Optimal Nonlinear Taxes in an Economy with Aggregate Shocks

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Abstract

We study dynamic optimal taxation in a class of economies with private information. We first use a variational approach to characterize the optimal nonlinear tax schedule in the stationary equilibrium and find the role of the endogenous distribution in the optimal long-run tax. We then introduce the aggregate shock to the heterogeneous agent model and employ stochastic control in infinite dimensions to solve the optimal nonlinear tax schedule in a Krusell-Smith economy. The paper also shows that machine learning could be a powerful tool for solving the complicated dynamic systems in macroeconomics.

JEL classification: D31, E32, H21

Keywords: nonlinear taxes, aggregate shocks, heterogeneous-agent economy, optimal control in infinite dimension, machine learning

1 Introduction

Government tax policy reacts to the change of inequality and taxes have impacts on the inequality. There are complicated intereactions between taxes and inequality in a dynamic model. To investigate optimal nonlinear taxes in an economy with aggregate shocks, we first use a variational approach to characterize the optimal nonlinear tax schedule in the stationary equilibrium and find the role of the endogeneous distribution in the optimal long-run tax policy. We then introduce the aggregate shock to the model and investigate the dynamic Mirrlessian taxation in the heterogeneous agent economy. We solve for a rational expectations equilibrium in which the agent's belief about the evolution of aggregate states are consistent with the dynamics that emerge in the economy.

For a heterogeneous agent model without aggregate shocks, we concentrate on the stationary equilibrium and investigate the nonlinear taxes that maximize social welfare in the steady state. We first use the Kolmogorov forward equation to characterize the stationary wealth distribution. Then we use the calculus of variations to find the formula for the optimal tax schedule.

For a heterogeneous agent model with aggregate shocks, we carefully characterize the state space, which permits us to write down the government's problem recursively. The nonlinear tax schedule changes with respect to the dynamic state variables of the economy. The state space consists of a joint distribution of individual wealth, promises, and the aggregate shock.

Our main idea to tackle the heterogeneous-agent Ramsey problem is to characterize the evolution of the joint distribution of wealth and promise. The problem is then isomorphic to an optimal transport problem (Kantorovich).

We employ stochastic optimal control in infinite dimensions to solve the optimal nonlinear tax schedule in a Krusell-Smith economy. The government's social wealth maximization problem is characterized by a dynamic programming problem with a stochastic partial differential equation (SPDE) as the constraint. The SPDE characterizes the evolution of the joint distribution, which describes the whole system.

The novel things of the paper are: first, we describe the government's optimal nonlinear tax problem as a recursive dynamic game of the heterogeneous agent model. We use the Kolmogorov forward equation to characterize the evolution of the economy, and the SPDE becomes the constraint of the government's social welfare maximization problem. Second, we use stochastic optimal control in infinite dimensions to solve the government's problem. Third, we try to use machine learning to implement the numerical exercise.

One major difficulty with working on these models is that the agent distribution becomes a state variable and so the state space becomes infinite dimensional. In this paper, we demonstrate how deep learning techniques can relax the "curse of dimensionality" for continuous time heterogeneous agent models and allow global numerical solutions to be computed. The paper also shows that machine learning could be a powerful tool for solving the complicated dynamic systems in macroeconomics. Specifically, we resort to reinforcement learning (RL) to solve the agent-environment interaction problem in infinite dimensions.

1.1 Literature review

Saez (2001) uses the variational approach to derive the formula of the nolinear tax in the Mirrleesian model. Sachs et al. (2020) extends the formula to a model with general equilibrium. Golosov et al. (2014) provides a good summary of the variational approach in the study of nonlinear taxes.

Chang and Park (2021) use the variational approach to investigate the optimal nonlinear tax in the Aiyagari model without aggregate shocks. The tax schedule in their paper does not change over time, even though it takes into account the transition path of the heterogeneous agent model. We permit the optimal nonlinear tax schedule to change along time. Chang and Park (2021) use simulations to calculate the Gateaux derivatives involved in variations.[1] Different from theire paper, we use numerical methods to solve the Kolmogorov forward equation directly and obtain the stationary wealth distribution. Then we calculate the Gateaux derivatives by using the difference method.

Marcet and Marimon (2019) investigate the recursive contract that can be applied to optimal policy design in dynamic macroeconomics. We extend the recursive method to the heterogeneous agent model by tracking the cross-section distribution of the state variables. Jiang et al. (2022) use a recursive method in a continuous-time model, and their model is a representative-agent model.

Golosov et al. (2016) and Farhi and Werning (2013) investigate the dynamic Mirrleesian

[1] Using seqential Monte Carlo methods to calculate the gradient is popular in reinforcement learning.

model and study the optimal nonlinear taxes. However, they do not have general equilibrium and thus omit the pecuniary externality of the taxes. We investigate the optimal nonlinear taxes in the heterogeneous agent model with incomplete financial markets.

Bhandari et al. (2021) solve the Ramsey problem in a heterogeneous agent model with aggregate shocks. They concentrate on the local dynamics around the steady state. And the steady state corresponds to a heterogeneous agent model without aggregate shocks.[2] We solve the full dynamics of the economy with aggregate shocks and we investigate the optimal nonlinear taxe.

Dyrda and Pedroni (2022) and Acikgoz et al. (2022) investigate the Ramsey problem in heterogeneous agent models. They concentrate on the transition of the economy without aggregate shocks. We investigate the optimal nonlinear taxe and we solve the full dynamics of the economy with aggregate shocks.

The paper is related to the heterogeneous agent continuous-time (HACT) model. Ahn et al. (2017) investigate the general equilibrium in a HACT model with aggregate shocks. Nuno and Moll (2018) investigate the contained efficiency problem in a HACT model.

Renner and Scheidegger (2018) solve large-scale infinite-horizon dynamic incentive problems with persistent hidden types, but they still haven't solved the general equilibrium endogenous price vector, so they don't need to track the endogenous distribution function. Nuno and Thomas (2022) and Dávila and Schaab (2022) investigate the dynamic game in HACT models. They concentrate on the open-loop solution. However, they do not find the recursive solution.

Schaab (2020) argues that the main challenge in numerically solving my model is the entire cross-sectional distribution of agents, an infinite-dimensional object, becomes part of the aggregate state space. Gu et al. (2023) use deep learning methods to solve the equilibrium of the Kresull-Smith model. They incoporate the wealth distribution into the aggregate state variables and use the Kolmogorov forward equation to describe the evolution of the wealth distributioh. However, they do not investigate the optimal policy.

Maliar et al. (2021) use machine learning to solve the general equilibrium of the Kresull-Smith model. Han et al. (2022) use reinforcement learning to study the constrained efficiency problem in a Krusell-Smith model. However, Maliar et al. (2021) and Han et al. (2022) do

[2]Reiter (2009) develops the algorithm to solve the local dynamics around the steady state of an Aiyagri model.

not investigate the implementation and the dynamic game of the heterogeneous agent model.

2 Model

Time is continuous and is indexed by $t \in [0, \infty)$. There is a continuum of infinitely-lived households with a fixed mass indexed by $j \in [0, 1]$. Each household consists of a worker and an entrepreneur. The worker provides his labor to the labor market, while the entrepreneur runs a private firm by hiring labor from the labor market.

Financial markets are incomplete. Households face the borrowing constraint $k_t \geq 0$ and they have accumulating capital within their own family businesses. The evolution of capital is given by households' budget constraint,

$$dk_t = d\pi_t + [w_t x_t l_t - c_t - \delta k_t - T_t(y_t)]\, dt, \tag{1}$$

where $d\pi_t$ is the profit earned from the family business, and w_t is the wage rate. The government tax schedule $T_t(y_t)$ is a twice-continuously differentiable function of income y_t and time t, where

$$y_t = r_t k_t + w_t x_t l_t. \tag{2}$$

Thus, the tax base includes both labor earnings and captial income.

Following Achdou et al.(2022), we assume that individual labor efficiency x_t follows a two-state Poisson process, $x_t \in \{x_1, x_2\}$ with $x_2 > x_1$. The process jumps from state 1 to state 2 with intensity λ_1 and vice versa with λ_2. In particular, the two states can be interpreted as high productivity and low productivity.

Following Angeletos (2007) and Angeletos and Panousi (2009), we assume that each household has a private business. The private firms have a Cobb-Douglas production function,

$$F(k_t, n_t) = e^{z_t} A k_t^{\alpha} n_t^{1-\alpha},$$

where $\alpha \in (0, 1)$. z_t represents the aggregate productivity shock, and A is the average productivity level. k_t is capital, and n_t is labor hired by the family business. The profit of each private firm is

$$d\pi_t = \left(e^{z_t} A k_t^{\alpha} n_t^{1-\alpha} - w_t n_t\right) dt + \sigma_1 k_t dB_t,$$

where B_t denotes a standard Brownian Motion on individual level. σ_1 measures the amount of undiversified idiosyncratic investment risk. Private firms hire labor in a competitive labor market. Wage rate w_t is determined by the labor-market equilibrium. dB_t represents the idiosyncratic investment return shock, which is a crucial mechanism generating the fat tail of the wealth distribution in the model.

We assume that z_t follows the Ornstein-Uhlenbeck process with $\bar{z} = 0$,

$$dz_t = -\eta z_t dt + \sigma_2 dW_t, \tag{3}$$

where dW_t is the innovation to a standard Brownian motion, η is the rate of mean reversion, and σ_2 captures the size of innovations. Before section 4, we ignore the process of z_t and let it be a constant. In section 4, we take z_t into consideration.

The firm chooses n_t to achieve

$$\max_{n_t} e^{z_t} A k_t^{\alpha} n_t^{1-\alpha} - w_t n_t.$$

The optimal labor hiring is

$$n_t = \left[\frac{(1-\alpha) e^{z_t} A}{w_t}\right]^{\frac{1}{\alpha}} k_t.$$

Therefore, we have

$$d\pi_t = r_t k_t dt + \sigma_1 k_t dB_t, \tag{4}$$

where

$$r_t = \alpha \left(e^{z_t} A\right)^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{w_t}\right)^{\frac{1}{\alpha}-1}.$$

The labor supply in the economy, however, is endogenous. The labor market equilibrium is an important channel through which the wealth distribution and the aggregate economy interact. The higher the aggregate capital is, the higher the equilibrium wage rate is. The high wage rate causes the low average rate of capital return. This mechanism decreases the dispersion of the wealth distribution, which influences the aggregate wealth since the saving function is nonlinear and thus the distribution matters.

We assume that each household has a GHH utility function, $u(\tilde{c}_t)$, where

$$\tilde{c}_t = c_t - m(l_t), \tag{5}$$

and a household has preferences over paths for consumption,

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u\left(\tilde{c}_t\right) dt \right] \tag{6}$$

where $\mathbb{E}_0$ is the expectation operator conditional on the information set at $t = 0$, and $u(\tilde{c}_t)$ is the instantaneous utility function with $u' > 0$ and $u'' < 0$. For simplicity, we assume that

$$u(\tilde{c}_t) = \frac{\tilde{c}_t^{\,1-\gamma}}{1-\gamma}, \tag{7}$$

where $\gamma > 1$. And

$$m(l_t) = \chi \frac{l_t^{1+\frac{1}{e}}}{1+\frac{1}{e}}, \tag{8}$$

where e is the Frisch elasticity of labor supply.

From (1), (2) and (4), we can redefine the household's budget constraint,

$$dk_t = [y_t - \tilde{c}_t - \delta k_t - T_t(y_t) - m(l_t)]dt + \sigma_1 k_t dB_t. \tag{9}$$

The household suffers from idiosyncratic labor income risk and idiosyncratic investment risk, and both of them cause the precautionary savings motive. The idiosyncratic labor income risks are represented by a two-state Poisson process. When the household wealth k is close to 0, the saving of a household in state x_1 is 0, while the counterpart in state x_2 is strictly larger than 0. The large savings of poor individuals in high-earning states are due to the precautionary savings motive. Further, the positive savings of the high income state cause the lower bound of the wealth space $\underline{k} = 0$ to act as a reflecting barrier of the wealth accumulation process $\{k_t\}_{t=0}^\infty$. Thus, process $\{k_t\}_{t=0}^\infty$ would not be stuck at zero and has a non-degenerating stationary distribution.

The government returns all revenues back to households as lump-sum redistribution and has a balanced budget in each period,

$$\int T_t(y_t)g_t(k)dk = R_t, \tag{10}$$

where $g_t(k)$ is the the capital distributions in period t, and R_t is the government revenue. Incorporating government debts into the model requires complicated techniques. For simplicity, we omit debts in this study. Debt management and tax smoothing in the heterogeneous-agents model are interesting topics, which we leave to future research.

3 The steady-state economy

The household chooses consumption at time t in order to maximize his welfare. The value function of the household at time t can be expressed as

$$v(k) = \max_{\{\tilde{c}(s)\}_{s=t}^{\infty}} \mathbb{E}_t \left[\int_t^{\infty} e^{-\rho(s-t)} u\left(\tilde{c}_s\right) ds \right]. \tag{11}$$

subject to the law of motion of individual capital (9) and borrowing limit $k_t \geq 0$. We use the shorthand notation $v_i(k)$ for the value function when household income is low ($i = 1$) or high ($i = 2$). The Hamilton-Jacobi-Bellman (HJB) equation corresponding to the problem above is

$$\rho v_i(k) = \max_{\tilde{c}_i, l_i} u(\tilde{c}_i) + v_i'(k)s_i(k) - \lambda_i \left(v_i(k) - v_j(k)\right) + \frac{1}{2}v_i''(k)\sigma_1^2 k^2, \tag{12}$$

for $i, j = 1, 2$, and $j \neq i$, where

$$s_i(k) = y_i - \tilde{c}_i - m(l_i) - T(y_i) - \delta k. \tag{13}$$

The first-order condition for consumption is

$$v_i'(k) = u'(\tilde{c}_i). \tag{14}$$

Therefore, household consumption falls with the slope of the value function. Intuitively, a steeper value function makes it more attractive to consume less and save more.

The first-order condition for labor supply is

$$(1 - T'(y_i))\, wx_i = m'(l_i). \tag{15}$$

The household problem is characterized by the HJB equation. We can compute $\tilde{c}_i = (v_i'(k))^{-\frac{1}{\gamma}}$ and $l_i = \left[\frac{(1-T'(y_i))wx_i}{\chi}\right]^e$ from the above functions, and we can obtain consumption function $c_i(k)$ and saving function $s_i(k)$. Detailed derivations of the HJB equation are in Appendix.

The cross-section capital distributions $g_{it}(k)$, $i = 1, 2$, are governed by the Kolmogorov Forward (KF) equation

$$\frac{\partial g_{it}(k)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma^2 k^2 g_{it}(k)\right] - \frac{\partial}{\partial k}\left[s_{it}(k) g_{it}(k)\right] - \lambda_i g_{it}(k) + \lambda_j g_{jt}(k), \tag{16}$$

for $j = 1, 2$ and $j \neq i$. Here we have $\sum_{i=1}^{2} \int_0^\infty g_{it}(k) dk = 1$.

The stationary distribution $g_i(k)$ satisfies

$$0 = \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma_1^2 k^2 g_i(k)\right] - \frac{\partial}{\partial k}\left[s_i(k) g_i(k)\right] - \lambda_i g_i(k) + \lambda_j g_j(k), \tag{17}$$

for $i, j = 1, 2$ and $j \neq i$.

3.1 Competitive equilibrium

The competitive equilibrium of the economy is standard.

Definition 1 *Given k_0, a competitive equilibrium is defined as sequences of prices $\{w_t, r_t\}_{t=0}^\infty$, aggregate allocations $\{C_t, N_t, K_t, Y_t\}$, government tax policy $\{T_t\}_{t=0}^\infty$ and individual plans $\{c_t, l_t\}_{t=0}^\infty$, such that the following conditions hold:*

1. given $\{w_t, r_t\}_{t=0}^\infty$ and $\{T_t\}_{t=0}^\infty$, the plans $\{c_t, l_t\}_{t=0}^\infty$ are optimal for each household;

2. the labor market clears: $\int n_t^j dj = N_t = \frac{\sum_{i=1}^{2} \int_0^\infty \lambda_i l_{it} x_i g_{it}(k) dk}{\lambda_1 + \lambda_2}$;

3. the government budget is balanced;

4. the aggregate variables are consistent with individual behaviors, $C_t = \int c_t^j dj, K_t = \int_0^1 k_t^j dj$, and $Y_t = \int e^{z_t} (k_t^j)^\alpha (n_t^j)^{1-\alpha} dj$, for all $t \geq 0$.

Besides, we find the relationship between w_t, r_t and K_t, N_t in the competitive equilibrium

$$w_t = (1-\alpha)e^{z_t}\left(\frac{K_t}{N_t}\right)^{\alpha}, \tag{18}$$

$$r_t = \alpha e^{z_t}\left(\frac{K_t}{N_t}\right)^{\alpha-1}. \tag{19}$$

3.2 Tax Incidence

We study different incicdences of the income taxation. Since the tax system T is a functional, we rely on the Gateaux difference.

Definition 2 *Let $J(a)$ be a functional and let h be arbitrary in $L^2(\Phi)$. If the limit*

$$dJ(a;h) = \lim_{\alpha\to 0}\frac{J(a+\alpha h)-J(a)}{\alpha}, \tag{20}$$

exists, it is called the Gateaux derivative of J at a in the direction h. If the limit exists for each $h \in L^2(\Phi)$, the functional J is said to be Gateaux differentiable at a.

Consider an income tax reform represented by a continuous differentiable function $h(\cdot)$ on $\mathbb{R}^+$. Then, a perturbation on tax reform is $T(\cdot)+\alpha h(\cdot)$, where $\alpha \in \mathbb{R}$ parameterizes the size of the tax reform. As in Sachs, Tsyvinski, and Wenquin (2020), the first-order effects of this perturbation can be formally represented by the Gateaux derivative in the direction h. For example, the incidence of the labor supply is

$$dl(T;h) = \lim_{\alpha\to 0}\frac{l(T+\alpha h)-l(T)}{\alpha}. \tag{21}$$

Similarly, we can define incidences for other variables such as wage rate w, interest rate r, and social welfare W. In this section, we focus on the elementary tax reforms,

$$h(y) = \frac{1}{1-F(y^*)}\mathbb{1}_{\{y\geq y^*\}}, \tag{22}$$

for a given level of income y^*. Under this tax reform, the tax payment of an individual with income above y^* increases by a constant amount $\frac{1}{1-F(y^*)}$, which can be obtained by the marginal perturbation $h'(y) = \frac{1}{1-F(y^*)}\delta_{y^*}(y)$ where δ_{y^*} is the Direc delta function at y^*.

