Adverse Selection and Income Inequality

Yaping Wu
Institute of Economics and Finance,
Nanjing Audit University

Shenghao Zhu*
School of International Trade and Economics,
University of International Business and Economics

November 24, 2021

Abstract

We investigate the inequality implications of the incentive-feasible contracts, including opitmal and non-optimal contracts, when there is a trade-off between rent extraction and efficiency. We show that information frictions cause inequality and the change of social norms influences inequality under the optimal contract. We then study the impacts of information structures on income inequality.

Keywords: Contracts, Income Inequality, Information frictions, change of social norms, Information structure

JEL classification: D31, D52, E21, J22

*We would like to thank Jess Benhabib, Jie Cai, Zhiyun Li, Thomas Sargent, and Yitong Wang for their comments and suggestions.

1 Introduction

We investigate the inequality implications of the incentive-feasible contracts, including opitmal and non-optimal contracts, when there is a trade-off between rent extraction and efficiency. Contracts pin down the agent's payment in our model. Thus, contracts transform their types to incomes. Thus, contracts, whether optiaml or not, have specifc implications of income inequality.

There are a continuum of agents with measure 1 in the economy. The agent's marginal cost of production is θ. Even though the distribution of Θ is common knowledge, the reliaztion of θ is unobservable to the principal and it is the agent's private information. We study the adverse selection problem in this paper and show that the information rent increases income inequality. We investigate the inequality implications of the incentive-feasible contracts. The incentive feasible contracts satisfy both incentive and participation constraints. We find that information rents increase income inequality for the incentive feasible contracts under certain conditions.

We find that output scheme determines the payment schedule for any feasible contracts. The payment includes two parts, the production cost and the information rent. Information rents can be expressed as a function of the output. Therefore, the payment is a function of the output level. We can characterize the payment schedule through investigating the properties of the output scheme. In this sense the output scheme contains sufficient clues to study the inequality implications of any feasible contract.

In this paper we investigate the income distribution when the agents' payments are not in line with the marginal cost. However, in the neo-classical theory of distribution, factors receive payments according to their marginal product. Information frictions cause the wedge between agents' payments and their marginal costs. We find that information frictions cause inequality and situations under which we can rank contracts according to their induced income inequality.

In order to investigate the impacts of asymmetric information on income inequality, we find the optimal contract under asymmetric information and the optimal contract under complete information. Each contract induces income inequality in the economy. We then compare the inequality under asymmetric information and that under complete information. We find that the inequality under asymmetric information less equal than that under complete information. We rank the income equality of these contracts using the Lorenz ordering. The Lorenz curves can be ranked without intersections. The optimal contract incurs a less equal output distribution under incomplete information than that under complete information. We show that asymmetric infromation causes income inequality through two channels. A less equal output distribution is the first channel. And the information rent exaggerates the income inequality further.

Piketty and Saez (2003) document that the top income shares in the United States display a rapid trend of increase since the 1970s. They propose that changing social norms regarding inequality partly explain the rise in top wage shares. Piketty (2020) argues that idiology is an important force determing the social inequality. Ideology is refered to "a set of a priori plausible ideas and discourses describing how society should be structured." We show that different contracts (mechanims) has distinct inequality implications in the economy.

We study the inequaity implication of the incentive feasible contracts. However, the pricipal's objective function of our model does not reflect the equity concern. We also find that the change of social norms influences inequality under the optimal contract. Baron and Myerson (1982) use a weighted sum of the expected gains to consumers the expected profit for the firm as the social welfare function. Specifically, they use a parameter to represent the relative weight between consumers and the firm. Following their paper, we introduce parameter $\alpha \in [0, 1]$, regarding the concern for the agent's utility, into the social welfare function. The change of α represnets the change of social norms. We find that inequality, measured by the Lorenz ordering, in our model increases if α decreases.

Finally, we then study the impacts of information structures on income inequality. We first study a more favorable information structure change, intruduce by Laffont and Tirole (1993). We find that a more favorable distribution type imply a less equal payment schedule. We then study how a mean-preserving spread influences the payment schedule.

1.1 Reltated literatures

Our model shares a common feature with Che and Gale (1998). The principals cares about their own surplus from the contract in both models and they do not have equity concern in the objective funtions. In Che and Gale (1998), the agents have two-dimentional private information. The agents in our model have one-dimentional private information. One component of private information in their model is wealth, which represents inequality of the economy. The income in our model is from the contract. Thus, inequality of their model is exogenous, while inquality of ours is endogenous. The other difference between their paper and ours, is that Che and Gale (1998) focus on allocation efficiency while we concentrate on inequality induced by the contracts. Che and Gale (1998) compare two mechnisms, the first-price auction and the second-price auction, through two aspects, expected revenue and expected social surplus. We compare different contracts from the inequality perspective. Che, Gale, and Kim (2013) and Condorelli (2013) compare market mechanisms and non-market mechanisms.

Our work is also related to Lazear and Rosen (1981). Both papers compare the income distributions under different contracts. Lazear and Rosen (1981) investigate how different incentive-inducing contracts under moral hazard generate different income distributions. Our model studies income distributions implied by different incentive-inducing contracts under adverse selection. Lazear and Rosen (1981) find that the tournament mechnism can produce the skewed income distribution. Similarly, we find that the optimal contract under asymmetric information can generate an income ditribution more dispersed than that under complete information. Lazear and Rosen (1981) investigate the aggregate welfare for two cases, the case of risk neutral agents and that of risk averse agents.[1] We concentrate on the situation of risk neutral agents and thus, the principal's objective function does not incorporate the risk-sharing incentive in our model.

Fernandez and Gali (1999) compare markets and tournaments in an

[1] Other literatures of tournament-based compensation schemes include Green and Stokey (1983) and Nalebuff and Stiglitz (1983). The tournament is a usual scheme in the moral hazard problem with many agents. As papers in the optimal taxation literature, our model concentrates on the independent contrats among agents.

economy with borrowing constraints. They concentrate on allocation efficiency of these two mechanims. Complementary to the literature which compares the efficiency of different mechanims, our paper compare the income distributions under different contracts. Even though the initial wealth distribution is exogenous in Fernandez and Gali (1999), the income distribution is endogenous and depends on the mechanism in their model. The income distribution in our model depends on contracts and is also endogenous.

Dworczak et al. (2021) investigate the role of redistribution of the price regulation in a market with private information. While Dworczak et al. (2021) study the optimal mechanism design, the policymakers in their model have equity concern. The principal in our model has no equity concern. Thus, we concentrate on the implications of optimal contracts which only reflects the production (allocation) dimension. We intentionally shut down the equity concern in the contract design. In this sense, the inequality is an "unintentional" product of the contract's incentive stimulus effects.

