it is found that the deep learning technique can reduce the complexity of the code significantly as it does not involve as many steps as the traditional method. Meanwhile, the computation speed and the accuracy of the results increase tremendously with the deep learning technique.

The algorithm of the deep learning is as follows. Firstly, two neural networks are set up: $nn_m(y)$ and $nn_W(y)$. The $nn_m(y)$ network consists of three linear layers and a Sigmoid activation layer at the following order: the first linear layer, the second linear layer, the Sigmoid activation layer, and the third linear layer. The $nn_W(y)$ network is composed of three linear layers and two activation layers at the following order: the first linear layer, the second linear layer, the Sigmoid activation layer, the third linear layer, and the Softplus activation layer such that the output of $nn_W(y)$ is larger than 0 as $W(y) > 0$.

Secondly, the function $\tilde{m}(y) = B(\bar{y} - y) + C(y - \underline{y}) + (\bar{y} - y)(y - \underline{y})nn_m(y)$ is defined, where $B = \frac{\underline{x}}{\bar{y}-\underline{y}}$, and $C = \frac{\bar{x}}{\bar{y}-\underline{y}}$. This ensures that $\tilde{m}(\underline{y}) = \underline{x}$ and $\tilde{m}(\bar{y}) = \bar{x}$, and that $\tilde{m}(y)$ is a smooth curve, enabling quick reduction of the losses shown below. Also, $\tilde{W}(y)$ is defined as the output of $nn_W(y)$.

Finally, the losses are obtained using the differential equations Equation (7) and Equation (11),

$$Loss_W\left(\Theta_W; y_j\right) = \frac{1}{N}\sum_{j=1}^{N}\left(\tilde{W}'(\Theta_W; y_j) - \frac{1-\beta}{\phi}\frac{y_j^{-\frac{1}{\alpha}}\tilde{W}(\Theta_W; y_j)}{\beta\tilde{m}(\Theta_m; y_j)^{\frac{\alpha-1}{\alpha}} + (1-\beta)y_j^{\frac{\alpha-1}{\alpha}}}\right)^2,$$

and

$$Loss_m\left(\Theta_m; y_j\right) = \frac{1}{N}\sum_{j=1}^{N}\left(\tilde{m}'(\Theta_m; y_j) - \frac{f_Y(y_j)}{L\left(\tilde{m}(\Theta_m; y_j)\right)f_X\left(\tilde{m}(\Theta_m; y_j)\right)}\right)^2,$$

where y_j represents the input data, and Θ_W and Θ_m denote the parameters defining the two networks. The optimal Θ_W and Θ_m are found by minimizing the total loss

$$Loss_{total}\left(\Theta_W, \Theta_m; y_j\right) = Loss_W\left(\Theta_W; y_j\right) + Loss_m\left(\Theta_m; y_j\right),$$

by using the Adam algorithm in Pytorch.

Figure 1 shows some numerical results of matching, wage, and firm size by using the deep learning technique. These results are consistent with those obtained by Eeckhout and Kircher (2018). In our model, it is assumed that X and Y follow uniform distributions $U(1, 10)$. The parameter setting is as follows: $\alpha = 0.5$, $\phi = 0.5$, and $A = 1$. In this figure, at $\beta = 0.5$, $y = m(x) = x$, thus, the firm size is constant. Meanwhile, the matching rules are symmetrical with the increase in β, and the variation trends of the employees' wage and firm size are approximately symmetrical about the lines at $\beta = 0.5$.