With this tax reform, the increased government revenue due to a mechanical increase in tax payments is equal to 1$.

We start with the study of labor supply. Consider the Gateaux difference on drift term

$$ds_i(T;h) = kdr(T;h) + x_i l_i dw(T;h) - d\tilde{c}_i(T;h) + T'(y_i)wx_i dl_i(T;h) - h. \tag{23}$$

In (23), $dl_i(T;h)$ denotes the Gateaux derivative of the labor supply in response to tax reform. Recall the first-order conditions of l_i,

$$(1 - T'(y_i))\, wx_i = m'\left(l_i\right). \tag{24}$$

We can derive $dl_i(T;h)$ from the perturbation of (24),

$$dl_i(T;h) = \epsilon_w^{l_i}\frac{l_i}{w}dw + \epsilon_r^{l_i}\frac{l_i}{r}dr - \epsilon_{1-T'}^{l_i}\frac{h'(y_i)}{1-T'(y_i)}, \tag{25}$$

and the elasticity of l ($\epsilon_w^{l_i}$, $\epsilon_r^{l_i}$, $\epsilon_{1-T'}^{l_i}$) can be seen in Appendix. Elasticities with respect to r, w, and $1 - T'$ are related to individual productivity i.

Given the incidence dw and dr, the incidence on the government revenue dR is

$$\begin{aligned} dR(T;h) =& \sum_{i=1}^{2}\int_{k^*}^{\infty}\frac{g_i(k)}{1-F(y^*)}dk + dr\sum_{i=1}^{2}\int_{0}^{\infty}\left[1+\frac{\xi_i(y)}{1-\xi_i(y)}\epsilon_r^{l_i}\right]kT'(y_i)g_i(k)dk \\ &+ dw\sum_{i=1}^{2}\int_{0}^{\infty}(1+\epsilon_w^{l_i})x_i l_i T'(y_i)g_i(k)dk - \frac{T'(y^*)}{1-T'(y^*)}\frac{y^*f(y^*)}{1-F(y^*)}\sum_{i=1}^{2}\frac{g_i(\frac{y^*-wx_il_i}{r})}{f(y^*)}\xi_i(y^*)\epsilon_{1-T'}^{l_i}(y^*) \end{aligned} \tag{26}$$

where

$$\xi_i(y) = \frac{wx_il_i}{y_i} = \frac{wx_il_i}{rk + wx_il_i}. \tag{27}$$

When computing dR, we dispose grids on income y. We let $F(y)$ denote the CDF of y and $f(y)$ the PDF. Then, we can introduce it into the incidence of tax reform on $\tilde{c}$

$$d\tilde{c}_i(T;h) = (1-T')(kdr + x_il_idw) - ds_i - h + dR. \tag{28}$$

3.3 Tax reform analysis and optimal tax formula

In this section, we investigate the normative aspect of the model. Specifically, we study the different channels through which income taxation has impacts on social welfare and the relative importance of each channel. We then find the formula for optimal nonlinear income taxation.

The social welfare function in the steay state is

$$W = \sum_{i=1}^{2} \int_0^\infty \int_0^\infty e^{-\rho t} u(\tilde{c}_{it}(k)) g_{it}(k) dk dt.$$

Based on the type of households, we can do the social welfare decomposition. Owing to

$$\begin{aligned} W =& \sum_{i=1}^{2} \int_0^\infty e^{-\rho t} \int_0^\infty u(\tilde{c}_{it}(k)) g_{it}(k) dk dt \\ =& \frac{1}{\rho} \sum_{i=1}^{2} \int_0^\infty u(\tilde{c}_i(k)) g_i(k) dk. \end{aligned} \tag{29}$$

We obtain

$$\begin{aligned} dW(T;h) =& \frac{1}{\rho} \sum_{i=1}^{2} \int_0^\infty \left[u'(\tilde{c}_i) d\tilde{c}_i(T;h) g_i(k) + u(\tilde{c}_i) dg_i(T;h) \right] dk \\ =& \frac{1}{\rho} \sum_{i=1}^{2} \int_0^\infty u'(\tilde{c}_i) [(1 - T'(y_i))(kdr + x_i l_i dw) + dR] g_i(k) dk \\ & - \frac{1}{\rho} \sum_{i=1}^{2} \int_0^\infty u'(\tilde{c}_i) [ds_i(T;h) + h] g_i(k) dk \\ & + \frac{1}{\rho} \sum_{i=1}^{2} \int_0^\infty u(\tilde{c}_i) dg_i(T;h) dk. \end{aligned} \tag{30}$$

By imposing $dW = 0$, we obtain the optimal marginal tax rate at income level y^*.

$$\begin{aligned}
\frac{T'(y^*)}{1-T'(y^*)} &= \frac{1-F(y^*)}{y^* f(y^*)} \times \Gamma(y^*) \times \left[\mathcal{A}(y^*) + \mathcal{B}(y^*) + \mathcal{C}(y^*) + \mathcal{D}(y^*) + \mathcal{E}(y^*) + \mathcal{F}(y^*)\right] \\
\text{where } \Gamma(y^*) &= \frac{1}{\sum_{i=1}^{2} \frac{g_i\left(\frac{y^*-wx_il_i}{r}\right)}{f(y^*)} \xi_i(y^*) \epsilon^{l_i}_{1-T'}(y^*)} \\
\mathcal{A}(y^*) &= \sum_{i=1}^{2} \int_{k^*}^{\infty} \left[1 - \frac{u'(\tilde{c}_i)}{\varphi}\right] \frac{g_i(k)}{1-F(y^*)} dk \\
\mathcal{B}(y^*) &= \frac{1}{\varphi} \sum_{i=1}^{2} \int_{0}^{\infty} u'(\tilde{c}_i)(1-T'(y_i))(kdr + x_il_idw)g_idk \\
\mathcal{C}(y^*) &= dw \sum_{i=1}^{2} \int_{0}^{\infty} (1+\epsilon^{l_i}_{w})x_il_iT'(y_i)g_i(k)dk \\
\mathcal{D}(y^*) &= dr \sum_{i=1}^{2} \int_{0}^{\infty} \left[1 + \frac{\xi_i(y)}{1-\xi_i(y)}\epsilon^{l_i}_{r}\right] kT'(y_i)g_i(k)dk \\
\mathcal{E}(y^*) &= -\frac{1}{\varphi} \sum_{i=1}^{2} \int_{0}^{\infty} u'(\tilde{c}_i)ds_i(T;h)g_idk \\
\mathcal{F}(y^*) &= \sum_{i=1}^{2} \int_{0}^{\infty} u(\tilde{c}_i)dg_i(T;h)dk
\end{aligned} \tag{31}$$

here $\varphi = \sum_{i=1}^{2} \int_0^{\infty} u'(\tilde{c}_i)g_i(k)dk$.

With this method, we implement the iteration procedure to find the marginal tax rate through a numerical exercise. Our results roughly match those in Chang and Park (2021), and Figure 1 shows it. Chang and Park (2021) take into account the transition path of the economy, while we only consider the steady state in this section.

3.4 Numerical results

We choose proper parameters to calculate the quantiles of the distributions of income and wealth in the model and compare them with U.S. data. In the real world, the data in Table 1 and Table 2 refer to Díaz-Giménez, Glover, and Ríos-Rull (2011), who calculated the data from the Survey of Consumer Finances (SCF). From the data, we find that the Gini of capital in the U.S. is 0.816 and the Gini of income is 0.575.

To approximate the optimal tax schedule numerically, we use the optimal tax formula as an updating rule to find a fixed point in the marginal tax schedule. We set grid points on income y and assume that the tax schedule $T(y)$ is piecewise linear on the grid points.

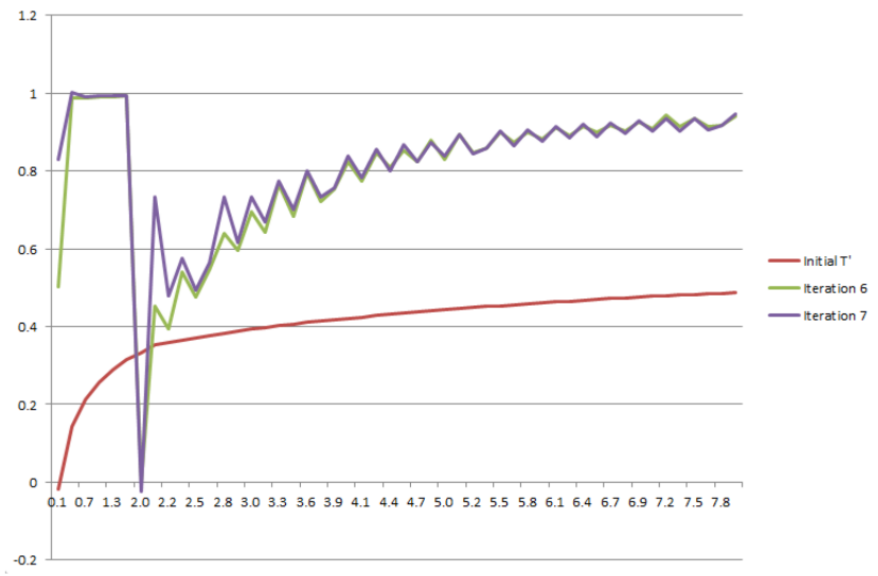


Figure 1: The optimal marginal tax rate T'

Step1 : Assume an initial tax schedule T^0.

Step2 : Given T^m, solve a steady-state equilibrium, in which agents choose optimal consumption and market clears.

Step3 : Use the optimal tax formula to compute an alternative marginal tax schedule, then we have T^{m+1}.

Step4 : Repeat **Step2** and **Step3**, until $|T^{m+1} - T^m| < \epsilon$.

The above is the basis algorithm of a numerical approximation of the optimal tax schedule, and it contains *inner* and *outer* parts. The inner part solves steady-state equilibrium, in which we deploy the *finite difference* scheme derived from Achdou et al. (2017). The outer part iterates $T'(y)$ to find a fixed-point of $T(y)$, and we use a loop algorithm similar to Chang and Park(2021). Following Sachs et al. (2020), we deploy a tax schedule with a constant rate of progressivity (CRP) as the initial guess of the iterative procedure.

The data derived from our model fit the capital in the U.S. well, with a Gini index of 0.809. Besides, the Top 1% groups share in the wealth distribution fits well. However, the counterpart of income is not fit enough, as the Gini index is 0.648.

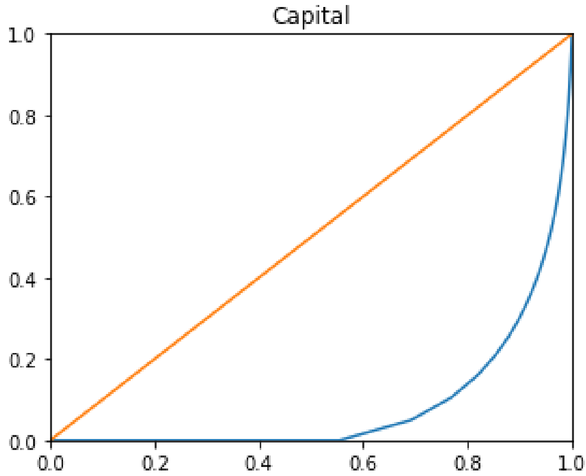


Figure 2: Lorenz curve of capital

Table 1: Wealth Distribution

Percentile	Wealth Partition							
	$0-20$	$20-40$	$40-60$	$60-80$	$80-90$	$90-95$	$95-99$	$99-100$
Wealth share (data)	-0.002	0.011	0.045	0.112	0.120	0.111	0.267	0.336
Wealth share (model)	0	0	0.049	0.109	0.170	0.154	0.298	0.217

Table 2: Income Distribution

Percentile	Income Partition							
	$0-20$	$20-40$	$40-60$	$60-80$	$80-90$	$90-95$	$95-99$	$99-100$
Income share (data)	0.028	0.067	0.113	0.183	0.138	0.102	0.159	0.210
Income share (model)	0.044	0	0.022	0.086	0.302	0.183	0.228	0.135

4 Introduce Aggregate Shocks into the Economy

A major innovation of our study is that we consider the aggregate shocks in an incomplete market. The expansion of the state space helps us deal with problems that have not been addressed in previous studies.

It should be noticed that, compared to its counterpart in Section 2, here the stationary state is expanded to take in the aggregate shock z_t. Besides, it is not a steady state but stationary instead, since z_t is not deterministic.

Our main idea to tackle the heterogeneous-agent Ramsey problem is to characterize the evolution of the joint distribution of wealth and promise. The problem is then isomorphic to an optimal transport problem (Kantorovich).

Now we turn to government problems. In an economic society, the government behaves as a social planner in order to guide and regulate economic operations. The government chooses an optimal taxation schedule $T(\cdot)$ to maximize the social welfare function. Notice that the government also "controls" the individual value function $v(\cdot)$. In fact, this is because the government needs to give "promise" to individuals for the sake of incentive compatibility. In mathematics terminology, we are solving mean field games with common noise.

4.1 Joint probability density distribution

Starting from the individual value function $v_i(k_t, r_t)$, it satisfies

$$v_i(k_t, r_t) = \max \mathbb{E}_t \left[\int_t^{\infty} e^{\rho(s-t)} u(\tilde{c}_{is}) ds \right]$$

with

$$dk_t = s_i(k_t)dt + \sigma_1 k_t dB_t, \tag{32}$$

and we derive

$$\begin{aligned} v_i(k_t, r_t) = & \max \mathbb{E}_t \left[\int_t^{t+\Delta t} e^{\rho(s-t)} u(\tilde{c}_{is}) ds + \int_{t+\Delta t}^{\infty} e^{\rho(s-t)} u(\tilde{c}_{is}) ds \right] \\ & = \max \left\{ u(\tilde{c}_{it})\Delta t + (1-\rho\Delta t)\mathbb{E}_t \left[\lambda_i \Delta t v_{-i}(k_{t+\Delta t}, r_{t+\Delta t}) + (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, r_{t+\Delta t}) \right] \right\} \\ & = \max \left[u(\tilde{c}_{it})\Delta t + \mathbb{E}_t \left[\lambda_i \Delta t v_{-i}(k_{t+\Delta t}, r_{t+\Delta t}) + (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, r_{t+\Delta t}) \right] \right\}, \end{aligned} \tag{33}$$

for $i = 1, 2$. We adopt the convention that $-i = 2$ when $i = 1$, and $-i = 1$ when $i = 2$. Now consider $\mathbb{E}_t[v_i(k_{t+\Delta t}, r_{t+\Delta t})]$. Different from the Section 2, the extra r_t here represents the environmental factor after the introduction of aggregate shock. With a increment of Δt, many variables in economy system has changed, of which the most anonyous are aggregate variables, such as price r_t and w_t. To calculate $\mathbb{E}_t[v_i(k_{t+\Delta t}, r_{t+\Delta t})]$, we take into account the evolution of r_t. Then (33) turns to

$$\begin{aligned} v_i(k_t, r_t) =& u(\tilde{c}_{it})\Delta t + \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, r_{t+\Delta t}) \psi_i(k_{t+\Delta t}, r_{t+\Delta t} | k_t, X_t) dk_{t+\Delta t} dr_{t+\Delta t} \\ &+ \iint (1 - \lambda_i \Delta t) v_i(k_{t+\Delta t}, r_{t+\Delta t}) \psi_i(k_{t+\Delta t}, r_{t+\Delta t} | k_t, r_t) dk_{t+\Delta t} dr_{t+\Delta t}, \end{aligned} \tag{34}$$

where $\psi_i(k_{t+\Delta t}, r_{t+\Delta t} | k_t, r_t)$ is the joint probability density function containing idiosyncratic labor shock i, idiosyncratic capital shock k, and aggregate shock X. And we have

$$\psi_i(k_{t+\Delta t}, r_{t+\Delta t} | k_t, r_t) = Pr_i(k_{t+\Delta t} | k_t) Pr(r_{t+\Delta t} | r_t), \tag{35}$$

for $i = 1, 2$. It can be derived in Appendix D.

Our approach here is to take the continuous-time limit. From (32) we have

$$\frac{dk_t}{k_t} = \tilde{s}_i(k_t) dt + \sigma_1 dB_t, \tag{36}$$

where $\tilde{s}_i(k_t) = \frac{s_i(k_t)}{k_t}$. By Itô's Lemma, we have

$$d \log k_t = \left[\tilde{s}_i(k_t) - \frac{1}{2} \sigma_1^2 \right] dt + \sigma_1 dB_t. \tag{37}$$

For $Pr_i(k_{t+\Delta t} | k_t)$, we have

$$Pr_i(k_{t+\Delta t} | k_t) = \frac{1}{k_{t+\Delta t} \hat{\sigma}_1 \sqrt{2\pi}} e^{-\frac{\left[\log\left(\frac{k_{t+\Delta t}}{k_t}\right) - \hat{s}_i(k_t)\right]^2}{2\hat{\sigma}_1^2}}, \tag{38}$$

where $\hat{s}_i(k_t) = \left[\tilde{s}_i(k_t) - \frac{1}{2}\sigma_1^2\right] \Delta t$, and $\hat{\sigma}_1 = \sigma_1 \sqrt{\Delta t}$, for $i = 1, 2$. Since $k_{t+\Delta t}$ is log normally distributed, we have $\log k_{t+\Delta t} \sim N\left(\log k_t + \hat{s}_i(k_t), \hat{\sigma}_1^2\right)$.

4.2 Envelope condition

Now we derive a necessary condition for incentive compatibility. First, we take derivative on (34) with respect to $\tilde{c}_{it}$

$$
\begin{aligned}
0 =& u'(\tilde{c}_{it})\Delta t + \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t}) \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial \tilde{c}_{it}} dk_{t+\Delta t} dX_{t+\Delta t} \\
&+ \iint (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, X_{t+\Delta t}) \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial \tilde{c}_{it}} dk_{t+\Delta t} dX_{t+\Delta t} \\
=& u'(\tilde{c}_{it})\Delta t + \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t}) \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) \Omega_i(k_t) \Phi_i(k_t) dk_{t+\Delta t} dX_{t+\Delta t} \\
&+ \iint (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, X_{t+\Delta t}) \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) \Omega_i(k_t) \Phi_i(k_t) dk_{t+\Delta t} dX_{t+\Delta t},
\end{aligned}
\tag{39}
$$

where

$$
\Omega_i(k_t) = \frac{\log\left(\frac{k_{t+\Delta t}}{k_t}\right) - \hat{s}_i(k_t)}{k_t \hat{\sigma}_1^2},
$$

and

$$
\Phi_i(k) = k_t \frac{\partial \hat{s}_{it}}{\partial \tilde{c}_{it}} = -\Delta t,
$$

for $i = 1, 2$.