Mirrlees (1971) and Saez (2001) investigate the optimal income taxation in models with unobservable productivity. The objective function of the principal (the government) is utilitarain. It is the sum of the utility function of all agents in the economy. The social welfare function incorporates the equity concern since the agent's utility function is risk-averse and has curvature. Our model differs from this literature in two aspects. First, the principal's objective function has no equity concern and only reflects the production dimension. Second, the optimal taxaion literature usually concentrates on the optimal tax scheme, while we investigate the inequality implications of the incentive-feasible contracts, including opitmal and non-optimal contracts. Wu and Zhu (2021) investigate the nonlinear tax incidence on inequality in a Mirrleessian framework.

The rest of the paper is organized as follows. We present our benchmark model in Section 2. We investigate the income distribution under the optimal contract in Section **??**. Section 4 contains an analysis of social norm change. Section 7 concludes the paper.

2 Feasible contracts and inequality

There are a continuum of agents with measure 1 in the economy. The agent's marginal cost of production is θ. Even though the distribution of Θ is common knowledge, the reliaztion of θ is unobservable to the principal and it is the agent's private information.

For the two-type case, we assume that

Assumption 1: Θ follows a two-type discrete probability distribution,

$$\Theta = \begin{cases} \underline{\theta}, & \text{with probability } v \\ \overline{\theta}, & \text{with probabiity } 1 - v \end{cases} .$$

Even though the distribution of Θ is common knowledge, the reliaztion of θ_s is unobservable to the principal. Let $\Delta\theta$ denote the spread of marginal cost,

$$\Delta\theta = \overline{\theta} - \underline{\theta} > 0.$$

There is no heterogeneity among principals. Principals run firms and hire agents. The principal offers the contract $\left\{ (t(\underline{\theta}), q(\underline{\theta})) ; \left(t\left(\overline{\theta}\right), q\left(\overline{\theta}\right)\right) \right\}$. The agent chooses to claim his type $\tilde{\theta}$. If $\tilde{\theta} = \underline{\theta}$, the agent receives payments $t(\underline{\theta})$ and provides output $q(\underline{\theta})$ to the principal's firm. If $\tilde{\theta} = \overline{\theta}$, the agent receives payments $t\left(\overline{\theta}\right)$ and provides output $q\left(\overline{\theta}\right)$ to the principal's firm. We view $t(\underline{\theta})$ and $t\left(\overline{\theta}\right)$ as agents' incomes.

The incentive compatibility constraints are

$$t(\underline{\theta}) - \underline{\theta} q(\underline{\theta}) \geq t\left(\overline{\theta}\right) - \underline{\theta} q\left(\overline{\theta}\right), \tag{1}$$

and

$$t\left(\overline{\theta}\right) - \overline{\theta} q\left(\overline{\theta}\right) \geq t(\underline{\theta}) - \overline{\theta} q(\underline{\theta}). \tag{2}$$

We call the optimal contract under incomplete information the second-best contract. Let $\underline{t} = t(\underline{\theta})$, $\underline{q} = q(\underline{\theta})$, $\bar{t} = t\left(\overline{\theta}\right)$, and $\overline{q} = q\left(\overline{\theta}\right)$. The incentive compatibility constraints implies that any pair of outputs $\left(\underline{q}, \overline{q}\right)$ that is implimentable must satisfy the implementability condition $\underline{q} \geq \overline{q}$.

Assumption 2 : The principal's firm has production function S,

$$S' > 0, S'' < 0, \lim_{q \to 0} S'(q) = \infty, \lim_{q \to 0} S'(q)q = 0.$$

Assumption 2 guarantees that

Each principal has the objective function,

$$\max_{\{(\underline{t},\underline{q});(\bar{t},\bar{q})\}} v\left[S\left(\underline{q}\right)-\underline{t}\right]+(1-v)\left[S\left(\bar{q}\right)-\bar{t}\right].$$

As in Laffont and Martimort (2002), we define information rents,

$$\underline{U}=\underline{t}-\underline{\theta}\underline{q},$$

for efficient agents and

$$\overline{U}=\bar{t}-\bar{\theta}\bar{q},$$

for inefficient agents. A feasible contract has to satisfy the incentive compatibility constraints (1) and (2), and the participation constraints,

$$\underline{U}\geq 0,$$

and

$$\overline{U}\geq 0.$$

The agent's payment scheme consists of

$$\underline{t}=\underline{\theta}\underline{q}+\underline{U},$$

and

$$\bar{t}=\bar{\theta}\bar{q}+\overline{U}.$$

Unless we have $\overline{U}=\underline{U}=0$, the agents' payments are not in line with the marginal cost. The incentive compatibility constraints cause the disparity between agents' payments and their marginal costs, which implies the income differences among agents. We then investigate the impacts of informational rents $\overline{U}$ and $\underline{U}$ on income inequality.

Theorem 1 *For the two-type case, we have* $\frac{\underline{\theta}\underline{q}+\underline{U}}{\bar{\theta}\bar{q}}\geq\frac{\underline{\theta}\underline{q}}{\bar{\theta}\bar{q}}$.

Let $F_X(\cdot)$ be the distribution function of a non-negative random variable X with a finite positive mean. Following Gastwirth (1971) we define the Lorenz curve as follows.

Definition 1 *The Lorenz curve of X, $L_X(p)$, is defined as*

$$L_X(p) = \frac{1}{E(X)} \int_0^p F_X^{-1}(r)dr, \forall p \in [0,1],$$

where $F_X^{-1}(r) = \inf \{x \geq 0 : F_X(x) \geq r\}$.

From the definition of the Lorenz curve, we know that, for any constant $\phi > 0$, X and ϕX share the same Lorenz curve. Thus, multiplying a random variable by a postive constant does not change its Lorenz curve. We then define the Lorenz ordering as follows.

Definition 2 *For two non-negative random variables X and Y, X Lorenz dominates Y if, and only if,*

$$L_X(p) \geq L_Y(p),$$

for all $p \in [0,1]$, denoted as $X \succeq_L Y$.

Thus, $X \succeq_L Y$ implies that Y is less equal than X. If $X \succeq_L Y$, then the Gini coefficient of X is smaller than Y.

Proposition 1 *Let*

$$X = \begin{cases} \underline{\theta}\underline{q}, & \textit{with probability } v \\ \overline{\theta}\overline{q}, & \textit{with probabiity } 1-v \end{cases},$$

where $x' < x$. And

$$Y = \begin{cases} \underline{\theta}\underline{q} + \underline{U}, & \textit{with probability } v \\ \overline{\theta}\overline{q}, & \textit{with probabiity } 1-v \end{cases},$$

then we have $X \succeq_L Y$.

Propostion implies that

insert a picture here!

insert a picture here!