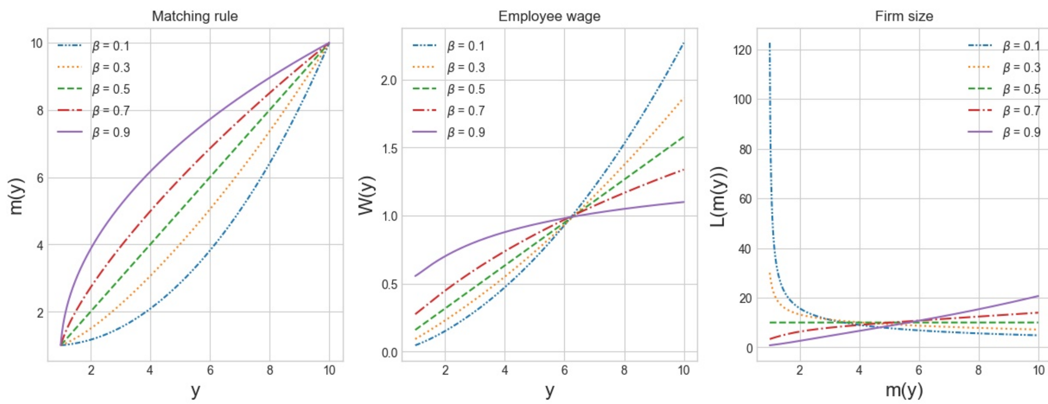

Figure 1: Matching, employee wage, and firm size.

When the weight of bosses, i.e., β, increases in production, the employees at the same level y can match bosses with more human capitals $m(y)$. Figure 1 shows the trade-off between quantity and quality of employees. If $\beta < 0.5$, the quantity dominates, the firm size exhibits a decreasing trend when y increases. This indicates that the boss hires more lower-skilled employees with lower wages, and the higher-skilled employees are paid with much higher wages. If $\beta > 0.5$, the quality dominates, the firm size exhibits an increasing trend when y increases. This indicates that the boss hires less lower-skilled employees, and the higher-skilled employees are paid with a little higher wages.

3 Perturbation on the matching

Following Costinot and Vogel (2010), comparative static analysis is conducted on the matching rule in the equilibrium. Since Equation (12) is nonlinear, it is impossible to analytically investigate the effect of β on the matching rule in the equilibrium. Therefore, this study uses the perturbation method to investigate the effects of β.[2] Besides, this paper uses hat to represent the derivative and the subscript to represent the variable with respect to which the derivative is calculated. For example, $\hat{m}_\beta(y)$ represents the derivative of $m(y)$ with respect to β.

[2]This perturbation method can be applied to other parameters as well. Due to the paper length requirement, only the effect of β on matching is presented in this paper.

Differentiating both sides of Equation (12) with respect to β yields

$$\hat{m}_\beta(y) = -\int_{\underline{y}}^{y} \frac{f_Y(z)dz}{L\left(m(z)\right) f_X(m(z))} \frac{\hat{L}_\beta(m(z))}{L\left(m(z)\right)} dz, \tag{13}$$

for all $y \in [\underline{y}, \bar{y}]$.

Substituting $x = m(y)$ into Equation (9) and then differentiating the result with respect to β, we have

$$\begin{aligned} \frac{\hat{L}_\beta(m(y))}{L\left(m(y)\right)} &= \frac{1}{1-\phi}\frac{\alpha}{\alpha-1}\frac{m(y)^{\frac{\alpha-1}{\alpha}} - y^{\frac{\alpha-1}{\alpha}}}{\beta m(y)^{\frac{\alpha-1}{\alpha}} + (1-\beta)y^{\frac{\alpha-1}{\alpha}}} - \frac{1}{1-\phi}\frac{\hat{W}_\beta(y)}{W(y)} \\ &\quad + \frac{1}{1-\phi}\frac{\beta m(y)^{\frac{\alpha-1}{\alpha}}}{\beta m(y)^{\frac{\alpha-1}{\alpha}} + (1-\beta)y^{\frac{\alpha-1}{\alpha}}}\frac{\hat{m}_\beta(y)}{m(y)}. \end{aligned} \tag{14}$$

Substituting Equation (14) into Equation (13) yields

$$\frac{\hat{m}_\beta(y)}{m(y)} = p(y) + q(y) + \frac{1}{\phi-1}\int_{\underline{y}}^{y} K(y,z)\frac{\hat{m}_\beta(z)}{m(z)}dz, \tag{15}$$

where

$$\begin{aligned} p(y) &= \frac{\alpha}{\alpha-1}\frac{1}{(\phi-1)m(y)}\int_{\underline{y}}^{y}\left[\frac{f_Y(z)}{L\left(m(z)\right) f_X(m(z))}\frac{m(z)^{\frac{\alpha-1}{\alpha}} - z^{\frac{\alpha-1}{\alpha}}}{\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta)z^{\frac{\alpha-1}{\alpha}}}\right]dz, \\ q(y) &= -\frac{1}{(\phi-1)m(y)}\int_{\underline{y}}^{y}\frac{f_Y(z)}{L\left(m(z)\right) f_X(m(z))}\frac{\hat{W}_\beta(z)}{W(z)}dz, \\ K(y,z) &= \frac{1}{m(y)}\frac{f_Y(z)}{L\left(m(z)\right) f_X(m(z))}\frac{\beta m(z)^{\frac{\alpha-1}{\alpha}}}{\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta)z^{\frac{\alpha-1}{\alpha}}}. \end{aligned}$$