Similarly, for l_{it} we have

$$
\begin{aligned}
0 =& \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, r_{t+\Delta t}) \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial l_{it}} dk_{t+\Delta t} dX_{t+\Delta t} \\
&+ \iint (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, r_{t+\Delta t}) \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial l_{it}} dk_{t+\Delta t} dX_{t+\Delta t} \\
=& \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, r_{t+\Delta t}) \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) \Omega_i(k_t) \Pi_i(k_t) dk_{t+\Delta t} dX_{t+\Delta t} \\
&+ \iint (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, r_{t+\Delta t}) \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) \Omega_i(k_t) \Pi_i(k_t) dk_{t+\Delta t} dX_{t+\Delta t},
\end{aligned}
\tag{40}
$$

where

$$
\Pi_i(k_t) = \left[w x_i \left(1 - T'(y_{it})\right) - \chi l_{it}^{\frac{1}{e}} \right] \Delta t,
$$

for $i = 1, 2$. (39) and (40) are the first-order conditions for the household's problem, and we can obtain optimal c^* and l^* from solving them.

Then, we take derivative on (34) with respect to k_t,

$$\begin{aligned}
&\frac{\partial v_i(k_t, X_t)}{\partial k_t}\\
=&u'(\tilde{c}_{it})\frac{\partial \tilde{c}_{it}}{\partial k_t}\Delta t + \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t})\frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial k_t}dk_{t+\Delta t}dX_{t+\Delta t}\\
&+\iint (1-\lambda_i\Delta t)v_i(k_{t+\Delta t}, X_{t+\Delta t})\frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial k_t}dk_{t+\Delta t}dX_{t+\Delta t}\\
=&\iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t})\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Omega_i(k_t)\Theta_i(k_t)dk_{t+\Delta t}dX_{t+\Delta t}\\
&+\iint (1-\lambda_i\Delta t)v_i(k_{t+\Delta t}, X_{t+\Delta t})\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Omega_i(k_t)\Theta_i(k_t)dk_{t+\Delta t}dX_{t+\Delta t}\\
&+\left\{\frac{u(\tilde{c}_{it})}{\partial \tilde{c}_{it}}\Delta t + \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t})\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Omega_i(k_t)\Phi_i(k_t)dk_{t+\Delta t}dX_{t+\Delta t}\right.\\
&\left.+\iint (1-\lambda_i\Delta t)v_i(k_{t+\Delta t}, X_{t+\Delta t})\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Omega_i(k_t)\Phi_i(k_t)dk_{t+\Delta t}dX_{t+\Delta t}\right\}\frac{\partial \tilde{c}_{it}}{\partial k_t}\\
&+\left\{\iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t})\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Omega_i(k_t)\Pi_i(k_t)dk_{t+\Delta t}dX_{t+\Delta t}\right.\\
&\left.+\iint (1-\lambda_i\Delta t)v_i(k_{t+\Delta t}, X_{t+\Delta t})\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Omega_i(k_t)\Pi_i(k_t)dk_{t+\Delta t}dX_{t+\Delta t}\right\}\frac{\partial l_{it}}{\partial k_t},
\end{aligned} \tag{41}$$

where

$$\Theta_i(k_t) = 1 + r_t\left(1 - T'(y_{it})\right) - \delta - \tilde{s}_i(k_t). \tag{42}$$

With (39) and (40), an envelope condition suggests that

$$\begin{aligned}
&\frac{\partial v_i(k_t, X_t)}{\partial k_t}\\
=&\iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, r_{t+\Delta t})\psi_i(k_{t+\Delta t}, r_{t+\Delta t}|k_t, r_t)\Omega_i(k_t)\Theta_i(k_t)dk_{t+\Delta t}dr_{t+\Delta t}\\
&+\iint (1-\lambda_i\Delta t)v_i(k_{t+\Delta t}, r_{t+\Delta t})\psi_i(k_{t+\Delta t}, r_{t+\Delta t}|k_t, r_t)\Omega_i(k_t)\Theta_i(k_t)dk_{t+\Delta t}dr_{t+\Delta t}.
\end{aligned}$$

4.3 Promise factor

We aim to solve the dynamic games between government and households. Since there are forward-looking constraints, the solution procedure for such games cannot be recursive. It is not measurable for the current physical measure; in fact, it corresponds to anticipating stochastic calculus in stochastic analysis. The solution is orbital dependent, and showing as history dependent. This is quite common in the inverse equations of finance. Therefore, we need to introduce a promise and expand the state space, in order to make solutions recursive.

We can regard Γ_{it} as components of the partial derivative of $v_i(k_t, r_t)$ with regard to k_t,

$$\begin{aligned}
\Gamma_{it} = & \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t}) \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) \Omega_i(k_t) \Theta_i(k_t) dk_{t+\Delta t} dX_{t+\Delta t} \\
& + \iint (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, X_{t+\Delta t}) \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) \Omega_i(k_t) \Theta_i(k_t) dk_{t+\Delta t} dX_{t+\Delta t} \\
= & \frac{\Theta_i(k_t)}{\Psi_i(k_t)} \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t}) \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial k_t} dk_{t+\Delta t} dX_{t+\Delta t} \\
& + \frac{\Theta_i(k_t)}{\Psi_i(k_t)} \iint (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, X_{t+\Delta t}) \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial k_t} dk_{t+\Delta t} dX_{t+\Delta t},
\end{aligned}$$

where

$$\Psi_i(k_t) = 1 + k_t \frac{\partial \hat{s}_i}{\partial k_t},$$

for $i = 1, 2$.

Now we look back to cross-sectional distribution,

$$\psi_i(k_{t+\Delta t}, r_{t+\Delta t}|k_t, r_t) = Pr(r_{t+\Delta t}|r_t) \frac{1}{k_{t+\Delta t} \hat{\sigma}_1 \sqrt{2\pi}} e^{-\frac{\left[\log\left(\frac{k_{t+\Delta t}}{k_t}\right) - \hat{s}_i(k_t)\right]^2}{2\hat{\sigma}_1^2}},$$

for $i = 1, 2$.

Compute the derivative of $\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)$ with respect to k_t and $k_{t+\Delta t}$, we find that

$$k_t \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial k_t} = -\Psi_i(k_t) \left[k_{t+\Delta t} \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial k_{t+\Delta t}} + \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) \right], \tag{43}$$

for $i = 1, 2$.

Then we can construct

$$\begin{aligned}
k_t \Gamma_{it} = & \frac{\Theta_i(k_t)}{\Psi_i(k_t)} \iint \lambda_i \Delta t v_{-i}(k_{t+\Delta t}, X_{t+\Delta t}) k_t \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial k_t} dk_{t+\Delta t} dX_{t+\Delta t} \\
& + \frac{\Theta_i(k_t)}{\Psi_i(k_t)} \iint (1-\lambda_i \Delta t) v_i(k_{t+\Delta t}, X_{t+\Delta t}) k_t \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial k_t} dk_{t+\Delta t} dX_{t+\Delta t} \\
= & \Theta_i(k_t) \iint k_{t+\Delta t} \Gamma_{i,t+\Delta t} \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) dk_{t+\Delta t} dX_{t+\Delta t},
\end{aligned}$$

for $i = 1, 2$. Here we use (43) to make it, and we consider the transition probability since there are idiosyncratic shocks. With Martingale Representation Theorem, we obtian

$$\Theta_i(k_t)k_{t+\Delta t}\Gamma_{i,t+\Delta t} - k_t\Gamma_{it} - \sigma_W\sigma_2 dW_t = \sigma_B\sigma_1 k_t dB_t,$$

which implies that, in the continuous-time limit, we can write

$$d(k_t\Gamma_{it}) = [(1-\Theta_i(k_t))(k_t\Gamma_{it})]dt + \sigma_B\sigma_1 k_t dB_t + \sigma_W\sigma_2 dW_t.$$

Applying Itô's Lemma, we infer that Γ solves the following stochastic differential equation

$$d\Gamma_{it} = \left\{\left[\delta - r_t\left(1-T'(y_{it})\right)\right]\Gamma_{it} - \sigma_1^2\hat{\sigma}_B\right\}dt + \sigma_1\hat{\sigma}_B dB_t + \sigma_2\hat{\sigma}_W dW_t,$$

where $\hat{\sigma}_B$ and $\hat{\sigma}_W$ are functions of the state variables $\{i, k_t, \Gamma_{it}\}$.

4.4 Incentive compatible by promise factor

Since we already have the definition and evoluntion of "promise factor" Γ_{it}, now we back to the first-order condition. For (39) we have

$$0 = u'(\tilde{c}_{it})\Delta t + \frac{\Phi_i(k)}{\Theta_i(k_t)}\Gamma_{it} = u'(\tilde{c}_{it})\Delta t - \frac{\Gamma_{it}}{\Theta_i(k_t)}\Delta t,$$

with $\Delta t \to 0$, we obtain

$$u'(\tilde{c}_{it}) = \frac{\Gamma_{it}}{\Theta_i(k_t)}. \tag{44}$$

Actually, it is homeomorphous with that in the household problem.

As for l_{it}, we introduce the definition of Γ_{it} into (40)

$$0 = \frac{\Pi(l_{it})}{\Theta_i(k_t)}\Gamma_{it}, \tag{45}$$

for any $\Delta t \to 0$ the equation holds, it requires

$$wx_i\left(1-T'(y_{it})\right) - \chi l_{it}^{\frac{1}{e}} = 0. \tag{46}$$

It is also compatible with that in the household problem.

4.5 Derivations of SPDE

Given the first-order condition, we can regard Γ_{it} as "promise factor" by solving optimal choices c_t and l_t from the first-order condition. The optimal taxation schedule, $\{T_{it}\}$, depends on an entire story of both idiosyncratic and aggregate shocks $\{B_t, W_t; t \geq 0\}$. Using the Martingale Representation Theorem, without loss of generality, we can represent the dynamics of the "promise factor" $\{\Gamma_{it}; t \geq 0\}$ in individual problems as

$$d\Gamma_{it} = \mu_{it}dt + \sigma_{\Gamma,B}dB_t + \sigma_{\Gamma,W}dW_t, \tag{47}$$

where $\mu_{it} = [\delta - r_t(1 - T'(y_{it})]\,\Gamma_{it} - \sigma_1\sigma_{\Gamma,B}$.

For simplicity, we let

$$\mathcal{S}_t = \begin{bmatrix} k_t \\ \Gamma_{it} \end{bmatrix}, \quad b_t = \begin{bmatrix} s_i(k_t) \\ \mu_{it} \end{bmatrix}, \tag{48}$$

and

$$\Sigma_{1,t} = \begin{bmatrix} \sigma_1 k_t \\ \sigma_{\Gamma,B} \end{bmatrix}, \quad \Sigma_{2,t} = \begin{bmatrix} 0 \\ \sigma_{\Gamma,W} \end{bmatrix}. \tag{49}$$

The evolution of $\mathcal{S}_t$ is

$$d\mathcal{S}_t = b_t dt + \Sigma_{1,t}dB_t + \Sigma_{2,t}dW_t. \tag{50}$$

Here, B_t is the representation of idiosyncratic shock, and W_t is said to be the common source of noise. The evolution of Γ_{it} in (47) suggests that the "economic promises" are constrained by idiosyncratic and aggregate shocks. When managing its taxation dynamics, the government also actively engages in idiosyncaratic and aggregate risk management by choosing hedging demands $\sigma_{\Gamma,B}$ and $\sigma_{\Gamma,W}$. The government chooses taxation T_{it}, idiosyncratic-risk hedging demand $\sigma_{\Gamma,B}$, and aggregate-risk hedging demand $\sigma_{\Gamma,W}$ to maximize the value function.

We apply Itô's formula and derive the evolution form of $\mathcal{G}_t(i,k,\Gamma)$, which is the joint distribution of state variables $\{i,k,\Gamma\}$

$$d\mathcal{G}_t(i,k,\Gamma) = \mathcal{M}(\mathcal{G}_t(i,k,\Gamma))dt + \mathcal{N}(\mathcal{G}_t(i,k,\Gamma))dW_t, \tag{51}$$

where

$$
\begin{aligned}
\mathcal{M}(\mathcal{G}_t) =& \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma_1^2 k_t^2 \mathcal{G}_t(i,k,\Gamma)\right] + \frac{1}{2}\frac{\partial^2}{\partial \Gamma^2}\left[(\sigma_{\Gamma,B}^2 + \sigma_{\Gamma,W}^2)\mathcal{G}_t(i,k,\Gamma)\right] \\
&- \frac{\partial}{\partial k}\left[s_{it}\mathcal{G}_t(i,k,\Gamma)\right] - \frac{\partial}{\partial \Gamma}\left[\mu_{it}\mathcal{G}_t(i,k,\Gamma)\right] - \lambda_i \mathcal{G}_t(i,k,\Gamma) + \lambda_{-i}\mathcal{G}_t(-i,k,\Gamma)
\end{aligned}
\tag{52}
$$

and

$$
\mathcal{N}(\mathcal{G}_t(i,k,\Gamma)) = \frac{\partial[\sigma_{\Gamma,W}\mathcal{G}_t(i,k,\Gamma)]}{\partial \Gamma}. \tag{53}
$$

for $i = 1, 2$. This is a stochastic partial differential equation (SPDE).

Combining $\mathcal{G}$ and z and letting $\mathcal{X} = \begin{bmatrix} \mathcal{G} \\ z \end{bmatrix}$, we have

$$
d\mathcal{X}_t = \mathcal{P}dt + \mathcal{Q}dW_t \tag{54}
$$

where

$$
\mathcal{P} = \begin{bmatrix} \mathcal{M} \\ -\eta z_t \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} \mathcal{N} \\ \sigma_2 \end{bmatrix}. \tag{55}
$$

4.6 Optimal taxation

For the government, the social welfare function is

$$
\mathcal{W}(\mathcal{G}_t, z_t) = \max_{T,\sigma_{\Gamma,B},\sigma_{\Gamma,W}} \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\left(\sum_{i=1}^{2}\iint u(\tilde{c}_s(i,k,\Gamma)\mathcal{G}_s(i,k,\Gamma)dkd\Gamma\right)ds\right]
$$

s.t.

$$
\begin{aligned}
d\mathcal{X}_t =& \mathcal{P}dt + \mathcal{Q}dW_t, \\
u'(\tilde{c}_{it}) =& \frac{\Gamma_{it}}{\Theta_i(k_t)}, \\
\chi l_{it}^{\frac{1}{e}} =& w x_i\left(1 - T'(y_{it})\right)
\end{aligned}
\tag{56}
$$

This is an optimal control problem in infinite dimension, since the evolution equation of $\mathcal{X}_t$ is stochastic partial differential equation. Notice that $u(\tilde{c}_s(i,k,\Gamma))$ is actually deterministic part, now the whole control focus on $\mathcal{G}_s(i,k,\Gamma)$.

Using a recursive argument, we can obtain the HJB equation that $\mathcal{W}(\mathcal{G}, z)$ satisfies

$$
\begin{aligned}
&\rho\mathcal{W}(\mathcal{G}, z) \\
=&\max_{T,\sigma_\Gamma,B,\sigma_\Gamma,W} \sum_{i=1}^{2} \int u(\tilde{c}(i,k,\Gamma))\mathcal{G}(i,k,\Gamma)dkd\Gamma + \left\langle \mathcal{P}, \frac{\partial \mathcal{W}}{\partial \mathcal{X}} \right\rangle + \frac{1}{2}Tr\left[\mathcal{Q}\mathcal{Q}^T \frac{\partial^2 \mathcal{W}}{\partial \mathcal{X}^2}\right] \\
=&\max_{T,\sigma_\Gamma,B,\sigma_\Gamma,W} \sum_{i=1}^{2} \int u(\tilde{c}(i,k,\Gamma))\mathcal{G}(i,k,\Gamma)dkd\Gamma + \sum_{i=1}^{2} \int \mathcal{B}\mathcal{W}_g dkd\Gamma - \eta z \mathcal{W}_z \\
&+ \frac{1}{2}\left[\sum_{i=1}^{2} \iint \mathcal{W}_{gg}(x,y)\mathcal{D}(x)\mathcal{D}(y)dydx + 2\sum_{i=1}^{2} \int \mathcal{W}_{gz}(x)\mathcal{D}\sigma_2 dx + \mathcal{W}_{zz}\sigma_2^2\right].
\end{aligned}
$$

The main challenge is to compute a series of choice functions. Comparing with Ahn et al. (2017), our methods have some advantages. First, Ahn et al. (2017) exert the first-order Taylor expansion on steady state, while we deploy the same things on the whole state space of the dynamic system. Second, the model of Ahn et al. (2017) only has one transition path, while we enlarge the state space to make the model recursive. Besides, we blaze a new trail to solve the social planner problem by recursive methods, which is quite different from Nuno (2018) and Dávila and Schaab (2022).

5 Numerical exercises

As of now, our problem is gigantic and systematic, mainly because of the introduction of environment. With the evolution of z in the environment, the equilibrium diverges, and the state space is infinite-dimensional. Besides, there are interactions between agents and environments. One major difficulty with working on these models is that the agent distribution becomes a state variable and so the state space becomes infinite dimensional. Traditional numerical computation methods are scarcely able to solve such problems, so we have to resort to reinforcement learning (RL).

The economics literature has traditionally used three main approaches for solving heterogeneous agent models with aggregate shocks. One approach is to fit a statistical approximation to the law of motion for the key aggregate state variables (e.g. Krusell and Smith (1998), Den Haan (1997), Fernández-Villaverde et al. (2018)). As has been extensively discussed in the literature, this approach works well when the law of motion for the key state variables can be well approximated as a function of key moments of the distribution (and so the economy is very close to permitting "aggregation"). By contrast, our approach can handle economies without near aggregation results. A second approach is to take a type of linear perturbation in the aggregate state and then solve the resulting linear problem with matrix algebra (e.g. Reiter (2002), Reiter (2008), Reiter (2010), Winberry (2018), Ahn et al. (2018), Auclert et al. (2021), Bilal (2021), Bhandari et al. (2023)). By contrast, we solve the model globally and so can handle partial differential equations with extensive non-linearity. A final approach is to take a low dimensional projection of the distribution (e.g. Prohl (2017), Schaab (2020)). Our approach is complementary to these papers in that it allows for more general, higher dimensional projections.

5.1 The RL method

Broadly speaking, reinforcement learning is a computational method by which agents interact with the environment to achieve goals. A round of interaction means that the agent makes an action decision in a state of the environment, applies this action to the environment, and the environment changes accordingly and gives the corresponding reward feedback and the next round of state back to the agent. The interaction is iterative, and the agent's goal is to maximize the expectation of cumulative rewards earned over the course of multiple rounds of interaction. Agents in reinforcement learning can not only perceive information about the environment around them, but also directly change the environment by making decisions, instead of just giving some predictive signals.