For the continous type, we assume

Assumption 1′: Θ follows a continuous probability distribution on $[\underline{\theta}, \overline{\theta}]$, with a probability density function $f(\theta)$, and

$$\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \geq 0,$$

where $F(\theta)$ is the cumulative distribution function.

Assumption 1′ implies the monotone hazard rate property.

The incentive compatibility condition is

$$t(\theta) - \theta q(\theta) \geq t(\tilde{\theta}) - \theta q(\tilde{\theta}),$$

for any $(\theta, \tilde{\theta})$ in Θ^2. The local incentive constraints and local second order condition imply $\dot{q}(\theta) \leq 0$.

We define

$$\varepsilon_{q(\theta)} = -\frac{\theta \dot{q}(\theta)}{q(\theta)}.$$

Theorem 2 *For the continous-type case, if*

$$q(\theta) + \theta \dot{q}(\theta) < 0 \tag{3}$$

and

$$\varepsilon_{q(\theta)} \leq \frac{\overline{\theta}}{\overline{\theta} - \theta}, \text{ or } \varepsilon_{q(\theta)} \leq \frac{\overline{\theta}}{\triangle\theta} \text{ (suffcient condition)}, \tag{4}$$

then we have

$$\frac{\partial}{\partial\theta}\left(\frac{\theta q(\theta) + U(\theta)}{\theta q(\theta)}\right) \leq 0. \tag{5}$$

Proof:

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\frac{\theta q(\theta) + U(\theta)}{\theta q(\theta)}\right) &= \frac{\partial}{\partial\theta}\left(\frac{t(\theta)}{\theta q(\theta)}\right) = \frac{t'(\theta)\theta q(\theta) - (q(\theta) + \theta q'(\theta))\, t(\theta)}{\theta^2 q(\theta)^2}, & (6)\\
&= \frac{\theta^2 q'(\theta) q(\theta) - (q(\theta) + \theta q'(\theta))\left(\theta q(\theta) + \int_\theta^{\overline{\theta}} q(\tau)\, d\tau\right)}{\theta^2 q(\theta)^2} & (7)\\
&= \frac{-\theta q(\theta)^2 - (q(\theta) + \theta q'(\theta)) \int_\theta^{\overline{\theta}} q(\tau)\, d\tau}{\theta^2 q(\theta)^2}. & (8)
\end{aligned}$$

The numerator

$$-\theta q(\theta)^2 - (q(\theta) + \theta q'(\theta)) \int_\theta^{\overline{\theta}} q(\tau)\, d\tau = -\theta q(\theta)^2 - (q(\theta) + \theta q'(\theta)) \left(\overline{\theta} - \theta\right) q(\xi), \tag{9}$$

where $\xi \in \left(\theta, \overline{\theta}\right)$. Since $\dot{q}(\theta) < 0$, $q(\theta) > q(\xi)$. If

$$q(\theta) + \theta \dot{q}(\theta) < 0, \tag{10}$$

then the numerator of $\frac{\partial}{\partial \theta}\left(\frac{\theta q(\theta) + U(\theta)}{\theta q(\theta)}\right)$ is negative if

$$-\theta q(\theta)^2 - (q(\theta) + \theta \dot{q}(\theta)) \left(\overline{\theta} - \theta\right) q(\theta) \leq 0, \tag{11}$$

which reduces to

$$-\frac{\theta \dot{q}(\theta)}{q(\theta)} \leq \frac{\overline{\theta}}{\overline{\theta} - \theta}. \tag{12}$$

A sufficient condition of the above inequality is

$$-\frac{\theta \dot{q}(\theta)}{q(\theta)} \leq \frac{\overline{\theta}}{\triangle \theta}. \tag{13}$$

■

Proposition 2 *Let*

$$X(\theta) = \theta q(\theta),$$

and

$$Y(\theta) = \theta q(\theta) + U(\theta),$$

then we have $X \succeq_L Y$.

Proposition implies that

insert a picture here!

We find that information rents increase income inequality for the incentive feasible contracts under certain conditions.

We find that information rents also depend on the output level. Thus, the output level plays an important role in understanding the income equality.

Theorem 3 *For the two-type case, if* $\underline{q}\,/\,\overline{q}$ *increases,* $\underline{t}\,/\,\overline{t}$ *increases.*

Proof: The relevant (binding) constraints are incentive compatibility constraint of the efficient type and the participation constraint of the inefficient type

$$\underline{t}-\underline{\theta}\underline{q}\geq\overline{t}-\underline{\theta}\overline{q}, \tag{14}$$
$$\overline{t}-\overline{\theta}\overline{q}\geq 0. \tag{15}$$

The two constraints imply

$$\underline{t}=\underline{\theta}\underline{q}+\left(\overline{\theta}-\underline{\theta}\right)\overline{q}, \tag{16}$$
$$\overline{t}=\overline{\theta}\overline{q}. \tag{17}$$

Therefore,

$$\underline{t}\,/\,\overline{t}=\left(\underline{\theta}\underline{q}+\left(\overline{\theta}-\underline{\theta}\right)\overline{q}\right)/\left(\overline{\theta}\overline{q}\right)=\left(\underline{\theta}\,/\,\overline{\theta}\right)\left(\underline{q}\,/\,\overline{q}\right)+\left(\overline{\theta}-\underline{\theta}\right)/\overline{\theta}. \tag{18}$$

Thus, $\underline{t}\,/\,\overline{t}$ increases with $\underline{q}\,/\,\overline{q}$. ■

Theorem 4 *For the continous-type case,* $\frac{\partial}{\partial\theta}\left(\frac{t(\theta)}{\widetilde{t}(\theta)}\right)\leq 0$ *if and only if*

$$\frac{\theta\widetilde{q}\left(\theta\right)+\int_{\theta}^{\overline{\theta}}\widetilde{q}\left(\tau\right)d\tau}{\widetilde{q}'\left(\theta\right)}\leq\frac{\theta q\left(\theta\right)+\int_{\theta}^{\overline{\theta}}q\left(\tau\right)d\tau}{q'\left(\theta\right)}. \tag{19}$$

A sufficient condition is that if $q\left(\underline{\theta}\right)\leq\widetilde{q}\left(\underline{\theta}\right)$ *and* $q'\left(\theta\right)\leq\widetilde{q}'\left(\theta\right)$*, then we have* $\frac{\partial}{\partial\theta}\left(\frac{t(\theta)}{\widetilde{t}(\theta)}\right)\leq 0$.