There are three parts in Equation (15): the first part $p(y)$ is the direct effect from β perturbation, the second part $q(y)$ is the effect from wage perturbation, and the last part is the effect of the matching rule. $K(y,z)$ denotes the effect of employees' type interaction, i.e., the employee cross-type effect. Even though Equation (12) is a nonlinear equation of $m(y)$, the perturbed version (15) is a linear equation of $\hat{m}_\beta(y)$.

Theorem 1 *The perturbation of the matching rule follows a Volterra integral equation, and the solution is*

$$\frac{\hat{m}_\beta(y)}{m(y)} = \underbrace{p(y) + \underbrace{\lambda \int_{\underline{y}}^{y} R(y,z;\lambda)p(z)dz}_{\textit{the cross-type direct effect}}}_{\textit{the direct effect}} + \underbrace{q(y) + \underbrace{\lambda \int_{\underline{y}}^{y} R(y,z;\lambda)q(z)\,dz}_{\textit{the cross-type wage effect}}}_{\textit{the wage effect}}, \tag{16}$$

where $R(y,z;\lambda)$ is the unique resolvent kernel,

$$R(y,z;\lambda) = \sum_{i=1}^{T} \lambda^{i-1} K_i(y,z), \tag{17}$$

for $T \to \infty$, with $\lambda = \frac{1}{\phi-1}$, $K_1(y,z) = K(y,z)$, and

$$K_i(y,z) = \int_z^y K_{i-1}(y,s)K(s,z)ds, \tag{18}$$

for $i \geq 2$. By induction, each iterated kernel satisfies the condition $K_i(y,z) \equiv 0$ if $y < z$.

Theorem 1 indicates that the matching rule perturbation can be decomposed into two parts: the direct effect of β perturbation and the channel of wage perturbation. In the right-hand side of Equation (16), the first two terms represent the direct effect, while the last two terms represent the wage effect, through which β perturbation causes the changes of the matching rule.

The resolvent kernel, defined by the series in Equation (17), expresses the total effect of the employee of type z on the employee of type y. In Equation (18), the first iterated kernel $(i = 1)$ is simply $K_1(y,z) = K(y,z)$, which accounts for the impact of the employee type z on the employee type y directly; the second iterated kernel $(i = 2)$ in Equation (18) accounts for the impact of employee type z on the employee type y indirectly through the behavior of third parties s. This term can be represented as

$$K_2(y,z) = \int_z^y K(y,s)K(s,z)ds.$$

From Equation (18), it can be seen that $K_i(y, z)$ captures the interaction effect of employees' types and thus represents a cross-type effect.[3] Since $m(\underline{y}) = \underline{x}$ and $m(\bar{y}) = \bar{x}$ for any β, we have $\frac{\hat{m}_\beta(\underline{y})}{m(\underline{y})} = \frac{\hat{m}_\beta(\bar{y})}{m(\bar{y})} = 0$. Therefore, according to Equation (15), the value of $\frac{\hat{m}_\beta(y)}{m(y)}$ for any y is not only related to the integral result from $\underline{y}$ to y but also related to the integral result from y to $\bar{y}$, to ensure $\frac{\hat{m}_\beta(\bar{y})}{m(\bar{y})} = 0$. That is, the value of $\frac{\hat{m}_\beta(y)}{m(y)}$ is related to the entire support, not just to the range below y.

Figure 2 demonstrates the perturbation result according to Equation (16) with β changing from 0.3 to 0.31.