Specifically, in each round of interaction, the agent senses the current state of the environment, calculates the action of the epicycle, and then acts on it in the environment. After the environment receives the action of the agent, the corresponding immediate reward signal is generated, and the corresponding state transition occurs. The agent perceives the new environmental state in the next round of interaction, and so on. In the dynamic environment, every time the agent interacts with the environment, the environment will generate corresponding reward signals, which are often represented by real scalar numbers. This reward signal is usually a timely feedback signal that interprets the current state or action, like the score value of an action during the course of playing a game. The reward signals obtained in each round of the interaction process can be added up to form the overall return of the agent, like the score value at the end of a game. According to the dynamic nature of the environment, we can know that even if the environment and the agent's strategy remain unchanged, the initial state of the agent will also remain unchanged, and the interaction between the agent and the environment will probably produce different results and corresponding returns. Therefore, in reinforcement learning, we pay attention to the expectation of return and define it as value, which is the optimization goal of the agent.

The overall idea of our RL algorithm is an iterative procedure starting from the initial guess of the government policy T^0, $\sigma^0_{\Gamma,B}$, and $\sigma^0_{\Gamma,W}$. Each iteration (in RL we call it an episode) includes three steps: (a) gerenate the training dataset with current policy; (b) use the dataset to train RL framework and refresh neural network parameters; (c) update the policy and start a new iteration with the refreshed policy. We have two sub-networks. The first approximates the new policies T, $\sigma_{\Gamma,B}$, and $\sigma_{\Gamma,W}$ with parameter Θ_{Pol}, and the second approximates the welfare function $\mathcal{W}$ with parameter $\Theta_{\mathcal{W}}$. Both sub-networks take $(\mathcal{G}, z)$ in training set as input.

To solve this infinite-dimensional optimal control problem in continuous time more efficiently, we focus on the PPO (Proximal Policy Optimization) algorithm. PPO is a kind of actor-critic framework, so it can handle continuous-time issues. Besides, PPO reduces the calculation time significantly in the following ways: First, it restricts in the objective function to ensure that the gap between the new parameter and the old parameter is not too large, by means of POP-penalty and POP-clip. Second, it reuses the data under the old policy to refresh the new policy by means of importance sampling.

Now we further explain the details of the three main steps of the T round.

Preparing the training dataset. We simulate the economy for N periods under current policies T^{t-1}, $\sigma_{\Gamma,B}^{t-1}$, and $\sigma_{\Gamma,W}^{t-1}$. Then we store enough samples of state variables $(i, k, \Gamma; \mathcal{G}, z)$, which will be used as the initial condition for later updating.

Update the welfare function. Given the policy, updating the welfare function can be formulated as a supervised learning problem. First, we use the training data to calculate a truncated real welfare function

$$\hat{\mathcal{W}} = \sum_{t=0}^{N_1} e^{-\rho t} \left(\sum_{i=1}^{2} \iint u(\tilde{c}_t(i, k, \Gamma)\mathcal{G}_t(i, k, \Gamma) dk d\Gamma \right) \tag{57}$$

Then we use $\mathcal{W}_{NN}(\Theta_{\mathcal{W}})$ to approximate the expected welfare function, and we can construct loss function

$$Loss_{\mathcal{W}} = \mathcal{W}_{NN}(\Theta_{\mathcal{W}}) - \hat{\mathcal{W}} \tag{58}$$

Until loss function converges, we update the welfare function $\mathcal{W}_{NN}$ with parameter $\Theta_{\mathcal{W}}$.

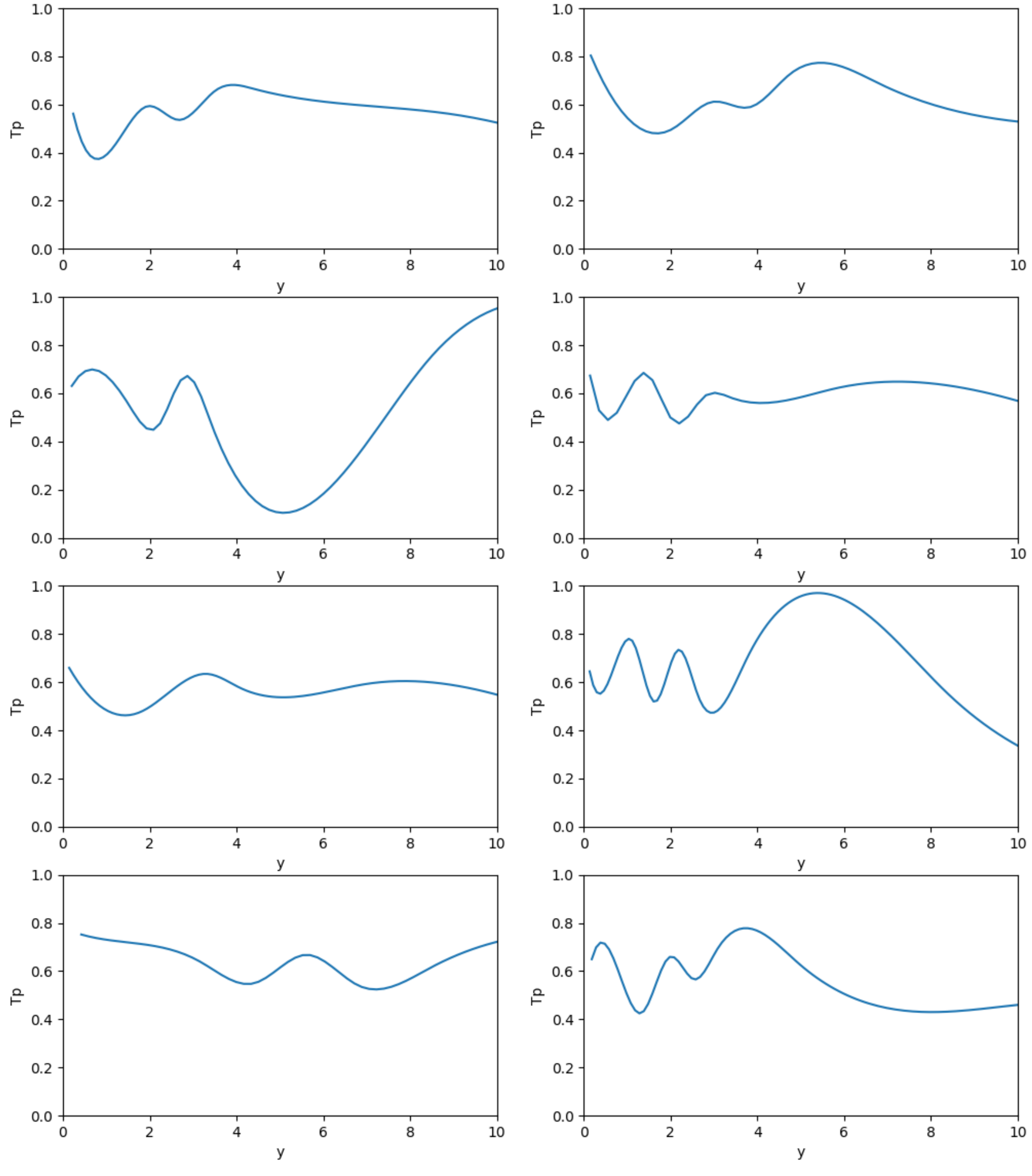
Optimize policy function. Now we need to optimize the parameters of the policy function neural network over simulated paths. Given the updated welfare function $\mathcal{W}_{NN}(\Theta_{\mathcal{W}})$, we update Θ_{Pol} in Pol_{NN} by solving such a maximum problem

$$\mathbb{E}_{\mathcal{G}^{t-1}} \left[\sum_{s=1}^{M_2} \beta^s \mathcal{W}_{NN}(\Theta_{Pol}) + \beta^{M_2} \mathcal{W}_{NN}(\Theta_{\mathcal{W}}) \right] \tag{59}$$

The Θ_{Pol} will be updated by the SGD (Stochastic Gradient Descent) algorithm. When the new policy Pol^k are generated, a new epoch starts. After T epochs, we believe the final policies Pol^T are acceptable, so we regard them as optimal policies. Unlike conventional machine learning methods, the convergence of RL is not judged by the loss function but is instead achieved when the rewards in each epoch turn out to be stable.

Here is the pesudo code of our RL algorithm.

Here are the policy curves under a random seed.


Tp
y
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10

Algorithm 1: Framework of RL algorithm on Policy Optimization

Input: Iteration epoch T, neural network with parameters $\Theta_{\mathcal{W}}$ and Θ_{Pol}

Output: Optimal policy function $Pol = \{T,\ \sigma_{\Gamma,B},\ \sigma_{\Gamma,W}\}$

for $t = 1, 2, ..., T$ **do**
  Given policy function Pol^{t-1}, prepare stationary distribution $\mathcal{G}^{t-1}$;
  for $i = 1, 2, ..., N_1$ **do**
    Simulate M_1 samples of state (s, X) from $\mathcal{G}^{t-1}$;
    Compute the realized welfare $\hat{\mathcal{W}}$ through these samples;
    Compute $Loss = |\hat{\mathcal{W}} - \mathcal{W}_{NN}|$ and generate the gradient $\nabla_{\Theta_{\mathcal{W}}}$;
    Update $\Theta_{\mathcal{W}}$ with $\nabla_{\Theta_{\mathcal{W}}}$
  end
  for $j = 1, 2, ..., N_2$ **do**
    Simulate M_2 samples of state (s, X) with new policy;
    Solve Maximum problem $\mathbb{E}_{\mathcal{G}^{t-1}}\left[\sum_{s=1}^{M_2}\beta^s\mathcal{W}(Pol_{NN}) + \beta^{M_2}\mathcal{W}_{NN}\right]$ and generate the gradient $\nabla_{\Theta_{Pol}}$;
    Update Θ_{Pol} with $\nabla_{\Theta_{Pol}}$
  end
  Define Pol^t
end

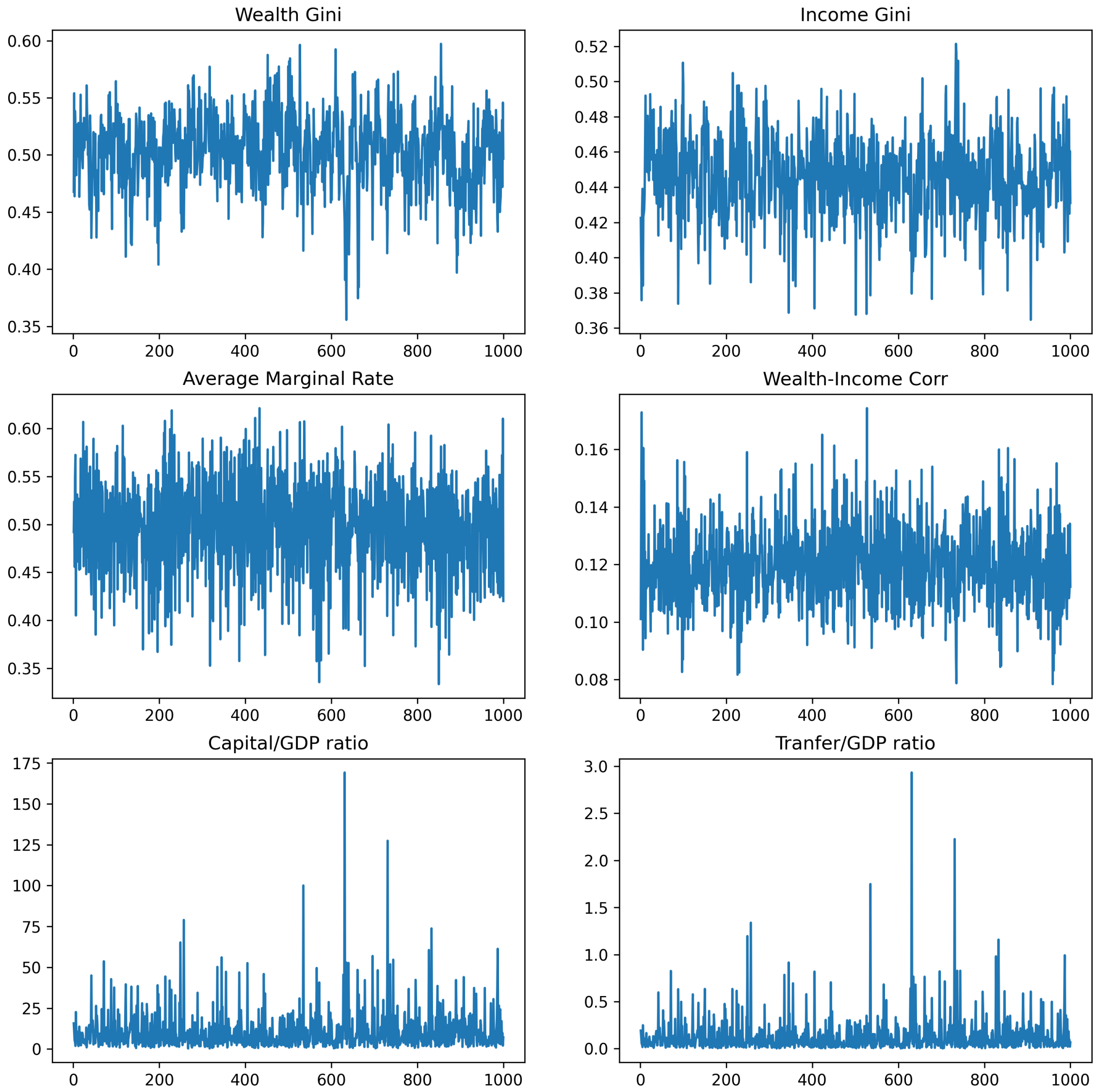
The following is the simulation under a random seed. We randomly generate a 1000-preriod $z-$series, under simulate the economy with an established and already trained marginal rate. Here we capture four index: wealth Gini index, income Gini index, average marginal rate, and ratio of government transfers to GDP.

With the above results, the ability to describe economic scenarios of our model is illustrated. Table 3 shows a comparison of the model data with realistic U.S. data or previous studies.

Table 3: Index Comparation

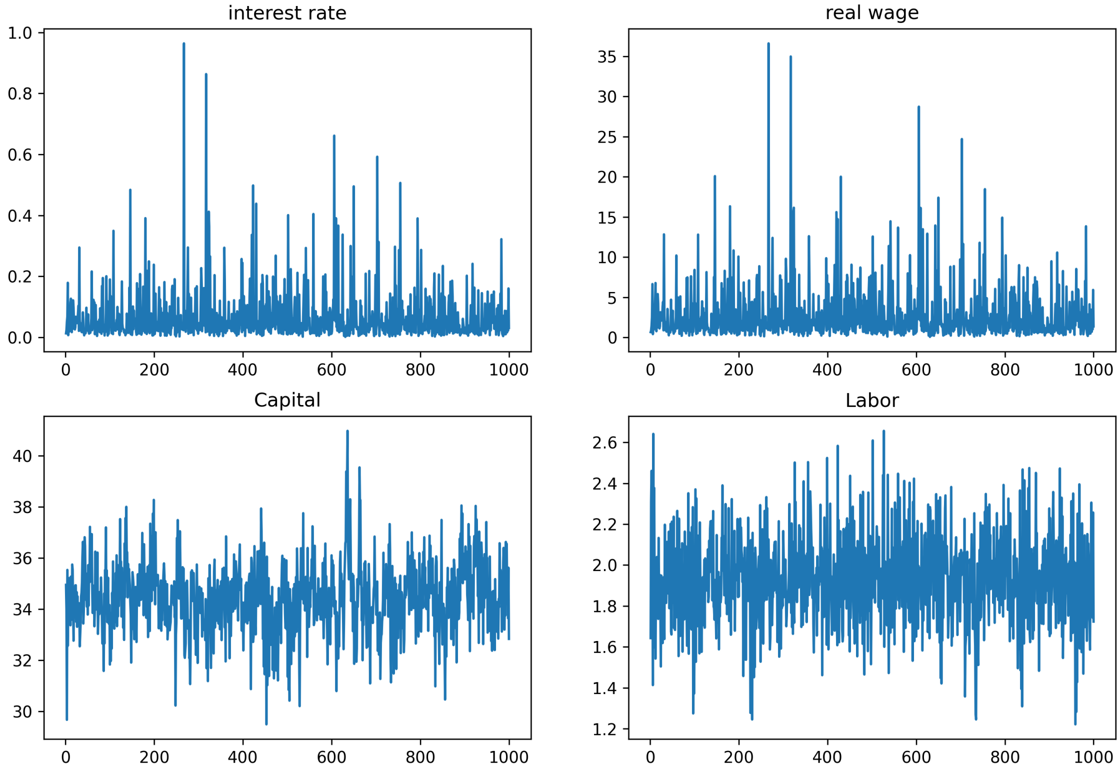
Index	U.S. data	Our model
Wealth Gini	0.816	0.521
Income Gini	0.575	0.454
Average MR	0.255	0.486
Wealth-Income Cor	0.430	0.128
Capital/GDP	2.540	5.58
Transfer/GDP	0.150	0.128

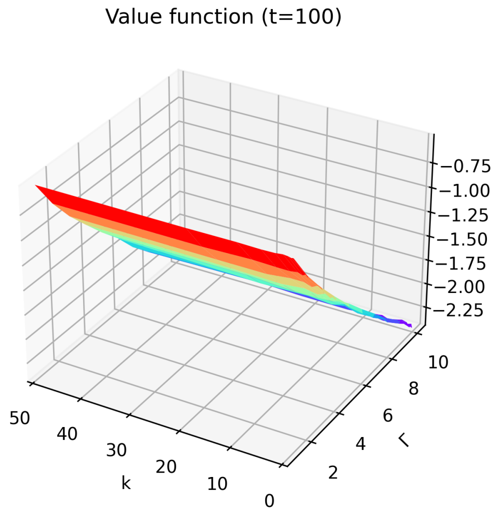
In particular, the U.S. data of the *Wealth Gini* index and *Income Gini* index is calculated by Díaz-Giménez et.al. (2011) from the Survey of Consumer Finances (SCF). The *Average MR* is from Chang, Y. and Y. Park (2021), and the *Transfer/GDP* ratio is calculated from BEA

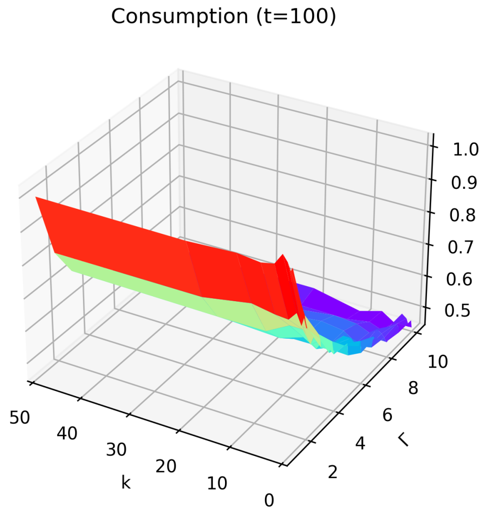

Wealth Gini
Income Gini
Average Marginal Rate
Wealth-Income Corr
Capital/GDP ratio
Tranfer/GDP ratio

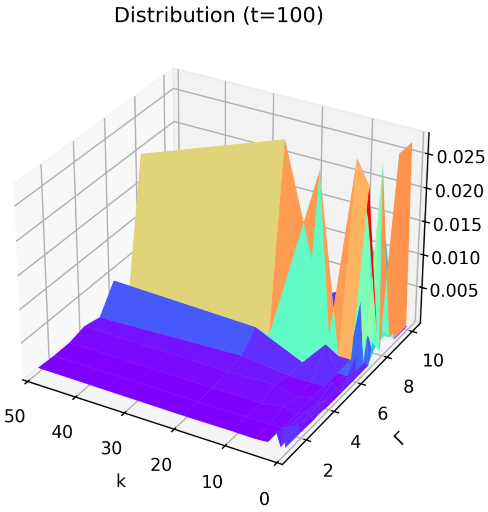
database. Data of *Wealth-Income Corr* and *Capital/GDP* are from Bhandari et.al. (2023).