Proof: Since

$$\frac{t'\left(\theta\right)}{\widetilde{t}'\left(\theta\right)}=\frac{q'\left(\theta\right)}{\widetilde{q}'\left(\theta\right)}, \tag{20}$$

and

$$\frac{t'\left(\theta\right)}{\widetilde{t}'\left(\theta\right)}\geq\frac{t\left(\theta\right)}{\widetilde{t}\left(\theta\right)} \tag{21}$$

is a necessary and sufficient condition for

$$\frac{\partial}{\partial\theta}\left(\frac{t\left(\theta\right)}{\widetilde{t}\left(\theta\right)}\right)=\frac{t'\left(\theta\right)\widetilde{t}\left(\theta\right)-t\left(\theta\right)\widetilde{t}'\left(\theta\right)}{\widetilde{t}\left(\theta\right)^{2}}\leq 0. \tag{22}$$

therefore, $\frac{t'(\theta)}{\widetilde{t}'(\theta)} \geq \frac{t(\theta)}{\widetilde{t}(\theta)}$ if and only if

$$\frac{q'(\theta)}{\widetilde{q}'(\theta)} \geq \frac{\theta q(\theta) + \int_{\theta}^{\overline{\theta}} q(\tau)\, d\tau}{\theta \widetilde{q}(\theta) + \int_{\theta}^{\overline{\theta}} \widetilde{q}(\tau)\, d\tau}, \tag{23}$$

or

$$\frac{\theta \widetilde{q}(\theta) + \int_{\theta}^{\overline{\theta}} \widetilde{q}(\tau)\, d\tau}{\widetilde{q}'(\theta)} \leq \frac{\theta q(\theta) + \int_{\theta}^{\overline{\theta}} q(\tau)\, d\tau}{q'(\theta)}. \tag{24}$$

Since $\dot{q}(\theta) < 0$, $q'(\theta) \leq \widetilde{q}'(\theta)$ implies

$$\frac{q'(\theta)}{\widetilde{q}'(\theta)} = \frac{|q'(\theta)|}{|\widetilde{q}'(\theta)|} \geq 1. \tag{25}$$

Since $q(\underline{\theta}) \leq \widetilde{q}(\underline{\theta})$ and $q'(\theta) \leq \widetilde{q}'(\theta)$,

$$q(\theta) \leq \widetilde{q}(\theta), \forall \theta \in [\underline{\theta}, \overline{\theta}]; \tag{26}$$

that is, the curve $q(\theta)$ is (weakly) below the curve $\widetilde{q}(\theta)$, $\forall \theta \in [\underline{\theta}, \overline{\theta}]$. If $q(\underline{\theta}) = \widetilde{q}(\underline{\theta})$, the two curves start at the same point when $\theta = \underline{\theta}$ and then, $q(\theta)$ decreases (weakly) faster than $\widetilde{q}(\theta)$, and thus, is (weakly) below the $\widetilde{q}(\theta)$, $\forall \theta \in [\underline{\theta}, \overline{\theta}]$. Thus, we obtain

$$\int_{\theta}^{\overline{\theta}} q(\tau)\, d\tau \leq \int_{\theta}^{\overline{\theta}} \widetilde{q}(\tau)\, d\tau. \tag{27}$$

$q(\theta)$ and $\widetilde{q}(\theta)$ are positive. Hence,

$$\frac{\theta q(\theta) + \int_{\theta}^{\overline{\theta}} q(\tau)\, d\tau}{\theta \widetilde{q}(\theta) + \int_{\theta}^{\overline{\theta}} \widetilde{q}(\tau)\, d\tau} \leq 1. \tag{28}$$

Therefore,

$$\frac{t(\theta)}{\widetilde{t}(\theta)} = \frac{\theta q(\theta) + \int_{\theta}^{\overline{\theta}} q(\tau)\, d\tau}{\theta \widetilde{q}(\theta) + \int_{\theta}^{\overline{\theta}} \widetilde{q}(\tau)\, d\tau} \leq 1 \leq \frac{q'(\theta)}{\widetilde{q}'(\theta)}. \tag{29}$$

In addition,

$$\frac{t'(\theta)}{\widetilde{t}'(\theta)} = \frac{\theta q'(\theta)}{\theta \widetilde{q}'(\theta)} = \frac{q'(\theta)}{\widetilde{q}'(\theta)}, \tag{30}$$

we obtain

$$\frac{t'(\theta)}{\widetilde{t}'(\theta)} \geq \frac{t(\theta)}{\widetilde{t}(\theta)}. \tag{31}$$

Since $t(\theta)$ and $\widetilde{t}(\theta)$ are positive, and $t'(\theta)$ and $\widetilde{t}'(\theta)$ are negative, the above inequality implies

$$\frac{t'(\theta)\,\widetilde{t}(\theta)}{\widetilde{t}'(\theta)} \geq t(\theta), \tag{32}$$

and then

$$t'(\theta)\,\widetilde{t}(\theta) \leq t(\theta)\,\widetilde{t}'(\theta). \tag{33}$$

Hence,

$$\frac{\partial}{\partial\theta}\left(\frac{t(\theta)}{\widetilde{t}(\theta)}\right) = \frac{t'(\theta)\,\widetilde{t}(\theta) - t(\theta)\,\widetilde{t}'(\theta)}{\widetilde{t}(\theta)^2} \leq 0. \tag{34}$$

■

We find that output scheme determines the payment schedule for any feasible contracts. The payment includes two parts, the production cost and the information rent. Information rents can be expressed as a function of the output. Therefore, the payment is a function of the output level. We can characterize the payment schedule through investigating the properties of the output scheme. In this sense the output scheme contains sufficient clues to study the inequality implications of any feasible contract.

3 Information frictions

In order to investigate impacts of information frictions on the income distribution, we present the income distribution under complete information. We compare the income distributin under incomplete information and that under complete information to find the impacts of asymmetric information on the income distribution.

3.1 Two types

With two types, the optimal contract problem is

$$\max_{\{(\underline{U},\underline{q});(\overline{U},\overline{q})\}} v\left[S\left(\underline{q}\right) - \underline{\theta}\underline{q}\right] + (1-v)\left[S\left(\overline{q}\right) - \overline{\theta}\overline{q}\right] - \left[v\underline{U} + (1-v)\overline{U}\right]$$

$$s.t.\ \underline{U} \geq \overline{U} + \Delta\theta\overline{q}, \tag{35}$$

$$\overline{U} \geq \underline{U} - \Delta\theta\underline{q}, \tag{36}$$

$$\underline{U} \geq 0, \tag{37}$$

$$\overline{U} \geq 0. \tag{38}$$

Constraints (35) and (36) are from the incentive compatibility constraints (1) and (2). Constraints (37) and (38) are the participation constraints.