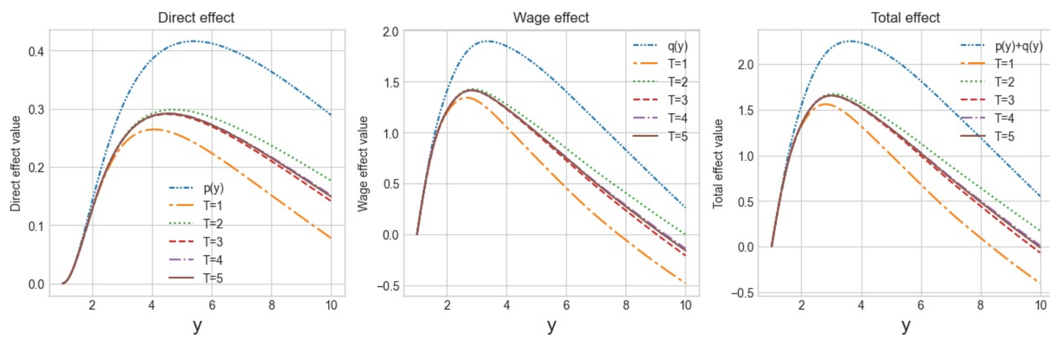

Figure 2: Perturbation result.

When $T = 1$, $p(z) + q(z)$ directly affects $\frac{\hat{m}_\beta(y)}{m(y)}$ through the effect of employee type z on employee type y. When $T = 2$, $p(z) + q(z)$ affects $\frac{\hat{m}_\beta(y)}{m(y)}$ with the indirect impact of employee type z on employee type y through the type s. The effects shown in Figure 2 fluctuate since $\lambda < 0$, and this is caused by the decreasing return of firm size in production, i.e., $\phi < 1$. The figure illustrates that the right-hand side of Equation (16) gradually converges to $\frac{\hat{m}_\beta(y)}{m(y)}$ as T increases from 1 to 5. Meanwhile, the left figure shows that the direct effect is always positive, while the middle figure shows that the wage effect could be negative when employee's human capital exceeds a certain level. In the right figure, the improvement of matching for the top skilled employees becomes smaller due to the negative impact from the wage channel.

[3] In a discrete approximation, $K(y, z)$ can be represented as a matrix. In this matrix, each row element represents a value based on employees' types y, and each column element represents a value based on employees' types z. Thus, in Equation (18), $K_{i-1}(y, s)K(s, z)$ is actually a matrix product.

4 Conclusion

Based on a stripped-down version of the benchmark study by Eeckhout and Kircher (2018), this study derived a Volterra integral equation for the perturbation of the matching rule and applied the latest deep learning technique to conduct numerical analysis on matching. It was found that the effect of the role of the boss in the production on matching can be decomposed into two channels: the direct effect and the wage effect. Besides, the experimental results indicate that the weight of bosses in the production affects the trade-off between the quality and quantity of employees. The deep learning technique facilitates the solving process, and it will benefit future research regarding the matching of many-to-one cases.

References

Becker, G. (1973). "A theory of marriage: Part I." *Journal of Political Economy*, 81, 813-846.

Eeckhout, J. and B. Jovanovic (2011). "Occupational choice and development." *Journal of Economic Theory*, 147 (2),657-683.

Eeckhout, J. and P. Kircher (2018). "Assortative matching with large firms." *Econometrica*,86(1), 85-132.

Gaubert, C. (2018). "Firm sorting and agglomeration." *American Economic Review*, 108 (11), 3117-53.

Kantorovich, L. V. (1942). "On the translocation of masses." *Dokl. Akad. Nauk USSR(NS)*, 37, 199-201.

Kelso Jr, A.S. and V. P. Crawford (1982). "Job matching, coalition formation, and gross substitutes." *Econometrica*, 50 (6), 1483-1504.

Koopmans, T. C. and M. Beckmann (1957). "Assignment problems and the location of economic activities." *Econometrica*, 25 (1) 53-76.

Lucas, R. E. (1978). "On the size distribution of business firms." *Bell Journal of Economics*, 9 (2), 508-523.

Orefice, G. and G. Peri (2020). "Immigration and worker-firm matching." *NBER Working Paper* 26860.

Raissi, M., P. Perdikaris, and G.E. Karniadakis (2019). "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations." *Journal of Computational physics*, 378, 686-707.

Shapley, L. and M. Shubik (1972). "The assignment game I: The core." *International Journal of Game Theory*, 1 (1),111-130.

Zemyan, S. M. (2012). *The Classical Theory of Integral Equations: A Concise Treatment.* Springer Science & Business Media.