Besides, we also report the aggregate variables.




Value function (t=100)
−0.75
−1.00
−1.25
−1.50
−1.75
−2.00
−2.25
50
40
30
20
10
0
k
2
4
6
8
10
l


Consumption (t=100)
1.0
0.9
0.8
0.7
0.6
0.5
50
40
30
20
10
0
k
2
4
6
8
10
l



A Derivations in Section 2

A.1 HJB equation

Consider the following income fluctuation problem in continuous time, in which periods are of length Δt. The value function is

$$v_i(k) = \max_{\tilde{c},l} \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} u(\tilde{c}_i(s))ds \tag{A.1}$$

for $i = 1, 2$. We will momentarily take $\Delta t \to 0$, then we have

$$\begin{aligned}
v_i(k) &= \max_{\tilde{c},l} \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} u(\tilde{c}_i(s))ds \\
&= \max_{\tilde{c},l} \mathbb{E}_t \left[\int_t^{t+\Delta t} e^{-\rho(s-t)} u(\tilde{c}_i(s))ds + \int_{t+\Delta t}^{\infty} e^{-\rho(s-t)} u(\tilde{c}_i(s))ds \right] \\
&= \max_{\tilde{c},l} \mathbb{E}_t \left[\int_t^{t+\Delta t} e^{-\rho(s-t)} u(\tilde{c}_i(s))ds + (1-\lambda_i \Delta t)\, v_i(k+\Delta k) + \lambda_i \Delta t v_{-i}(k+\Delta k) \right] \\
&= \max_{\tilde{c},l} \mathbb{E}_t \left[u(\tilde{c}_i)\Delta t + (1-\lambda_i \Delta t)\, v_i(k+\Delta k) + (\lambda_i \Delta t) v_{-i}(k+\Delta k) \right] \\
&= \max_{\tilde{c},l} \mathbb{E}_t [u(\tilde{c}_i)\Delta t + (1-\lambda_i \Delta t)(v_i(k) + v_{i,t}\Delta t + v_{i,k}\Delta k + \frac{1}{2} v_{i,tt}(\Delta t)^2 + \frac{1}{2} v_{i,kk}(\Delta k)^2) \\
&\quad + \lambda_i \Delta t (v_{-i}(k) + v_{-i,t}\Delta t + v_{-i,k}\Delta k + \frac{1}{2} v_{-i,tt}(\Delta t)^2 + \frac{1}{2} v_{-i,kk}(\Delta k)^2)] \\
&= \max_{\tilde{c},l} \mathbb{E}_t \left[u(\tilde{c}_i)\Delta t + v_i(k) - \lambda_i \Delta t\, (v_i(k) - v_j(k)) - \rho v_i(k)\Delta t + v_i'(k) s_i(k)\Delta t + \frac{1}{2} v_i''(k)\sigma_1^2 k^2 \Delta t \right]
\end{aligned} \tag{A.2}$$

$$\Leftrightarrow 0=\max_{\tilde{c},l} u(\tilde{c}_i)-\rho v_i(k)+v_i'(k)s_i(k)-\lambda_i\left(v_i(k)-v_j(k)\right)+\frac{1}{2}v_{-i}''(k)\sigma_1^2k^2 \tag{A.3}$$

We have the HJB function

$$\rho v_i(k)=\max_{\tilde{c},l} u(\tilde{c}_i)+v_i'(k)s_i(k)-\lambda_i\left(v_i(k)-v_j(k)\right)+\frac{1}{2}v_i''(k)\sigma_1^2k^2 \tag{A.4}$$

A.2 The KF equation

We know that

$$dk_t=s(k_t)dt+\sigma_1 k_t dB_t \tag{A.5}$$

For all functions $\varphi(k)$, we have

$$E[\varphi(k_{t+\Delta t})]=\int_{\underline{k}}^{\infty}\varphi(k)[p_i\Delta t g_i(k_{t+\Delta t})+(1-p_i\Delta t)g_{-i}(k_{t+\Delta t})]dk \tag{A.6}$$

On the other hand, we have

$$\begin{aligned} d\varphi(k_t)=&\varphi'(k_t)dk_t+\frac{1}{2}\varphi''(k_t)(dk_t)^2\\ =&\left[\varphi'(k_t)s(k_t)+\frac{1}{2}\varphi''(k_t)\sigma_1^2k_t^2\right]dt+\varphi'(k_t)\sigma_1k_tdB_t \end{aligned} \tag{A.7}$$

by Itô' s lemma.

We have

$$\begin{aligned} &\frac{\int_{\underline{k}}^{\infty}\varphi(k_t)\left(p_i\Delta t g_i(k_{t+\Delta t})+\left(1-p_i\Delta t\right)g_{-i}(k_{t+\Delta t})\right)dk-\int_{\underline{k}}^{\infty}\varphi(k_t)g_i(k_t)dk}{\Delta t}\\ =&\frac{E_t[\varphi(k_{t+\Delta t})]-E_t[\varphi(k_t)]}{\Delta t}\\ =&E_t\left[\varphi'(k_t)s(k_t)+\frac{1}{2}\varphi''(k_t)\sigma_1^2k_t^2\right]\\ =&\int_{\underline{k}}^{\infty}\left[\varphi'(k_t)s(k_t)+\frac{1}{2}\varphi''(k_t)\sigma_1^2k_t^2\right]g(k_t)dk \end{aligned} \tag{A.8}$$

On the other hand, we have

$$\begin{aligned}
&\frac{\int_{\underline{k}}^{\infty} \varphi(k_t)\left[p_i \Delta t g_i(k_{t+\Delta t}) + (1-p_i\Delta t)\, g_{-i}(k_{t+\Delta t})\right] dk - \int_{\underline{k}}^{\infty} \varphi(k_t) g_i(k_t) dk}{\Delta t} \\
=&\frac{\int_{\underline{k}}^{\infty} \varphi(k_t)\left[(1-\lambda_i \Delta t)\, g_i(k_{t+\Delta t}) + \lambda_i \Delta t g_{-i}(y_{t+\Delta t})\right] dk - \int_{\underline{k}}^{\infty} \varphi(k_t) g_i(k_t) dk}{\Delta t} \\
=&\int_{\underline{k}}^{\infty} \varphi(k_t)\left[\frac{g_i(k_{t+\Delta t}) - g_i(k_t)}{\Delta t} - \lambda_i g_i(k_{t+\Delta t}) + \lambda_{-i} g_{-i}(k_{t+\Delta t})\right] dk \\
=&\int_{\underline{k}}^{\infty} \varphi(k_t)\left[\frac{\partial}{\partial t} g_i(k_t) - \lambda_i \left(g_i(k_t) - g_{-i}(k_t)\right)\right] dk
\end{aligned} \tag{A.9}$$

Therefore, let $\varphi(0) = \varphi(\infty) = 0$ and we have

$$\begin{aligned}
&\int_{\underline{k}}^{\infty} \varphi(k_t)[\frac{\partial}{\partial t} g_i(k_t) - \lambda_j \left(g_i(k_t) - g_{-i}(k_t)\right)] dk \\
=&\int_{\underline{k}}^{\infty} \left[\varphi'(k_t) s_i(k_t) + \frac{1}{2}\varphi''(k_t)\sigma_1^2 k_t^2\right] g_i(k_t) \\
=&-\int_{\underline{k}}^{\infty} \varphi(k_t) \frac{\partial}{\partial k}\left[s_i(k_t) g_i(k_t)\right] dk + \frac{1}{2}\int_{\underline{k}}^{\infty} \varphi(k_t) \frac{\partial^2}{\partial k^2}\left[\sigma_1^2 k_t^2 g_i(k_t)\right] dk
\end{aligned} \tag{A.10}$$

Thus, we have

$$0 = \int_{\underline{k}}^{\infty} \varphi(k_t)\left[\frac{\partial}{\partial t} g_i(k_t) + \frac{\partial}{\partial k}[s_i(k_t) g_i(k_t)] - \lambda_i (g_i(k_t) - g_{-i}(k_t)) - \frac{1}{2}\frac{\partial^2}{\partial k^2}[\sigma_1^2 k_t^2 g_i(k_t)]\right] dk \tag{A.11}$$

Since $\varphi(k)$ is arbitrary, we have

$$0 = \frac{\partial}{\partial t} g_i(k_t) + \frac{\partial}{\partial k}[s_i(k_t) g_i(k_t)] - \lambda_i (g_i(k_t) - g_{-i}(k_t)) - \frac{1}{2}\frac{\partial^2}{\partial k^2}[\sigma_1^2 k_t^2 g_i(k_t)] \tag{A.12}$$

Thus we have

$$0 = \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma^2 k^2 g_i(k)\right] - \frac{\partial}{\partial k}\left[s_i(k) g_i(k)\right] - \lambda_i g_i(k) + \lambda_{-i} g_{-i}(k) \tag{A.13}$$

B Derivations in Section 3

B.1 Incidence on Labor Supply

The standard labor supply elasticity with respect to the retention rate, $1-T'(y)$, is defined as

$$e = \frac{m'(l)}{lm''(l)} \tag{B.14}$$

which only considers the linear constraint and thus the direct effect on the labor supply of an exogenous increase in the retention rate. However, there are also indirect effects under a nonlinear tax system T. An adjustment on the labor supply l leads to an endogenous change on the marginal tax rate $T'(y)$, leading to a further adjustment on the labor supply.

Let's start from first-order condition (15). Consider the perturbed first-order condition.

1. Hold w and r constant and change the retention rate to $1-T'$, then

$$\begin{aligned} m'(l_i + \alpha dl_i) =& wx_i\{1 - T'[(rk + wx_i(l_i + \alpha dl_i)] + \alpha d(1 - T'(y_i))\} \\ m'(l_i) + m''(l_i)\cdot \alpha dl_i =& wx_i[1 - T'(y_i) - T''(y_i)\cdot \alpha wx_i dl_i + \alpha d(1 - T'(y_i))] \\ m''(l_i)dl_i =& - w^2x_i^2T''(y_i)dl_i + wx_i d(1 - T'(y_i)) \\ [m''(l_i) + w^2x_i^2T''(y_i)]dl_i =& wx_i d(1 - T'(y_i)) \end{aligned} \tag{B.15}$$

Then

$$\begin{aligned} \epsilon^{l_i}_{1-T'} =& \frac{dl_i}{d(1 - T'(y_i))}\frac{1 - T'(y_i)}{l_i} = \frac{wx_i}{m''(l_i) + w^2x_i^2T''(y_i)}\frac{1 - T'(y_i)}{l_i} \\ =& \frac{\frac{wx_i(1-T'(y_i))}{l_i\cdot m''(l_i)}}{1 + \frac{w^2x_i^2T''(y_i)}{m''(l_i)}} = \frac{\frac{m'(l_i)}{l_i\cdot m''(l_i)}}{1 + \frac{wx_il_i}{y_i}\frac{m'(l_i)}{l_im''(l_i)}\frac{y_iT''(y_i)}{1-T'(y_i)}} = \frac{e}{1 + ep_i(y)\xi_i(y)} \end{aligned} \tag{B.16}$$

where

$$p_i(y) = \frac{y_iT''(y_i)}{1 - T'(y_i)} \tag{B.17}$$

and

$$\xi_i(y) = \frac{wx_il_i}{y_i} = \frac{wx_il_i}{rk + wx_il_i} \tag{B.18}$$

It denotes the local rate of progressivity of the tax schedule.

2. Hold r and rentention rate constant and change the w, then

$$\begin{aligned}
m'(l_i+\alpha dl_i) =& (w+\alpha dw)x_i\{1-T'[rk+(w+\alpha dw)x_i(l_i+\alpha dl_i)]\} \\
m'(l_i)+\alpha m''(l_i)dl_i =& (w+\alpha dw)x_i[1-T'(y_i+\alpha x_i l_i dw+\alpha w x_i dl_i+\alpha^2 dw\cdot x_i dl_i)] \\
m''(l_i)dl_i =& -wx_iT''(y_i)(wx_idl_i+x_il_idw)+[1-T'(y_i)]x_idw \\
[m''(l_i)+w^2x_i^2T''(y_i)]dl_i =& [(1-T'(y_i))x_i-T''(y_i)wx_i^2l_i]dw
\end{aligned} \tag{B.19}$$

Then

$$\begin{aligned}
\epsilon_w^{l_i} =& \frac{dl_i}{dw}\frac{w}{l_i} = \frac{[1-T'(y_i)]x_i-T''(y_i)wx_i^2l_i}{m''(l_i)+w^2x_i^2T''(y_i)}\frac{w}{l_i} \\
=& \frac{\frac{wx_i(1-T'(y_i))-w^2x_i^2l_iT''(y_i)}{l_i\cdot m''(l_i)}}{1+\frac{w^2x_i^2T''(y_i)}{m''(l_i)}} = \frac{\frac{m'(l_i)}{l_im''(l_i)}\left(1-\frac{wx_il_i}{y_i}\frac{y_iT''(y_i)}{1-T'(y_i)}\right)}{1+\frac{wx_il_i}{y_i}\frac{m'(l_i)}{l_im''(l_i)}\frac{y_iT''(y_i)}{1-T'(y_i)}} = \frac{e(1-p_i(y)\xi_i(y))}{1+ep_i(y)\xi_i(y)}
\end{aligned} \tag{B.20}$$

It denotes the labor income share.

3. Hold w and rentention rate constant and change the r, then

$$\begin{aligned}
m'(l_i+\alpha dl_i) =& wx_i\{1-T'[(r+\alpha dr)k+wx_i(l_i+\alpha dl_i)]\} \\
m'(l_i)+m''(l_i)\alpha dl_i =& wx_i[1-T'(y_i)-T''(y_i)(\alpha kdr+\alpha wx_idl_i)]\} \\
m''(l_i)dl_i =& -wx_iT''(y_i)(kdr+wx_idl_i) \\
[m''(l_i)+w^2x_i^2T''(y_i)]dl_i =& -wx_iT''(y_i)kdr
\end{aligned} \tag{B.21}$$

Then

$$\epsilon_r^{l_i} = \frac{dl_i}{dr}\frac{r}{l_i} = -\frac{wxT''(y_i)k}{m''(l_i)+w^2x_i^2T''(y_i)}\frac{r}{l_i} = \frac{\frac{m'(l_i)}{l_im''(l_i)}\frac{y_iT''(y_i)}{1-T'(y_i)}\frac{rk}{y_i}}{1+\frac{wx_il_i}{y_i}\frac{m'(l_i)}{l_im''(l_i)}\frac{y_iT''(y_i)}{1-T'(y_i)}} = -\frac{ep_i(y)(1-\xi_i(y))}{1+ep_i(y)\xi_i(y)} \tag{B.22}$$

4. Change the tax formula, then

$$m'(l_i+\alpha dl_i) = (w+\alpha dw)x_i\{1-T'[k(r+\alpha dr)+(w+\alpha dw)x_i(l_i+dl_i)]-\alpha h[k(r+\alpha dr)+(w+\alpha dw)x_i(l_i+dl_i)]\} \tag{B.23}$$

A first-order taylor expansion implies

$$m'(l_i)+m''(l_i)\alpha dl_i = wx_i[1-T'(y_i)-\alpha T''(y_i)dy_i-\alpha h'(y_i)]+\alpha x_i dw[1-T'(y_i)-\alpha T''(y_i)dy_i-\alpha h'(y_i)] \tag{B.24}$$

Then

$$[m''(l_i) + T''(y_i)w^2x_i^2]dl_i = -[T''(y_i)(kdr + x_il_idw) + h'(y_i)]wx_i + [1 - T'(y_i)]x_idw \tag{B.25}$$

This yields the solution for $dl_i(T; h)$ as

$$\begin{aligned}
dl_i(T; h) =& \frac{(1 - T'(y_i))x_idw - T''(y_i)(kdr + x_il_idw)wx_i - h'(y_i)wx_i}{m''(l_i) + T''(y_i)w^2x_i^2} \\
=& \frac{\frac{1-T'(y_i)}{m''(l_i)}x_idw - \frac{T''(y_i)}{1-T'(y_i)}\frac{m'(l_i)}{m''(l_i)}(kdr + x_il_idw) - \frac{h'(y_i)}{1-T'(y_i)}\frac{m'(l_i)}{m''(l_i)}}{1 + \frac{T''(y_i)}{1-T'(y_i)}\frac{m'(l_i)}{m''(l_i)}wx_i} \\
=& \frac{\frac{m'(l_i)}{m''(l_i)l_i}\frac{x_idw}{wx_i}l_i - \frac{T''(y_i)y_i}{1-T'(y_i)}\frac{m'(l_i)}{m''(l_i)l_i}\frac{kdr+x_il_idw}{y_i}l_i - \frac{h'(y_i)}{1-T'(y_i)}\frac{m'(l_i)}{m''(l_i)l_i}l_i}{1 + \frac{T''(y_i)y_i}{1-T'(y_i)}\frac{m'(l_i)}{m''(l_i)l_i}\frac{wx_il_i}{y_i}} \\
=& \epsilon_w^{l_i}\frac{l_i}{w}dw + \epsilon_r^{l_i}\frac{l_i}{r}dr - \epsilon_{1-T'}^{l_i}\frac{h'(y_i)l_i}{1 - T'(y_i)}
\end{aligned} \tag{B.26}$$