Proposition 3 *Under Assumptions 1 and 2, the optimal contract under asymmetric information satisfies*

$$\begin{aligned} S'(\underline{q}^{SB}) &= \underline{\theta}, &(39)\\ S'\left(\overline{q}^{SB}\right) &= \overline{\theta} + \frac{v}{1-v}\Delta\theta, &(40) \end{aligned}$$

and the agents' incomes under the second-best contract are

$$\underline{t}^{SB} = \underline{\theta}\underline{q}^{SB} + \Delta\theta\overline{q}^{SB},$$

and

$$\overline{t}^{SB} = \overline{\theta}\overline{q}^{SB}.$$

The optimal contract under asymmetric information is influenced by the information structure and the production function. The ouput level and transfers are determined by the optimal contract. To extract rents from the agents, principals distort the output level of the inefficient agents. To implement $\underline{q}^{SB}$ and $\overline{q}^{SB}$, principals offer transfers $\underline{t}^{SB}$ and $\overline{t}^{SB}$. These transfers determine the income distribution in the economy under the second-best contract.

The contract under complete information only has to satisfy the participation constraints and does not have to obey the incentive compatibility constraints. The optimal contract problem under complete information is

$$\max_{\left\{(\overline{U},\overline{q});(\underline{U},\underline{q})\right\}} \ v\left[S\left(\underline{q}\right) - \underline{\theta}\underline{q}\right] + (1-v)\left[S\left(\overline{q}\right) - \overline{\theta}\overline{q}\right] - \left[v\underline{U} + (1-v)\overline{U}\right]$$

$$s.t.\ \underline{U} \geq 0,$$

$$\overline{U} \geq 0.$$

We call the optimal contract under complete information the first-best contract.

Lemma 1 *Under Assumptions 1 and 2, we have*

$$\underline{q}^{FB}/\overline{q}^{FB} < \underline{q}^{SB}/\overline{q}^{SB}.$$

The second-best contract incurs a less equal output distribution than the first-best contract. The $\underline{\theta}-$type agents under the second-best contract have the same output level as those under the first-best contract, $\underline{q}^{SB} = \underline{q}^{FB}$. Due to the incentive compatibility constraints, the $\overline{\theta}-$type agents under the second-best contract have the lower output level than those under the first-best contract, $\overline{q}^{SB} < \overline{q}^{FB}$. The incomplete information casues a downward output distortion for the $\overline{\theta}-$type agents. This distortion induces an efficiency loss and a less equal output distribution. Therefore, we have $\underline{q}^{FB}/\overline{q}^{FB} < \underline{q}^{SB}/\overline{q}^{SB}$.

And we have

$$\underline{t}^{FB} = \underline{\theta}\underline{q}^{FB},$$

and

$$\overline{t}^{FB} = \overline{\theta}\overline{q}^{FB}.$$

Theorem 5 *Under Assumptions 1 and 2, we have*

$$\underline{t}^{FB}/\overline{t}^{FB} < \underline{t}^{SB}/\overline{t}^{SB}.$$

Let Y^{SB} be the income distribution of the economy under the second-best contract,

$$Y^{SB} = \begin{cases} \underline{t}^{SB}, & \text{with probability } v \\ \overline{t}^{SB}, & \text{with probabiity } (1-v) \end{cases}.$$

Thus, income per capita is

$$E\left(Y^{SB}\right) = v\underline{t}^{SB} + (1-v)\overline{t}^{SB}. \tag{41}$$

Let $\hat{Y}^{SB}$ be the normalized income distribution,

$$\hat{Y}^{SB} = \frac{Y^{SB}}{E\left(Y^{SB}\right)} = \begin{cases} \underline{t}^{SB}/E\left(Y^{SB}\right), & \text{with probability } v \\ \bar{t}^{SB}/E\left(Y^{SB}\right), & \text{with probabiity } (1-v) \end{cases}.$$

Obviously, we have $E\left(\hat{Y}^{SB}\right) = 1$.

Let Y^{FB} be the income distribution of the economy with complete information,

$$Y^{FB} = \begin{cases} \underline{t}^{FB}, & \text{with probability } v \\ \bar{t}^{FB}, & \text{with probabiity } (1-v) \end{cases}.$$

Thus, income per capita is $E\left(Y^{FB}\right) = v\underline{t}^{FB} + (1-v)\bar{t}^{FB}$. Let $\hat{Y}^{FB} = \frac{Y^{FB}}{E(Y^{FB})}$. Thus, we have $E\left(\hat{Y}^{FB}\right) = 1$.

We know that

$$\begin{aligned} \underline{t}^{FB}/\bar{t}^{FB} &= \left(\underline{\theta}\underline{q}^{FB}\right)/\left(\bar{\theta}\bar{q}^{FB}\right) \\ &< \left(\underline{\theta}\underline{q}^{SB}\right)/\left(\bar{\theta}\bar{q}^{SB}\right) \\ &< \left(\underline{\theta}\underline{q}^{SB} + \underline{U}^{SB}\right)/\left(\bar{\theta}\bar{q}^{SB}\right) \\ &= \underline{t}^{SB}/\bar{t}^{SB}, \end{aligned}$$

where the first inequality is due to the output distortion and the second inequality comes from the information rent. The fact $\underline{t}^{FB}/\bar{t}^{FB} < \underline{t}^{SB}/\bar{t}^{SB}$ implies that asymmeric information causes a larger relative difference between income of the $\underline{\theta}-$type agents and that of the $\bar{\theta}-$type agents.

Theorem 6 *Under Assumptions 1 and 2, we have $W^{FB} \succeq_L W^{SB}$.*

Under complete information, agents produce the output levels that are determined by the marginal cost of production. And they receive the compensation for the production cost. They do not receive information rents. Under incomplete information, the $\underline{\theta}-$type agents produce the output levels according to the marginal cost of production, while the $\bar{\theta}-$type agents suffer from a downward output distortion. A less equal output distribution is the first channel through which information frictions cause income inequality.

In order to keep the incentive compatibility constraint, the pricipals have to provide sufficient incentives to the $\underline{\theta}-$type agents. Thus, the

$\underline{\theta}-$type agents receive information rents under incomplete information. Information rents are tranfers beyond the compensation for the production cost. The $\overline{\theta}-$type agents do not receive information rents. Thus, information rents increase income inequality. This is the second channel through which information frictions cause income inequality.

The income share of agents in the total income under complete information is γ, which is the same as that under incomplete information. However, the income distribution within agents under incomplete information is less equal than that under complete information. The output distortion due to asymmetric information cause the income inequality within agents. And the information rent exaggerates the income inequality within agents further.