B.2 Incidence on consumption

Recall the drift term

$$s_i = y_i - \tilde{c}_i - \delta k - T(y_i) - m(l_i) \tag{B.27}$$

And the perturbation on it shows that

$$s_i+\alpha ds_i = y_i+\alpha dy_i-(\tilde{c}_i+\alpha d\tilde{c}_i)-\delta k-[T(y_i+\alpha dy_i)+\alpha h(y_i+\alpha dy_i)]-m(l_i+\alpha dl_i)+dR \tag{B.28}$$

Let (B.27) minus (B.2), then

$$\begin{aligned}
ds_i =& dy_i - d\tilde{c}_i - T'(y_i)dy_i - h(y_i) - m'(l_i)dl_i + dR \\
=& (1 - T'(y_i))(kdr + x_il_idw) - d\tilde{c}_i - h + dR
\end{aligned} \tag{B.29}$$

here we use

$$dy_i = kdr(T; h) + x_il_idw(T; h) + wx_idl_i(T; h) \tag{B.30}$$

then we obtain

$$d\tilde{c}_i = (1 - T'(y_i))(kdr + x_i l_i dw) - ds_i - h + dR \tag{B.31}$$

B.3 Incidence on government revenue

The government revenue is shown in (10), and the perurbation is

$$R(T + \alpha h) = R(T) + \alpha dR \tag{B.32}$$

Then

$$\begin{aligned}
\alpha dR =& \sum_{i=1}^{2} \int \{T[(r + \alpha dr)k + (w + \alpha dw)x_i(l_i + \alpha l_i)] + \alpha h[(r + \alpha dr)k + (w + \alpha dw)x_i(l_i + \alpha l_i)]\} g_i(k)dk \\
& - \sum_{i=1}^{2} \int T(rk + wx_i l_i) g_i(k)dk \\
=& \sum_{i=1}^{2} \int [\alpha T'(y_i)(kdr + x_i l_i dw + wx_i dl_i) + \alpha h(y_i)] g_i(k)dk
\end{aligned} \tag{B.33}$$

Then

$$
\begin{aligned}
dR &= \sum_{i=1}^{2} \int [T'(y_i)(kdr + x_i l_i dw + w x_i dl_i) + h(y_i)] g_i(k) dk \\
&= \sum_{i=1}^{2} \int_{k^*}^{\infty} \frac{g_i(k)}{1 - F(y^*)} dk + dr \sum_{i=1}^{2} \int T'(y_i) k g_i(k) dk + dw \sum_{i=1}^{2} \int T'(y_i) x_i l_i g_i(k) dk \\
&\quad + \sum_{i=1}^{2} \int T'(y_i) \left[\epsilon_w^{l_i} \frac{l_i}{w} dw + \epsilon_r^{l_i} \frac{l_i}{r} dr - \epsilon_{1-T'}^{l_i} \frac{h'(y_i) l_i}{1 - T'(y_i)} \right] w x_i g_i(k) dk \\
&= \sum_{i=1}^{2} \int_{k^*}^{\infty} \frac{g_i(k)}{1 - F(y^*)} dk + dr \sum_{i=1}^{2} \int_{0}^{\infty} \left[1 + \frac{\xi_i(y)}{1 - \xi_i(y)} \epsilon_r^{l_i} \right] k T'(y_i) g_i(k) dk \\
&\quad + dw \sum_{i=1}^{2} \int_{0}^{\infty} (1 + \epsilon_w^{l_i}) x_i l_i T'(y_i) g_i(k) dk - \sum_{i=1}^{2} \int_{0}^{\infty} \epsilon_{1-T'}^{l_i} \frac{T'(y_i)}{1 - T'(y_i)} \frac{\delta_{y^*}(y_i)}{1 - F(y^*)} w x_i l_i g_i(k) dk \\
&= \sum_{i=1}^{2} \int_{k^*}^{\infty} \frac{g_i(k)}{1 - F(y^*)} dk + dr \sum_{i=1}^{2} \int_{0}^{\infty} \left[1 + \frac{\xi_i(y)}{1 - \xi_i(y)} \epsilon_r^{l_i} \right] k T'(y_i) g_i(k) dk \\
&\quad + dw \sum_{i=1}^{2} \int_{0}^{\infty} (1 + \epsilon_w^{l_i}) x_i l_i T'(y_i) g_i(k) dk - \sum_{i=1}^{2} \epsilon_{1-T'}^{l_i}(y^*) \frac{T'(y^*)}{1 - T'(y^*)} \frac{g_i(\frac{y^* - w x_i l_i}{r})}{1 - F(y^*)} w x_i l_i \\
&= \sum_{i=1}^{2} \int_{k^*}^{\infty} \frac{g_i(k)}{1 - F(y^*)} dk + dr \sum_{i=1}^{2} \int_{0}^{\infty} \left[1 + \frac{\xi_i(y)}{1 - \xi_i(y)} \epsilon_r^{l_i} \right] k T'(y_i) g_i(k) dk \\
&\quad + dw \sum_{i=1}^{2} \int_{0}^{\infty} (1 + \epsilon_w^{l_i}) x_i l_i T'(y_i) g_i(k) dk - \frac{T'(y^*)}{1 - T'(y^*)} \frac{y^* f(y^*)}{1 - F(y^*)} \sum_{i=1}^{2} \frac{g_i(\frac{y^* - w x_i l_i}{r})}{f(y^*)} \xi_i(y^*) \epsilon_{1-T'}^{l_i}(y^*)
\end{aligned}
\tag{B.34}
$$

where $F(y)$ is the CDF of total income, and $f(y)$ is the PDF. $\xi_i(y)$, $\epsilon_{1-T'}^{l_i}$, $\epsilon_l^{l_i}$, $\epsilon_r^{l_i}$ can be seen in A.1.

B.4 Incidence on welfare function

The incidence on agents' consumption is

$$d\tilde{c}_i = (1 - T')(kdr + x_i l_i dw) - ds_i - h + dR \tag{B.35}$$

Recalling the expression of dW, we have

$$\begin{aligned}
dW(T;h) =& \frac{1}{\rho}\sum_{i=1}^{2}\int_0^\infty \left[u'(\tilde{c}_i)d\tilde{c}_i(T;h)g_i(k) + u(\tilde{c}_i)dg_i(T;h)\right] dk \\
=& \frac{1}{\rho}\sum_{i=1}^{2}\int_0^\infty u'(\tilde{c}_i)[(1-T')(kdr + x_i l_i dw) + dR]g_i(k)dk \\
&- \frac{1}{\rho}\sum_{i=1}^{2}\int_0^\infty u'(\tilde{c}_i)[ds_i(T;h) + h]g_i(k)dk \\
&+ \frac{1}{\rho}\sum_{i=1}^{2}\int_0^\infty u(\tilde{c}_i)dg_i(T;h)dk
\end{aligned} \tag{B.36}$$

By imposing $dW = 0$, we can obtain the optimal income tax rate on y^*. By imposing $dW = 0$, we can obtain the optimal marginal tax rate at income level y^*. Note that $h'(y) = \frac{1}{1-F(y^*)}\delta_{y^*}(y)$, move the part contains $h'(y)$ to the left side and compute integration involving the Dirac function

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{k^*}^\infty \frac{u'(\tilde{c}_i)g_i(k)}{1-F(y^*)}dk + \sum_{i=1}^{2}\int_0^\infty u'(\tilde{c}_i)ds_i(T;h)g_i(k)dk \\
=& \sum_{i=1}^{2}\int_0^\infty [u'(\tilde{c}_i)(1-T'(y_i))(kdr + x_i l_i dw)g_i(k) + u(\tilde{c}_i)dg_i(T;h)]dk + dR\sum_{i=1}^{2}\int_0^\infty u'(\tilde{c}_i)g_i dk \\
=& \sum_{i=1}^{2}\int_0^\infty [u'(\tilde{c}_i)(1-T'(y_i))(kdr + x_i l_i dw)g_i(k) + u(\tilde{c}_i)dg_i(T;h)]dk \\
&+ \varphi\sum_{i=1}^{2}\int_{k^*}^\infty \frac{g_i(k)}{1-F(y^*)}dk + \varphi dr\sum_{i=1}^{2}\int_0^\infty \left[1 + \frac{\xi_i(y)}{1-\xi_i(y)}\epsilon_r^{l_i}\right] kT'(y_i)g_i(k)dk \\
&+ \varphi dw\sum_{i=1}^{2}\int_0^\infty (1+\epsilon_w^{l_i})x_i l_i T'(y_i)g_i(k)dk - \varphi\frac{T'(y^*)}{1-T'(y^*)}\frac{y^* f(y^*)}{1-F(y^*)}\sum_{i=1}^{2}\frac{g_i(\frac{y^*-wx_i l_i}{r})}{f(y^*)}\xi_i(y^*)\epsilon_{1-T'}^{l_i}(y^*)
\end{aligned} \tag{B.37}$$

where $\varphi = \sum_{i=1}^{2}\int_{0}^{\infty} u'(\tilde{c}_i)g_i(k)dk$. Then we obtain

$$
\begin{aligned}
&\varphi\frac{T'(y^*)}{1-T'(y^*)}\frac{y^*f(y^*)}{1-F(y^*)}\sum_{i=1}^{2}\frac{g_i(\frac{y^*-wx_il_i}{r})}{f(y^*)}\xi_i(y^*)\epsilon^{l_i}_{1-T'}(y^*)\\
=&\varphi\sum_{i=1}^{2}\int_{k^*}^{\infty}(1-\frac{u'(\tilde{c}_i)}{\varphi})\frac{g_i(k)}{1-F(y^*)}dk\\
&+\sum_{i=1}^{2}\int_{k^*}^{\infty}u'(\tilde{c}_i)(1-T')(kdr+x_il_idw)g_idk\\
&+\varphi dw\sum_{i=1}^{2}\int_{0}^{\infty}(1+\epsilon^{l_i}_{w})x_il_iT'(y_i)g_i(k)dk\\
&+\varphi dw\sum_{i=1}^{2}\int_{0}^{\infty}(1+\epsilon^{l_i}_{w})x_il_iT'(y_i)g_i(k)dk\\
&+\varphi dr\sum_{i=1}^{2}\int_{0}^{\infty}\left[1+\frac{\xi_i(y)}{1-\xi_i(y)}\epsilon^{l_i}_{r}\right]kT'(y_i)g_i(k)dk\\
&-\sum_{i=1}^{2}\int_{0}^{\infty}u'(\tilde{c}_i)g_i(k)ds_i(T;h)dk\\
&+\sum_{i=1}^{2}\int_{0}^{\infty}u(\tilde{c}_i)dg_i(T;h)dk
\end{aligned}
\tag{B.38}
$$

After simplifying we have

$$
\begin{aligned}
\frac{T'(y^*)}{1-T'(y^*)} &= \frac{1-F(y^*)}{y^* f(y^*)} \times \Gamma(y^*) \times \left[\mathcal{A}(y^*) + \mathcal{B}(y^*) + \mathcal{C}(y^*) + \mathcal{D}(y^*) + \mathcal{E}(y^*) + \mathcal{F}(y^*)\right] \\
\text{where } \Gamma(y^*) &= \frac{1}{\sum_{i=1}^{2} \frac{g_i(\frac{y^*-wx_il_i}{r})}{f(y^*)} \xi_i(y^*) \epsilon^{l_i}_{1-T'}(y^*)} \\
\mathcal{A}(y^*) &= \sum_{i=1}^{2} \int_{k^*}^{\infty} \left[1 - \frac{u'(\tilde{c}_i)}{\varphi}\right] \frac{g_i(k)}{1-F(y^*)} dk \\
\mathcal{B}(y^*) &= \frac{1}{\varphi} \sum_{i=1}^{2} \int_{0}^{\infty} u'(\tilde{c}_i)(1-T'(y_i))(kdr + x_il_idw) g_i dk \\
\mathcal{C}(y^*) &= dw \sum_{i=1}^{2} \int_{0}^{\infty} (1+\epsilon^{l_i}_{w}) x_il_iT'(y_i) g_i(k) dk \\
\mathcal{D}(y^*) &= dr \sum_{i=1}^{2} \int_{0}^{\infty} \left[1 + \frac{\xi_i(y)}{1-\xi_i(y)} \epsilon^{l_i}_{r}\right] kT'(y_i) g_i(k) dk \\
\mathcal{E}(y^*) &= -\frac{1}{\varphi} \sum_{i=1}^{2} \int_{0}^{\infty} u'(\tilde{c}_i) ds_i(T;h) g_i dk \\
\mathcal{F}(y^*) &= \sum_{i=1}^{2} \int_{0}^{\infty} u(\tilde{c}_i) dg_i(T;h) dk
\end{aligned}
\tag{B.39}
$$

C A brief description of the algorithm

1. Guess a initial marginal tax schedule. Here, we use a CRP form.
2. Given that initial tax schedule, calculate the lump-sum transfer that satisfies the government budget constraint.
3. Given the tax schedule, solve a steady-state equilibrium.
4. Consider a tax reform of increasing marginal tax rate at each grid point, and compute the new steady state at each grid point.
5. Then, use the tax formula to compute the right-hand side (RHS) and thus an alternative marginal tax schedule.
6. Check if $||\frac{T'}{1-T'} - RHS|| < tol$. If not, go to step 3 with an alternative tax schedule. This loop is repeated until a fixed-point optimal tax schedule is found.

D Derivations in Section 4

D.1 Joint probability distribution

The joint probability distribution of idiosyncratic k_t and aggregate X_t goes as following

$$
\begin{aligned}
Pr(k_t, X_t)\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) =& Pr(k_{t+\Delta t}, X_{t+\Delta t}, k_t, X_t) \\
=& Pr(k_{t+\Delta t}|X_{t+\Delta t}, k_t, X_t) Pr(X_{t+\Delta t}, k_t, X_t) \\
=& Pr(k_{t+\Delta t}|X_{t+\Delta t}, k_t, X_t) Pr(X_{t+\Delta t}|k_t, X_t) Pr(k_t, X_t) \\
=& Pr(k_{t+\Delta t}|k_t) Pr(X_{t+\Delta t}|X_t) Pr(k_t, X_t)
\end{aligned}
\tag{D.40}
$$

so we have

$$
\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) = Pr(k_{t+\Delta t}|k_t) Pr(X_{t+\Delta t}|X_t) \tag{D.41}
$$

We do this split in order to illustrate the genuine law of motion after the introduce of aggregate shock.

D.2 Geometric Brownian diffusion in continuous time

Start from (32)

$$
dk_t = s_{it}dt + \sigma_1 k_t dB_t \tag{D.42}
$$

Rearrange

$$
\frac{dk_t}{k_t} = \tilde{s}_{it}dt + \sigma_1 dB_t \tag{D.43}
$$

here we denote

$$
\tilde{s}_{it} = \frac{s_{it}}{k_t} \tag{D.44}
$$

Obviously, it is an *Itô Process*. We let $f = \log k_t$ and apply Itô Lemma

$$
df = d\log k_t = \left(\tilde{s}_{it}(k_t) - \frac{1}{2}\sigma_1^2\right) dt + \sigma_1 dB_t \tag{D.45}
$$

With the properties of Brownian motion, we know that within any time increment Δt, the evolution of $\log k_t$ follows normal distribution

$$
\log k_{t+\Delta t} \sim N\left[\log k_t + \hat{s}_{it}(k_t), \hat{\sigma}_1^2\right] \tag{D.46}
$$

with the denotation

$$\hat{s}_{it}(k_t) = \left[\tilde{s}_{it}(k_t) - \frac{1}{2}\sigma_1^2\right]\Delta t, \quad \hat{\sigma}_1 = \sigma_1\sqrt{\Delta t} \tag{D.47}$$

then with the definition of its probability density function, we obtain

$$Pr(k_{t+\Delta t}|k_t) = \frac{1}{k_{t+\Delta t}\hat{\sigma}_1\sqrt{2\pi}} e^{-\frac{\left(\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)\right)^2}{2\hat{\sigma}_1^2}} \tag{D.48}$$

The above derivations and treatments are refered to the counterparts in Farhi (2013).

By the way, we can proof that when $\Delta t \to 0$, the exponential function goes to 1. It is obvious that the numerator and the denominator of the exponent part are both infinitesimal when $\Delta t \to 0$, with L'Hospital's rule we obtain

$$-\frac{2\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{2\sigma_1^2} e^{\hat{s}_{it}}\left(-\frac{d\hat{s}_{it}}{d\Delta t}\right) = \frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right) e^{\hat{s}_{it}}\left[\tilde{s}_{it}(k_t) - \frac{1}{2}\sigma_1^2\right]}{\sigma_1^2} = 0 \tag{D.49}$$

where

$$\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right) = \log\frac{k_{t+\Delta t}}{k_t} - \left[\tilde{s}_{it}(k_t) - \frac{1}{2}\sigma_1^2\right]\Delta t = 0 \tag{D.50}$$

So we obtain

$$e^{-\frac{\left(\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_t(k_t)}}\right)\right)^2}{2\hat{\sigma}_1^2}} \to 1, \quad \text{when } \Delta t \to 0 \tag{D.51}$$

D.3 Envelope condition

For first-order condition of $\tilde{c}_t$, the derivative of ψ is as follows

$$\begin{aligned}
&\frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial \tilde{c}_t} \\
=& Pr(X_{t+\Delta t}|X_t)\frac{\partial Pr(k_{t+\Delta t}|k_t)}{\partial \tilde{c}_t} \\
=& Pr(X_{t+\Delta t}|X_t)\frac{1}{k_{t+\Delta t}\hat{\sigma}_1\sqrt{2\pi}} e^{-\frac{\left(\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)\right)^2}{2\hat{\sigma}_1^2}} \frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{k_t\hat{\sigma}_1^2} k_t \frac{\partial \hat{s}_{it}}{\partial \tilde{c}_t} \\
=& Pr(X_{t+\Delta t}|X_t) Pr(k_{t+\Delta t}|k_t) \frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{k_t\hat{\sigma}_1^2} k_t \frac{\partial \hat{s}_{it}}{\partial \tilde{c}_t} \\
=& \psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Omega_i(k_t)\Phi_i(k_t)
\end{aligned} \tag{D.52}$$

here we let

$$\Omega_i(k_t) = \frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_i(k_t)}}\right)}{k_t\hat{\sigma}_1^2} \tag{D.53}$$

and

$$\Phi_i(k_t) = k_t\frac{\partial \hat{s}_{it}}{\partial \tilde{c}_{it}} = -\Delta t, \tag{D.54}$$

where

$$\begin{aligned}\frac{\partial \hat{s}_{it}}{\partial \tilde{c}_{it}} =& \frac{\partial(\tilde{s}_{it} - \frac{1}{2}\sigma_1^2)\Delta t}{\partial \tilde{c}_{it}} \\ =&\partial(\frac{y_{it} - \tilde{c}_{it} - m(l_{it}) - T_{it} - \delta k_t}{k_t} - \frac{1}{2}\sigma_1^2)/\partial \tilde{c}_{it} \cdot \Delta t \\ =& -\frac{1}{k_t}\Delta t.\end{aligned} \tag{D.55}$$

Recall that

$$s_{it} = y_{it} - \tilde{c}_{it} - m(l_{it}) - T(y_{it}) - \delta k_t. \tag{D.56}$$

By the way, here we can proof that when Δt, the limit of Ω is finite. With L'Hospital's rule we obtain

$$\frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{k_t\hat{\sigma}_1^2} = \frac{\log k_{t+\Delta t} - \log k_t - \hat{s}_{it}}{k_t\sigma_1^2\Delta t} = -\frac{\tilde{s}_{it} - \frac{1}{2}\sigma_1^2}{k_t\sigma_1^2} \neq 0 \tag{D.57}$$

For first-order condition of l_t, the derivative of g is as follows

$$\begin{aligned}&\frac{\partial\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)}{\partial l_{it}} \\ =&Pr(X_{t+\Delta t}|X_t)\frac{\partial Pr(k_{t+\Delta t}|k_t)}{\partial l_{it}} \\ =&Pr(X_{t+\Delta t}|X_t)\frac{1}{k_{t+\Delta t}\hat{\sigma}_1\sqrt{2\pi}}e^{-\frac{\left(\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)\right)^2}{2\hat{\sigma}_1^2}}\frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{k_t\hat{\sigma}_1^2}k_t\frac{\partial \hat{s}_{it}}{\partial l_{it}} \\ =&Pr(X_{t+\Delta t}|X_t)Pr(k_{t+\Delta t}|k_t)\frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{k_t\hat{\sigma}_1^2}k_t\frac{\partial \hat{s}_{it}}{\partial l_{it}} \\ =&\psi_i(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Omega_i(k_t)\Pi_i(k_t)\end{aligned} \tag{D.58}$$

where

$$\Pi_i(k_t) = k_t\frac{\partial \hat{s}_{it}}{\partial l_{it}} = \left[wx_i\left(1 - T'(y_{it})\right) - \chi l_{it}^{\frac{1}{e}}\right]\Delta t \tag{D.59}$$

with

$$
\begin{aligned}
\frac{\partial \hat{s}_{it}}{\partial l_{it}} =& \frac{\partial(\tilde{s}_{it} - \frac{1}{2}\sigma_1^2)\Delta t}{\partial l_{it}} \\
=& \partial \left[\frac{r_t k_t + wx_i l_{it} - \tilde{c}_{it} - m(l_{it}) - T(r_t k_t + wx_i l_{it}) - \delta k_t}{k_t} - \frac{1}{2}\sigma_1^2 \right] / \partial l_{it} \cdot \Delta t \\
=& \frac{1}{k_t} \left[wx_i - \chi l_{it}^{\frac{1}{e}} - \frac{\partial T(r_t k_t + wx_i l_{it})}{\partial(r_t k_t + wxl_t)} \frac{\partial(r_t k_t + wxl_t)}{\partial l_t} \right] \Delta t \\
=& \frac{1}{k_t} \left[wx_i - \chi l_{it}^{\frac{1}{e}} - wx_i \frac{\partial T(r_t k_t + wx_i l_{it})}{\partial(r_t k_t + wx_i l_{it})} \right] \Delta t
\end{aligned}
\tag{D.60}
$$

Then, we take derivative on (34) with respect to k_t

$$
\begin{aligned}
&\frac{\partial v(k_t, X_t)}{\partial k_t} \\
=& u'(\tilde{c}_{it}) \frac{\partial \tilde{c}_{it}}{\partial k_t} \Delta t + (1 - \rho \Delta t) \iint v(k_{t+\Delta t}, X_{t+\Delta t}) \frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t} | k_t, X_t)}{\partial k_t} dk_{t+\Delta t} dX_{t+\Delta t} \\
=& u'(\tilde{c}_{it}) \frac{\partial \tilde{c}_{it}}{\partial k_t} \Delta t + (1 - \rho \Delta t) \iint v(k_{t+\Delta t}, X_{t+\Delta t}) Pr(X_{t+\Delta t} | X_t) \frac{\partial Pr(k_{t+\Delta t} | k_t)}{\partial k_t} dk_{t+\Delta t} dX_{t+\Delta t} \\
=& u'(\tilde{c}_{it}) \frac{\partial \tilde{c}_{it}}{\partial k_t} \Delta t + (1 - \rho \Delta t) \iint v(k_{t+\Delta t}, X_{t+\Delta t}) \psi_i(k_{t+\Delta t}, X_{t+\Delta t} | k_t, X_t) \Omega_i(k_t) \Psi_i(k_t) dk_{t+\Delta t} dX_{t+\Delta t}
\end{aligned}
$$

The derivative derives from

$$
\begin{aligned}
&\frac{\partial \psi_i(k_{t+\Delta t}, X_{t+\Delta t} | k_t, X_t)}{\partial k_t} \\
=& Pr(X_{t+\Delta t} | X_t) \frac{\partial Pr(k_{t+\Delta t} | k_t)}{\partial k_t} \\
=& Pr(X_{t+\Delta t} | X_t) \frac{1}{k_{t+\Delta t} \hat{\sigma}_1 \sqrt{2\pi}} e^{-\frac{\left(\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)\right)^2}{2\hat{\sigma}_1^2}} \frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{k_t \hat{\sigma}_1^2} \left(1 + k_t \frac{\partial \hat{s}_{it}}{\partial k_t}\right) \\
=& Pr(X_{t+\Delta t} | X_t) Pr(k_{t+\Delta t} | k_t) \Omega_i(k_t) \Psi_i(k_t) \\
=& \psi_i(k_{t+\Delta t}, X_{t+\Delta t} | k_t, X_t) \Omega_i(k_t) \Psi_i(k_t)
\end{aligned}
\tag{D.61}
$$

where

$$
\Psi_i(k_t) = 1 + k_t \frac{\partial \hat{s}_{it}}{\partial k_t} \tag{D.62}
$$

with

$$
\begin{aligned}
&\frac{\partial \hat{s}_{it}}{\partial k_t}\\
=&\frac{\partial(\tilde{s}_{it}-\frac{1}{2}\sigma_1^2)\Delta t}{\partial k_t}\\
=&\partial\left[\frac{r_tk_t+wx_il_{it}-\tilde{c}_{it}-m(l_{it})-T(r_tk_t+wx_il_{it})-\delta k_t}{k_t}-\frac{1}{2}\sigma_1^2\right]/\partial k_t\cdot\Delta t\\
=&\frac{1}{k_t^2}\left[wx_i\left(\frac{\partial l_{it}}{\partial k_t}k_t-l_{it}\right)-\left(\frac{\partial \tilde{c}_{it}}{\partial k_t}k_t-\tilde{c}_{it}\right)-\left(\frac{\partial m(l_{it})}{\partial l_{it}}\frac{\partial l_{it}}{\partial k_t}k_t-m(l_{it})\right)\right.\\
&\left.-\left(\frac{\partial T(y_{it})}{\partial(r_tk_t+wx_il_{it})}\left(r_t+wx_i\frac{\partial l_{it}}{\partial k_t}\right)k_t-T(y_{it})\right)\right]\Delta t\\
=&\frac{1}{k_t}\left[wx\frac{\partial l_{it}}{\partial k_t}-T'(y_{it})\left(r_t+wx_i\frac{\partial l_{it}}{\partial k_t}\right)-\frac{\partial \tilde{c}_{it}}{\partial k_t}-\frac{\partial m(l_{it})}{\partial l_{it}}\frac{\partial l_{it}}{\partial k_t}\right]\Delta t-\frac{1}{k_t^2}[wx_il_{it}-\tilde{c}_{it}-m(l_{it})-T(y_{it})]\Delta t\\
=&\frac{1}{k_t}\left[wx_i\frac{\partial l_{it}}{\partial k_t}-\left(r_t+wx_i\frac{\partial l_{it}}{\partial k_t}\right)T'(y_{it})-\frac{\partial \tilde{c}_t}{\partial k_t}-\chi l_{it}^{\frac{1}{e}}\frac{\partial l_t}{\partial k_t}-\frac{1}{k_t}(wx_il_{it}-\tilde{c}_{it}-m(l_{it})-T(y_{it}))\right]\Delta t\\
=&\frac{1}{k_t}\left[r_t\left(1-T'(y_{it})\right)-\delta-\tilde{s}_{it}(k_t)+\left(wx_i\left(1-T'(y_{it})\right)-\chi l_{it}^{\frac{1}{e}}\right)\frac{\partial l_{it}}{\partial k_t}-\frac{\partial \tilde{c}_{it}}{\partial k_t}\right]\Delta t
\end{aligned}
$$

D.4 Derivations of recursive Γ_t

First, we take derivative of (35) with respect to k_t

$$
\begin{aligned}
&\psi_ik_t(k_{t+\Delta t},X_{t+\Delta t}|k_t,X_t)\\
=&Pr(X_{t+\Delta t}|X_t)\frac{1}{k_{t+\Delta t}\hat{\sigma}_1\sqrt{2\pi}}e^{-\frac{\left(\log\left(\frac{k_{t+\Delta t}}{k_te^{\hat{s}_{it}(k_t)}}\right)\right)^2}{2\hat{\sigma}_1^2}}\frac{\log\left(\frac{k_{t+\Delta t}}{k_te^{\hat{s}_{it}(k_t)}}\right)}{k_t\hat{\sigma}_1^2}\left(1+k_t\frac{\partial \hat{s}_{it}}{\partial k_t}\right)\\
=&\psi_i(k_{t+\Delta t},X_{t+\Delta t}|k_t,X_t)\frac{\log\left(\frac{k_{t+\Delta t}}{k_te^{\hat{s}_{it}(k_t)}}\right)}{k_t\hat{\sigma}_1^2}\left(1+k_t\frac{\partial \hat{s}_{it}}{\partial k_t}\right)\\
=&\psi_i(k_{t+\Delta t},X_{t+\Delta t}|k_t,X_t)\frac{\log\left(\frac{k_{t+\Delta t}}{k_te^{\hat{s}_{it}(k_t)}}\right)}{k_t\hat{\sigma}_1^2}\Psi_i(k_t)
\end{aligned}
\tag{D.63}
$$

then we take derivative of (35) with respect to $k_{t+\Delta t}$

$$
\begin{aligned}
&\psi_{k_{t+\Delta t}}(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) \\
=&-Pr(X_{t+\Delta t}|X_t)\frac{1}{k_{t+\Delta t}\hat{\sigma}_1\sqrt{2\pi}}e^{-\frac{\left(\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)\right)^2}{2\hat{\sigma}_1^2}}\left(1+\frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{\hat{\sigma}_1^2}\right)\frac{1}{k_{t+\Delta t}} \\
=&-\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\left(1+\frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{\hat{\sigma}_1^2}\right)\frac{1}{k_{t+\Delta t}}
\end{aligned}
\tag{D.64}
$$

From (D.64) we have

$$
\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)+k_{t+\Delta t}\psi_{t+\Delta t}(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) = -\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\frac{\log\left(\frac{k_{t+\Delta t}}{k_t e^{\hat{s}_{it}(k_t)}}\right)}{\hat{\sigma}_1^2} \tag{D.65}
$$

introduce (D.65) into (D.63) and rearrange, we have

$$
k_t\psi_{k_t}(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t) = -\,\Psi_i(k_t)\left[k_{t+\Delta t}\psi_{k_{t+\Delta t}}(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)+\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\right] \tag{D.66}
$$

Then we can construct

$$
\begin{aligned}
k_t\Gamma_t =&\frac{\Theta(k_t)}{\Psi(k_t)}\iint v(k_{t+\Delta t}, X_{t+\Delta t})k_t\psi_{k_t}(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)dk_{t+\Delta t}dX_{t+\Delta t} \\
=&-\frac{\Theta(k_t)}{\Psi(k_t)}\iint v(k_{t+\Delta t}, X_{t+\Delta t})\Psi(k_t)\Delta t\left(k_{t+\Delta t}\psi_{k_{t+\Delta t}}+\psi\right)dk_{t+\Delta t}dX_{t+\Delta t} \\
=&-\Theta(k_t)\iint v(k_{t+\Delta t}, X_{t+\Delta t})\left(k_{t+\Delta t}\psi_{k_{t+\Delta t}}+\psi\right)dk_{t+\Delta t}dX_{t+\Delta t} \\
=&-\Theta(k_t)\iint v(k_{t+\Delta t}, X_{t+\Delta t})d[k_{t+\Delta t}\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)] \\
=&-\Theta(k_t)v(k_{t+\Delta t}, X_{t+\Delta t})k_{t+\Delta t}\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\Big|_{k_{min}}^{k_{max}} \\
&+\Theta(k_t)\iint k_{t+\Delta t}\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)dv(k_{t+\Delta t}, X_{t+\Delta t}) \\
=&\Theta(k_t)\iint k_{t+\Delta t}\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)dv(k_{t+\Delta t}, X_{t+\Delta t}) \\
=&\Theta(k_t)\iint k_{t+\Delta t}\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)\frac{\partial v_{t+\Delta t}}{\partial k_{t+\Delta t}}dk_{t+\Delta t}dX_{t+\Delta t} \\
=&\Theta(k_t)\iint k_{t+\Delta t}\Gamma_{t+\Delta t}\psi(k_{t+\Delta t}, X_{t+\Delta t}|k_t, X_t)dk_{t+\Delta t}dX_{t+\Delta t}.
\end{aligned}
$$

D.5 Martingale representation theorem

Start from

$$k_t\Gamma_t = \int \left[\int \Theta(k_t) k_{t+\Delta t}\Gamma_{t+\Delta t} \Pr(k_{t+\Delta t}|k_t) dk_{t+\Delta t} \right] Pr(X_{t+\Delta t}|X_t) dX_{t+\Delta t} \tag{D.67}$$

Transpose

$$\int \left[\int \Theta(k_t) k_{t+\Delta t}\Gamma_{t+\Delta t} \Pr(k_{t+\Delta t}|k_t) dk_{t+\Delta t} - k_t\Gamma_t \right] Pr(X_{t+\Delta t}|X_t) dX_{t+\Delta t} = 0 \tag{D.68}$$

Consider the outer integration, it is measurable on the set of r_t, so we can apply martingale representation theorem and write

$$\int \Theta(k_t) k_{t+\Delta t}\Gamma_{t+\Delta t} \Pr(k_{t+\Delta t}|k_t) dk_{t+\Delta t} - k_t\Gamma_t = \sigma_W \sigma_2 dW_t \tag{D.69}$$

where W_t is the Brownian motion of aggregate shock. Then we continue to transpose

$$\int \left[\Theta(k_t) k_{t+\Delta t}\Gamma_{t+\Delta t} \Pr(k_{t+\Delta t}|k_t) - k_t\Gamma_t - \sigma_W \sigma_2 dW_t \right] dk_{t+\Delta t} = 0 \tag{D.70}$$

This kind of integration is measurablehe on the set of k_t, so we can apply martingale representation theorem again and obtain

$$\Theta(k_t) k_{t+\Delta t}\Gamma_{t+\Delta t} - k_t\Gamma_t - \sigma_W \sigma_2 dW_t = \sigma_B \sigma_1 k_t dB_t \tag{D.71}$$

here σ_B and σ_W are some functions of state variables (k_t, Γ_t).

D.6 Derivations of $d\Gamma_t$

Start from

$$\Theta(k_t) k_{t+\Delta t}\Gamma_{t+\Delta t} - k_t\Gamma_t - \sigma_W \sigma_2 dW_t = \sigma_B \sigma_1 k_t dB_t \tag{D.72}$$

Rearrange

$$k_{t+\Delta t}\Gamma_{t+\Delta t} - k_t\Gamma_t = k_{t+\Delta t}\Gamma_{t+\Delta t} - \Theta(k_t) k_{t+\Delta t}\Gamma_{t+\Delta t} + \sigma_B \sigma_1 k_t dB_t + \sigma_W \sigma_2 dW_t \tag{D.73}$$

then

$$d(k_t\Gamma_t) = [(1-\Theta(k_t))(k_t\Gamma_t)]dt + \sigma_B\sigma_1 k_t dB_t + \sigma_W\sigma_2 dW_t \tag{D.74}$$

WIth Itô Lemma, we have

$$d(k_t\Gamma_t) = k_t d\Gamma_t + \Gamma_t dk_t + dk_t d\Gamma_t \tag{D.75}$$

Here we suppose

$$d\Gamma_t = \mu_\Gamma dt + \sigma_{\Gamma,B} dB_t + \sigma_{\Gamma,W} dW_t \tag{D.76}$$

So we have

$$\begin{aligned} d(k_t\Gamma_t) =& k_t(\mu_\Gamma dt + \sigma_{\Gamma,B} dB_t + \sigma_{\Gamma,W} dW_t) + \Gamma_t(s_t dt + \sigma_1 k_t dB_t) \\ &+ (s_t dt + \sigma_1 k_t dB_t)(\mu_{\Gamma_t} dt + \sigma_{\Gamma,B} dB_t + \sigma_{\Gamma,W} dW_t) \\ =& k_t\mu_\Gamma dt + k_t\sigma_{\Gamma,B} dB_t + k_t\sigma_{\Gamma,W} dW_t + s_t\Gamma_t dt + \sigma_1 k_t\Gamma_t dB_t + \sigma_1 k_t\sigma_{\Gamma,B} dt \\ =& (s_t\Gamma_t + k_t\mu_\Gamma + \sigma_1 k_t\sigma_{\Gamma,B})dt + (\sigma_1 k_t\Gamma_t + k_t\sigma_{\Gamma,B})dB_t + k_t\sigma_{\Gamma,W} dW_t \end{aligned} \tag{D.77}$$

Compare (D.77) with (D.74) , we obtain

$$\begin{aligned} (1-\Theta(k_t))(k_t\Gamma_t) =& s_t\Gamma_t + k_t\mu_\Gamma + \sigma_1 k_t\sigma_{\Gamma,B} \\ \sigma_B\sigma_1 k_t =& \sigma_1 k_t\Gamma_t + k_t\sigma_{\Gamma,B} \\ \sigma_W\sigma_2 =& k_t\sigma_{\Gamma,W} \end{aligned} \tag{D.78}$$

we can solve that

$$\begin{aligned} \sigma_{\Gamma,B} =& \sigma_1(\sigma_B - \Gamma_t) = \sigma_1\hat{\sigma}_B \\ \sigma_{\Gamma,W} =& \frac{\sigma_W\sigma_2}{k_t} = \sigma_2\hat{\sigma}_W \end{aligned} \tag{D.79}$$

and

$$\begin{aligned} \mu_\Gamma &= [\delta - r_t(1 - T'(y_t))]\,\Gamma_t - \sigma_1\sigma_{\Gamma,B} \\ &= [\delta - r_t(1 - T'(y_t))]\,\Gamma_t - \sigma_1^2\hat{\sigma}_B \end{aligned} \tag{D.80}$$

Finally, we obtain

$$d\Gamma_t = \left\{[\delta - r_t(1 - T'(y_t))]\,\Gamma_t - \sigma_1^2\hat{\sigma}_B\right\}dt + \sigma_1\hat{\sigma}_B dB_t + \sigma_2\hat{\sigma}_W dW_t \tag{D.81}$$

where $\hat{\sigma}_B$ and $\hat{\sigma}_W$ are functions of the state variables (Γ_t, k_t).