3.2 Continuous types

With the continous type, using the rent variable $U(\theta) = t(\theta) - \theta q(\theta)$, the optimization problem of the principal becomes

$$\max_{\{(U(.),q(.))\}} \int_{\underline{\theta}}^{\overline{\theta}} \left[S(q(\theta)) - \theta q(\theta) - U(\theta)\right] f(\theta)\, d\theta,$$

$$subject \quad to$$

$$\dot{U}(\theta) = -q(\theta), \tag{42}$$

$$\dot{q}(\theta) \leq 0, \tag{43}$$

$$U(\theta) \geq 0. \tag{44}$$

where (42) is the incentive compatibility constraint, (43) is the implementability condition and (44) is the participation constraint.

We assume that

Assumption 3: The principal's firm has production function,

$$S(q) = \frac{1}{\gamma} q^{\gamma}, \gamma \in (0, 1).$$

Assumption 3 implies that the marginal product is infinity at $q = 0$. Thus, there is no shutdown for $\overline{\theta}-$type agents and q is always greater than 0.

Proposition 4 *Under Assumptions 1′ and 2, the optimal contract under asymmetric information satisfies*

$$S'\left(q^{SB}\left(\theta\right)\right) = \theta + \frac{F\left(\theta\right)}{f\left(\theta\right)},$$

and the second best income and rent are

$$t^{SB}\left(\theta\right) = \theta q^{SB}\left(\theta\right) + U^{SB}\left(\theta\right), \tag{45}$$

$$U^{SB}\left(\theta\right) = \int_{\theta}^{\overline{\theta}} q^{SB}\left(\tau\right) d\tau, \tag{46}$$

$$U^{SB}\left(\overline{\theta}\right) = 0. \tag{47}$$

The second-best output, rent and income are

$$\begin{aligned}
q^{SB}\left(\theta\right) &= \left(\theta + \frac{F(\theta)}{f\left(\theta\right)}\right)^{\frac{1}{\gamma-1}}, \\
U^{SB}\left(\theta\right) &= \int_{\theta}^{\overline{\theta}} q^{SB}\left(\tau\right) d\tau = \int_{\theta}^{\overline{\theta}} \left(\tau + \frac{F(\tau)}{f\left(\tau\right)}\right)^{\frac{1}{\gamma-1}} d\tau, \\
t^{SB}\left(\theta\right) &= \theta q^{SB}\left(\theta\right) + U^{SB}\left(\theta\right) = \theta q^{SB}\left(\theta\right) + \int_{\theta}^{\overline{\theta}} q^{SB}\left(\tau\right) d\tau, \\
&= \theta\left(\theta + \frac{F(\theta)}{f\left(\theta\right)}\right)^{\frac{1}{\gamma-1}} + \int_{\theta}^{\overline{\theta}} \left(\tau + \frac{F(\tau)}{f\left(\tau\right)}\right)^{\frac{1}{\gamma-1}} d\tau.
\end{aligned}$$

We have investigated the income distribution when the agents' payments are not in line with the marginal cost. In the neoclassical theory of distribution, factors receive payments according to their marginal product. Some literatures use the neoclassical theory to explain the observed income distribution. Sattinger (1975) investgates how comparative advantage connects the ability distribution and the income distribution in the Roy model. Heckman and Honoré (1990) extends the classical Roy model. Gabaix and Landier (2008) and Terviö (2008, 2009) use the sorting mechanism in assignment models to investigate the income distriubiton. However, they find that the sorting mechanism itself is not enough to generate the fat tail of the income distribution. Geerolf (2017) uses an assigment model with complementarities to generate a Pareo tail of the income distribution.

The first-best output and income are

$$q^{FB}(\theta) = \theta^{\frac{1}{\gamma-1}}, \tag{48}$$

$$t^{FB}(\theta) = \theta q^{FB}(\theta) = \theta^{\frac{\gamma}{\gamma-1}}. \tag{49}$$

Assumption 4: $\frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{\theta f(\theta)}\right) \geq 0.$

Lemma 2 *Under assumption 4, we have*

$$\frac{\partial}{\partial\theta}\left(\frac{q^{SB}(\theta)}{q^{FB}(\theta)}\right) \leq 0. \tag{50}$$

Proof:

$$\frac{q^{SB}(\theta)}{q^{FB}(\theta)} = \frac{\left(\theta + \frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}}}{\theta^{\frac{1}{\gamma-1}}} = \left(1 + \frac{F(\theta)}{\theta f(\theta)}\right)^{\frac{1}{\gamma-1}}, \tag{51}$$

under assumption 4, $\frac{q^{SB}(\theta)}{q^{FB}(\theta)}$ decreases with θ. ■

Lemma 2 is equivalent to

$$q^{FB}(\theta')/q^{FB}(\theta) \leq q^{SB}(\theta')/q^{SB}(\theta),$$

for $\underline{\theta} \leq \theta' < \theta < \overline{\theta}$.

Theorem 7 *If Assumption 4 and*

$$\frac{1}{1-\gamma}\theta\left(\theta + \frac{F(\theta)}{f(\theta)}\right)^{-1}\left(1 + \frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{f(\theta)}\right)\right) \leq \frac{\overline{\theta}}{\triangle\theta}, \tag{52}$$

hold, then we have

$$\frac{\partial}{\partial\theta}\left(\frac{t^{SB}(\theta)}{t^{FB}(\theta)}\right) \leq 0. \tag{53}$$

Proof:

$$\frac{\partial}{\partial\theta}\left(\frac{t^{SB}(\theta)}{t^{FB}(\theta)}\right) = \frac{\partial}{\partial\theta}\left(\frac{\theta q^{SB}(\theta)}{t^{FB}(\theta)}\frac{t^{SB}(\theta)}{\theta q^{SB}(\theta)}\right) \tag{54}$$

$$= \frac{\partial}{\partial\theta}\left(\frac{\theta q^{SB}(\theta)}{t^{FB}(\theta)}\right)\frac{t^{SB}(\theta)}{\theta q^{SB}(\theta)} + \frac{\theta q^{SB}(\theta)}{t^{FB}(\theta)}\frac{\partial}{\partial\theta}\left(\frac{t^{SB}(\theta)}{\theta q^{SB}(\theta)}\right) \tag{55}$$

If $\frac{\partial}{\partial\theta}\left(\frac{q^{SB}(\theta)}{q^{FB}(\theta)}\right)\leq 0$, $\frac{\partial}{\partial\theta}\left(\frac{\theta q^{SB}(\theta)}{t^{FB}(\theta)}\right)=\frac{\partial}{\partial\theta}\left(\frac{\theta q^{SB}(\theta)}{\theta q^{FB}(\theta)}\right)=\frac{\partial}{\partial\theta}\left(\frac{q^{SB}(\theta)}{q^{FB}(\theta)}\right)\leq 0$.

The elasticity of $q^{SB}(\theta)$ is

$$
\begin{aligned}
-\frac{\theta q^{SB\prime}(\theta)}{q^{SB}(\theta)} &= -\frac{\theta\frac{1}{\gamma-1}\left(\theta+\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}-1}\left(1+\frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{f(\theta)}\right)\right)}{\left(\theta+\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}}}, && (56)\\
&= \frac{1}{1-\gamma}\theta\left(\theta+\frac{F(\theta)}{f(\theta)}\right)^{-1}\left(1+\frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{f(\theta)}\right)\right). && (57)
\end{aligned}
$$

From

$$\frac{\partial}{\partial\theta}\left(\frac{q^{SB}(\theta)}{q^{FB}(\theta)}\right)=\frac{q^{SB\prime}(\theta)\,q^{FB}(\theta)-q^{FB\prime}(\theta)\,q^{SB}(\theta)}{q^{FB}(\theta)^{2}}\leq 0, \qquad (58)$$

it implies

$$
\begin{aligned}
& q^{SB\prime}(\theta)\,q^{FB}(\theta)-q^{FB\prime}(\theta)\,q^{SB}(\theta) && (59)\\
= & \frac{1}{\gamma-1}\left(\theta+\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}-1}\left(1+\frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{f(\theta)}\right)\right)\theta^{\frac{1}{\gamma-1}} && (60)\\
& -\frac{1}{\gamma-1}\theta^{\frac{1}{\gamma-1}-1}\left(\theta+\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}} && (61)\\
\leq & \ 0, && (62)
\end{aligned}
$$

or

$$\left(\theta+\frac{F(\theta)}{f(\theta)}\right)^{-1}\left(1+\frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{f(\theta)}\right)\right)-\theta^{-1}\geq 0, \qquad (63)$$

or

$$\theta\left(\theta+\frac{F(\theta)}{f(\theta)}\right)^{-1}\left(1+\frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{f(\theta)}\right)\right)\geq 1. \qquad (64)$$

Then

$$\frac{1}{1-\gamma}\theta\left(\theta+\frac{F(\theta)}{f(\theta)}\right)^{-1}\left(1+\frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{f(\theta)}\right)\right)>1. \qquad (65)$$

Thus, the elasticity of $q^{SB}(\theta)$

$$-\frac{\theta q^{SB\prime}(\theta)}{q^{SB}(\theta)}>1. \qquad (66)$$

Therefore, the first condition of Theorem 2 is satisfied

$$q^{SB}(\theta) + \theta q^{SB\prime}(\theta) < 0. \tag{67}$$

Thus, the numerator of $\frac{\partial}{\partial\theta}\left(\frac{t^{SB}(\theta)}{\theta q^{SB}(\theta)}\right)$ is negative if the second condition of Theorem 2 is satisfied; that is, if

$$\frac{1}{1-\gamma}\theta\left(\theta + \frac{F(\theta)}{f(\theta)}\right)^{-1}\left(1 + \frac{\partial}{\partial\theta}\left(\frac{F(\theta)}{f(\theta)}\right)\right) \leq \frac{\overline{\theta}}{\triangle\theta}. \tag{68}$$

■

Theorem 6 is equivalent to

$$t^{FB}(\theta')/t^{FB}(\theta) < t^{SB}(\theta')/t^{SB}(\theta),$$

for $\underline{\theta} \leq \theta' < \theta < \overline{\theta}$.

Next, we examine the example of continous type following uniform distribution on $\Theta = [\underline{\theta}, \overline{\theta}]$ and $\Theta = [0, \overline{\theta}]$.

Corollary 8 *For continous type following uniform distribution on* $\Theta = [\underline{\theta}, \overline{\theta}]$, $\frac{\partial}{\partial\theta}\left(\frac{q^{SB}(\theta)}{q^{FB}(\theta)}\right) \leq 0$. *For* $\Theta = [0, \overline{\theta}]$, $\frac{\partial}{\partial\theta}\left(\frac{t^{SB}(\theta)}{t^{FB}(\theta)}\right) \leq 0$; *for* $\Theta = [\underline{\theta}, \overline{\theta}]$, *a sufficient condition for* $\frac{\partial}{\partial\theta}\left(\frac{t^{SB}(\theta)}{t^{FB}(\theta)}\right) \leq 0$ *is* $\underline{\theta} \geq \frac{\gamma+1}{2}\overline{\theta}$.

Proof: For continous type following uniform distribution on $\Theta = [\underline{\theta}, \overline{\theta}]$, $\frac{F(\theta)}{f(\theta)} = \theta - \underline{\theta}$. Thus,

$$\begin{aligned}
q^{FB}(\theta) &= \theta^{\frac{1}{\gamma-1}}, && (69)\\
t^{FB}(\theta) &= \theta^{\frac{\gamma}{\gamma-1}}, && (70)\\
q^{SB}(\theta) &= (2\theta - \underline{\theta})^{\frac{1}{\gamma-1}}, && (71)\\
\int_{\theta}^{\overline{\theta}} q^{\alpha}(\tau)\, d\tau &= \frac{1}{2}\frac{\gamma-1}{\gamma}\left[\left(2\overline{\theta} - \underline{\theta}\right)^{\frac{\gamma}{\gamma-1}} - (2\theta - \underline{\theta})^{\frac{\gamma}{\gamma-1}}\right], && (72)\\
t^{SB}(\theta) &= \theta(2\theta - \underline{\theta})^{\frac{1}{\gamma-1}} + \frac{1}{2}\frac{\gamma-1}{\gamma}\left[\left(2\overline{\theta} - \underline{\theta}\right)^{\frac{\gamma}{\gamma-1}} - (2\theta - \underline{\theta})^{\frac{\gamma}{\gamma-1}}\right] && (73)
\end{aligned}$$

Condition (52) reduces to

$$2\theta^2 - 2\gamma\theta\overline{\theta} - (1-\gamma)\underline{\theta}\overline{\theta} \geq 0, \tag{74}$$

which requires (a sufficient condition)

$$\underline{\theta} \geq \frac{\gamma\overline{\theta} + \sqrt{\gamma^2\overline{\theta}^2 + 2(1-\gamma)\underline{\theta}\overline{\theta}}}{2}; \tag{75}$$