D.7 Derivations of two-dimensional KFE

We already know that

$$\begin{aligned} d\Gamma_t =& \left\{ \left[\delta - r_t \left(1 - T'(y_t) \right) \right] \Gamma_t - \sigma_1^2 \hat{\sigma}_B \right\} dt + \sigma_1 \hat{\sigma}_B dB_t + \sigma_2 \hat{\sigma}_W dW_t \\ =& \mu_t dt + \sigma_{\Gamma,B} dB_t + \sigma_{\Gamma,W} dW_t \end{aligned} \tag{D.82}$$

If idiosyncratic labor productivity shock $x_t = x_i$, we have

$$dk_t = s_{it} dt + \sigma_1 k_t dB_t \tag{D.83}$$

where $s_{it} = y_{it} - \tilde{c}_{it} - m(l_{it}) - T(y_{it}) - \delta k_t$, for $i = 1, 2$. Let

$$\phi_1(k, \Gamma) = \phi(x_1, k, \Gamma), \quad \phi_2(k, \Gamma) = \phi(x_2, k, \Gamma) \tag{D.84}$$

We assume that those functions are twice continuously differentiable functions with compact support in $[0, \infty)$. For function $\phi(x, k, \Gamma)$, we have

$$\begin{aligned} &\mathbb{E}[\phi(x_t, k_t, \Gamma_t)] \\ =& \int \phi_1(k, \Gamma) \mathcal{G}_t(i = 1, k, \Gamma) dk d\Gamma + \int \phi_2(k, \Gamma) \mathcal{G}_t(i = 2, k, \Gamma) dk d\Gamma, \end{aligned} \tag{D.85}$$

and

$$\begin{aligned} &\mathbb{E}[\phi(x_{t+\Delta t}, k_{t+\Delta t}, \Gamma_{t+\Delta t})] \\ =& \int \phi_1(k, \Gamma) \mathcal{G}_{t+\Delta t}(i = 1, k, \Gamma) dk d\Gamma + \int \phi_2(k, \Gamma) \mathcal{G}_{t+\Delta t}(i = 2, k, \Gamma) dk d\Gamma \end{aligned} \tag{D.86}$$

On the one hand, we have

$$
\begin{aligned}
&\frac{\mathbb{E}[\phi(x_{t+\Delta t}, k_{t+\Delta t}, \Gamma_{t+\Delta t}) - \phi(x_t, k_t, \Gamma_t)]}{\Delta t} \\
=&\frac{1}{\Delta t}\left\{ \begin{array}{c} \int \phi_1(k,\Gamma)\mathcal{G}_{t+\Delta t}(i=1,k,\Gamma)dkd\Gamma + \int \phi_2(k,\Gamma)\mathcal{G}_{t+\Delta t}(i=2,k,\Gamma)dkd\Gamma \\ -\int \phi_1(k,\Gamma)\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma - \int \phi_2(k,\Gamma)\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma \end{array} \right\} \\
=&\frac{1}{\Delta t}\left\{ \begin{array}{c} \int \phi_1(k,\Gamma)[\mathcal{G}_{t+\Delta t}(i=1,k,\Gamma) - \mathcal{G}_t(i=1,k,\Gamma)]dkd\Gamma \\ +\int \phi_2(k,\Gamma)[\mathcal{G}_{t+\Delta t}(i=2,k,\Gamma) - \mathcal{G}_t(i=2,k,\Gamma)]dkd\Gamma \end{array} \right\} \\
=&\int \phi_1(k,\Gamma)\frac{\mathcal{G}_{t+\Delta t}(i=1,k,\Gamma) - \mathcal{G}_t(i=1,k,\Gamma)}{\Delta t}dkd\Gamma \\
&+\int \phi_2(k,\Gamma)\frac{\mathcal{G}_{t+\Delta t}(i=2,k,\Gamma) - \mathcal{G}_t(i=2,k,\Gamma)}{\Delta t}dkd\Gamma \\
=&\int \phi_1(k,\Gamma)\frac{\partial \mathcal{G}_t(i=1,k,\Gamma)}{\partial t}dkd\Gamma + \int \phi_2(k,\Gamma)\frac{\partial \mathcal{G}_t(i=2,k,\Gamma)}{\partial t}dkd\Gamma
\end{aligned}
\tag{D.87}
$$

On the other hand, by Itô's Lemma we have

$$
\begin{aligned}
d\phi_i(k_t,\Gamma_t) =&\phi_k(i,k_t,\Gamma_t)dk + \phi_\Gamma(i,k_t,\Gamma_t)d\Gamma + \frac{1}{2}\phi_{kk}(i,k_t,\Gamma_t)(dk)^2 + \frac{1}{2}\phi_{\Gamma\Gamma}(i,k_t,\Gamma_t)(d\Gamma)^2 \\
=&\varphi_k(i,k_t,\Gamma_t)(s_{it}dt + \sigma_1 k_t dB_t) + \phi_\Gamma(i,k_t,\Gamma_t)(\mu_{it}dt + \sigma_{\Gamma,B}dB_t + \sigma_{\Gamma,W}dW_t) \\
&+ \frac{1}{2}\phi_{kk}(i,k_t,\Gamma_t)(s_{it}dt + \sigma_1 k_t dB_t)^2 + \frac{1}{2}\varphi_{\Gamma\Gamma}(i,k_t,\Gamma_t)(\mu_{it}dt + \sigma_{\Gamma,B}dB_t + \sigma_{\Gamma,W}dW_t)^2 \\
=&\phi_k(i,k_t,\Gamma_t)s_{it}dt + \phi_k(i,k_t,\Gamma_t)\sigma_1 k_t dB_t \\
&+ \phi_\Gamma(i,k_t,\Gamma_t)\mu_{it}dt + \phi_\Gamma(i,k_t,\Gamma_t)\sigma_{\Gamma,B}dB_t + \phi_\Gamma(i,k_t,\Gamma_t)\sigma_{\Gamma,W}dW_t \\
&+ \frac{1}{2}\phi_{kk}(i,k_t,\Gamma_t)\sigma_1^2 k_t^2 dt + \frac{1}{2}\phi_{\Gamma\Gamma}(i,k_t,\Gamma_t)\sigma_{\Gamma,B}^2 dt + \frac{1}{2}\phi_{\Gamma\Gamma}(i,k_t,\Gamma_t)\sigma_{\Gamma,W}^2 dt \\
=&\left[\phi_k(i,k_t,\Gamma_t)s_{it} + \phi_\Gamma(i,k_t,\Gamma_t)\mu_{it} + \frac{1}{2}\phi_{kk}(i,k_t,\Gamma_t)\sigma_1^2 k_t^2 + \frac{1}{2}\phi_{\Gamma\Gamma}(i,k_t,\Gamma_t)(\sigma_{\Gamma,B}^2 + \sigma_{\Gamma,W}^2)\right]dt \\
&+ \left[\phi_k(i,k_t,\Gamma_t)\sigma_{k,B} + \phi_\Gamma(i,k_t,\Gamma_t)\sigma_{\Gamma,B}\right]dB_t + \phi_\Gamma(i,k_t,\Gamma_t)\sigma_{\Gamma,W}dW_t
\end{aligned}
\tag{D.88}
$$

Thus we have

$$
\begin{aligned}
&\frac{\mathbb{E}[\phi(x_{t+\Delta t},k_{t+\Delta t},\Gamma_{t+\Delta t})-\phi(x_t,k_t,\Gamma_t)]}{\Delta t}\\
=&\frac{1}{\Delta t}\left\{\mathbb{E}(\mathbb{E}[\phi(x_{t+\Delta t},k_{t+\Delta t},\Gamma_{t+\Delta t}|x_t,k_t,\Gamma_t)])-\mathbb{E}[\phi(x_t,k_t,\Gamma_t)]\right\}\\
=&\frac{1}{\Delta t}\mathbb{E}\left\{\mathbb{E}[\phi(x_{t+\Delta t},k_{t+\Delta t},\Gamma_{t+\Delta t}|k_t,\Gamma_t)]-\phi(x_t,k_t,\Gamma_t)\right\}\\
=&\frac{1}{\Delta t}\left\{\begin{array}{c}\int\left\{\mathbb{E}[\phi(k_{t+\Delta t},\Gamma_{t+\Delta t}|i=1,k_t,\Gamma_t)]-\phi_1(k,\Gamma)\right\}\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma\\+\int\left\{\mathbb{E}[\phi(k_{t+\Delta t},\Gamma_{t+\Delta t}|i=2,k_t,\Gamma_t)]-\phi_2(k,\Gamma)\right\}\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma\end{array}\right\}\\
=&\int\frac{[\phi(k_{t+\Delta t},\Gamma_{t+\Delta t}|i=1,k_t,\Gamma_t)-\phi_1(k,\Gamma)}{\Delta t}\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma\\
&+\int\frac{[\phi(k_{t+\Delta t},\Gamma_{t+\Delta t}|i=2,k_t,\Gamma_t)-\phi_2(k,\Gamma)}{\Delta t}\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma\\
=&\int d\phi_1(k_t,\Gamma_t)\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma+\int d\phi_2(k_t,\Gamma_t)\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma\\
=&\int\left[\phi^1_k s_{1t}+\phi^1_\Gamma\mu_{1t}+\frac{1}{2}\phi^1_{kk}\sigma_1^2k_t^2+\frac{1}{2}\phi^1_{\Gamma\Gamma}(\sigma^2_{\Gamma,B}+\sigma^2_{\Gamma,W})-\lambda_1\phi^1(k,\Gamma)+\lambda_1\phi^2(k,\Gamma)\right]\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma\\
&+\int\left[\phi^2_k s_{2t}+\phi^2_\Gamma\mu_{2t}+\frac{1}{2}\phi^2_{kk}\sigma_1^2k_t^2+\frac{1}{2}\phi^2_{\Gamma\Gamma}(\sigma^2_{\Gamma,B}+\sigma^2_{\Gamma,W})-\lambda_2\phi^2(k,\Gamma)+\lambda_2\phi^1(k,\Gamma)\right]\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma
\end{aligned}
\tag{D.89}
$$

Therefore, we have

$$
\begin{aligned}
&\int\phi_1(k,\Gamma)\frac{\partial\mathcal{G}_t(i=1,k,\Gamma)}{\partial t}dkd\Gamma+\int\phi_2(k,\Gamma)\frac{\partial\mathcal{G}_t(i=2,k,\Gamma)}{\partial t}dkd\Gamma\\
=&\int d\phi_1(k_t,\Gamma_t)\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma+\int d\phi_2(k_t,\Gamma_t)\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma
\end{aligned}
\tag{D.90}
$$

If we pick $\phi_2(k,\Gamma)=0$ for all k and Γ, we have

$$
\begin{aligned}
&\int\phi_1(k,\Gamma)\frac{\partial\mathcal{G}_t(i=1,k,\Gamma)}{\partial t}dkd\Gamma\\
=&\int\left[\phi^1_k s_{1t}+\phi^1_\Gamma\mu_{1t}+\frac{1}{2}\phi^1_{kk}\sigma_1^2k_t^2+\frac{1}{2}\phi^1_{\Gamma\Gamma}(\sigma^2_{\Gamma,B}+\sigma^2_{\Gamma,W})-\lambda_1\phi^1(k,\Gamma)\right]\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma\\
&+\int\lambda_2\phi^1(k,\Gamma)\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma\\
=&-\int\phi_1\frac{\partial}{\partial k}\left[s_{1t}\mathcal{G}_t(i=1,k,\Gamma)\right]dkd\Gamma-\int\phi_1\frac{\partial}{\partial\Gamma}\left[\mu_{1t}\mathcal{G}_t(i=1,k,\Gamma)\right]dkd\Gamma\\
&+\frac{1}{2}\int\phi_1\frac{\partial^2}{\partial k^2}\left[\sigma_1^2k_t^2\mathcal{G}_t(i=1,k,\Gamma)\right]dkd\Gamma+\frac{1}{2}\int\phi_1\frac{\partial^2}{\partial\Gamma^2}\left[(\sigma^2_{\Gamma,B}+\sigma^2_{\Gamma,W})\mathcal{G}_t(i=1,k,\Gamma)\right]dkd\Gamma\\
&-\int\lambda_1\phi_1(k,\Gamma)\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma+\int\lambda_2\phi_1(k,\Gamma)\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma
\end{aligned}
\tag{D.91}
$$

Since $\phi_1(k,\Gamma)$ is arbitrary, we obtain

$$\begin{aligned}\frac{\partial \mathcal{G}_t(i=1,k,\Gamma)}{\partial t} =& \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma_1^2 k_t^2 \mathcal{G}_t(i=1,k,\Gamma)\right] + \frac{1}{2}\frac{\partial^2}{\partial \Gamma^2}\left[(\sigma_{\Gamma,B}^2+\sigma_{\Gamma,W}^2)\mathcal{G}_t(i=1,k,\Gamma)\right] \\ &- \frac{\partial}{\partial k}\left[s_{1t}\mathcal{G}_t(i=1,k,\Gamma)\right] - \frac{\partial}{\partial \Gamma}\left[\mu_{1t}\mathcal{G}_t(i=1,k,\Gamma)\right] \\ &- \lambda_1 \mathcal{G}_t(i=1,k,\Gamma) + \lambda_2 \mathcal{G}_t(i=2,k,\Gamma)\end{aligned} \tag{D.92}$$

Similarly, Iif we pick $\phi_1(k,\Gamma)=0$ for all k and Γ, we have

$$\begin{aligned}&\int \phi_2(k,\Gamma)\frac{\partial \mathcal{G}_t(i=2,k,\Gamma)}{\partial t} dkd\Gamma \\ =&\int \left[\phi_k^2 s_{2t} + \phi_\Gamma^2 \mu_{2t} + \frac{1}{2}\phi_{kk}^2\sigma_1^2 k_t^2 + \frac{1}{2}\phi_{\Gamma\Gamma}^2(\sigma_{\Gamma,B}^2+\sigma_{\Gamma,W}^2) - \lambda_2\phi^2(k,\Gamma)\right]\mathcal{G}_t(2=1,k,\Gamma)dkd\Gamma \\ &+ \int \lambda_1\phi^2(k,\Gamma)\mathcal{G}_t(i=1,k,\Gamma)dkd\Gamma \\ =& -\int \phi_2 \frac{\partial}{\partial k}\left[s_{2t}\mathcal{G}_t(i=2,k,\Gamma)\right]dkd\Gamma - \int \phi_2\frac{\partial}{\partial\Gamma}\left[\mu_{2t}\mathcal{G}_t(i=2,k,\Gamma)\right]dkd\Gamma \\ &+ \frac{1}{2}\int \phi_2\frac{\partial^2}{\partial k^2}\left[\sigma_1^2 k_t^2\mathcal{G}_t(i=2,k,\Gamma)\right]dkd\Gamma + \frac{1}{2}\int\phi_2\frac{\partial^2}{\partial\Gamma^2}\left[(\sigma_{\Gamma,B}^2+\sigma_{\Gamma,W}^2)\mathcal{G}_t(i=2,k,\Gamma)\right]dkd\Gamma \\ &- \int \lambda_2\phi_2(k,\Gamma)\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma + \int\lambda_1\phi_2(k,\Gamma)\mathcal{G}_t(i=2,k,\Gamma)dkd\Gamma\end{aligned} \tag{D.93}$$

Since $\phi_2(k,\Gamma)$ is arbitrary, we obtain

$$\begin{aligned}\frac{\partial \mathcal{G}_t(i=2,k,\Gamma)}{\partial t} =& \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma_1^2 k_t^2 \mathcal{G}_t(i=2,k,\Gamma)\right] + \frac{1}{2}\frac{\partial^2}{\partial \Gamma^2}\left[(\sigma_{\Gamma,B}^2+\sigma_{\Gamma,W}^2)\mathcal{G}_t(i=2,k,\Gamma)\right] \\ &- \frac{\partial}{\partial k}\left[s_{2t}\mathcal{G}_t(i=2,k,\Gamma)\right] - \frac{\partial}{\partial \Gamma}\left[\mu_{2t}\mathcal{G}_t(i=2,k,\Gamma)\right] \\ &- \lambda_2 \mathcal{G}_t(i=2,k,\Gamma) + \lambda_1 \mathcal{G}_t(i=1,k,\Gamma)\end{aligned} \tag{D.94}$$

Combine (D.92) and (D.94), we finally obtain

$$\begin{aligned}\frac{\partial \mathcal{G}_t(i,k,\Gamma)}{\partial t} =& \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma_1^2 k_t^2 \mathcal{G}_t(i,k,\Gamma)\right] + \frac{1}{2}\frac{\partial^2}{\partial \Gamma^2}\left[(\sigma_{\Gamma,B}^2+\sigma_{\Gamma,W}^2)\mathcal{G}_t(i,k,\Gamma)\right] \\ &- \frac{\partial}{\partial k}\left[s_{it}\mathcal{G}_t(i,k,\Gamma)\right] - \frac{\partial}{\partial \Gamma}\left[\mu_{it}\mathcal{G}_t(i,k,\Gamma)\right] - \lambda_i\mathcal{G}_t(i,k,\Gamma) + \lambda_{-i}\mathcal{G}_t(-i,k,\Gamma)\end{aligned} \tag{D.95}$$

for $i=1,2$.

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