that is,

$$\underline{\theta} \geq \frac{\gamma+1}{2}\overline{\theta}. \tag{76}$$

For $\Theta = [0, \overline{\theta}]$,

$$\begin{aligned}
\frac{t^{SB}(\theta)}{t^{FB}(\theta)} &= \frac{\theta(2\theta)^{\frac{1}{\gamma-1}} + \frac{1}{2}\frac{\gamma-1}{\gamma}\left[\left(2\overline{\theta}\right)^{\frac{\gamma}{\gamma-1}} - (2\theta)^{\frac{\gamma}{\gamma-1}}\right]}{\theta^{\frac{\gamma}{\gamma-1}}}, &(77)\\
&= \frac{2^{\frac{1}{\gamma-1}}\theta^{\frac{\gamma}{\gamma-1}} + \frac{1}{2}\frac{\gamma-1}{\gamma}2^{\frac{\gamma}{\gamma-1}}\left(\overline{\theta}^{\frac{\gamma}{\gamma-1}} - \theta^{\frac{\gamma}{\gamma-1}}\right)}{\theta^{\frac{\gamma}{\gamma-1}}}, &(78)\\
&= 2^{\frac{1}{\gamma-1}} - \frac{1}{2}\frac{\gamma-1}{\gamma}2^{\frac{\gamma}{\gamma-1}} + \frac{1}{2}\frac{\gamma-1}{\gamma}2^{\frac{\gamma}{\gamma-1}}\left(\frac{\overline{\theta}}{\theta}\right)^{\frac{\gamma}{\gamma-1}}, &(79)
\end{aligned}$$

which decreases with θ. ■

4 The social norms

Piketty and Saez (2003) propose that changing social norms regarding inequality plays an important role in raising inequality in the United Stattes since the 1970s. We then introduce a parameter of social norms regarding inequality into our benchmark model. The principal maximizes a weighted average of its surplus and of the agent's rent U with $\alpha \in [0,1]$ for the agent's rent. For two-type case, the outputs are given by

$$\begin{aligned}
\underline{q}^{\alpha} &= \underline{q}^{*},\\
S'\left(\overline{q}^{\alpha}\right) &= \overline{\theta} + \frac{v}{1-v}(1-\alpha)\triangle\theta.
\end{aligned}$$

If $\alpha_2 < \alpha_1$, $\overline{q}^{\alpha_2} < \overline{q}^{\alpha_1}$, thus $\underline{q}^{\alpha_2}/\overline{q}^{\alpha_2} > \underline{q}^{\alpha_1}/\overline{q}^{\alpha_1}$. By theorem 3, $\underline{t}^{\alpha_2}/\overline{t}^{\alpha_2} > \underline{t}^{\alpha_1}/\overline{t}^{\alpha_1}$.

For continous type, the principal's program writes now as follows.

$$
\begin{aligned}
\max_{\{(U(.),q(.))\}} \quad & \int_{\underline{\theta}}^{\overline{\theta}} \left[S\left(q\left(\theta\right)\right) - \theta q\left(\theta\right) - (1-\alpha)U\left(\theta\right)\right] f\left(\theta\right) d\theta, \\
& subject \quad to \\
& \dot{U}\left(\theta\right) = -q\left(\theta\right), & (80) \\
& \dot{q}\left(\theta\right) \leq 0, & (81) \\
& U\left(\theta\right) \geq 0. & (82)
\end{aligned}
$$

The output is given by

$$
S'\left(q\left(\theta\right)\right) = \theta + (1-\alpha)\frac{F\left(\theta\right)}{f\left(\theta\right)}.
$$

Lemma 3 *Under assumption 3 and 4, for* $\alpha_2 < \alpha_1$,

$$
\frac{\partial}{\partial\theta}\left(\frac{q^{\alpha_2}\left(\theta\right)}{q^{\alpha_1}\left(\theta\right)}\right) \leq 0. \tag{83}
$$

Proof:

$$
\begin{aligned}
\frac{q^{\alpha_2}\left(\theta\right)}{q^{\alpha_1}\left(\theta\right)} &= \frac{\left(\theta + (1-\alpha_2)\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}}}{\left(\theta + (1-\alpha_1)\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}}} = \left(\frac{\theta + (1-\alpha_2)\frac{F(\theta)}{f(\theta)}}{\theta + (1-\alpha_1)\frac{F(\theta)}{f(\theta)}}\right)^{\frac{1}{\gamma-1}} & (84) \\
&= \left(\frac{1 + (1-\alpha_2)\frac{F(\theta)}{\theta f(\theta)}}{1 + (1-\alpha_1)\frac{F(\theta)}{\theta f(\theta)}}\right)^{\frac{1}{\gamma-1}}. & (85)
\end{aligned}
$$

Denote $G\left(\theta\right) \equiv \frac{F(\theta)}{\theta f(\theta)}$, by assumption 4,

$$
\begin{aligned}
& \frac{\partial}{\partial\theta}\left(\frac{1 + (1-\alpha_2)G\left(\theta\right)}{1 + (1-\alpha_1)G\left(\theta\right)}\right) & (86) \\
= & \frac{1}{\left[1 + (1-\alpha_1)G\left(\theta\right)\right]^2}\left[\left(\alpha_1 - \alpha_2\right) G'\left(\theta\right)\right] \geq 0, & (87)
\end{aligned}
$$

thus

$$
\frac{\partial}{\partial\theta}\left(\frac{q^{\alpha_2}\left(\theta\right)}{q^{\alpha_1}\left(\theta\right)}\right) \leq 0. \tag{88}
$$

■

Since

$$
\begin{aligned}
q^{\alpha}(\theta) &= \left(\theta + (1-\alpha)\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}}, &(89)\\
q^{\alpha\prime}(\theta) &= \frac{1}{\gamma-1}\left(\theta + (1-\alpha)\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}-1}\left(1 + (1-\alpha)\frac{\partial}{\partial\theta}\frac{F(\theta)}{f(\theta)}\right), &(90)
\end{aligned}
$$

we have the following theorem.

Theorem 9 *For the continous-type case, for $\alpha_2 < \alpha_1$, $\frac{\partial}{\partial\theta}\left(\frac{t^{\alpha_2}(\theta)}{t^{\alpha_1}(\theta)}\right) \leq 0$ if and only if*

$$
\frac{\partial}{\partial\alpha}\left(\frac{\theta\left(\theta + (1-\alpha)\frac{F(\theta)}{f(\theta)}\right)^{\frac{1}{\gamma-1}} + \int_{\underline{\theta}}^{\overline{\theta}}\left(\theta + (1-\alpha)\frac{F(\tau)}{f(\tau)}\right)^{\frac{1}{\gamma-1}}d\tau}{\frac{1}{\gamma-1}\left(\theta + (1-\alpha)\frac{F(\theta)}{f(\theta)}\right)^{\frac{2-\gamma}{\gamma-1}}\left(1 + (1-\alpha)\frac{\partial}{\partial\theta}\frac{F(\theta)}{f(\theta)}\right)}\right) \leq 0. \quad (91)
$$

Proof: This is given by the necessary and sufficient condition in Theorem 4:

$$
\frac{\theta q^{\alpha_1}(\theta) + \int_{\underline{\theta}}^{\overline{\theta}} q^{\alpha_1}(\tau)\,d\tau}{q^{\alpha_1\prime}(\theta)} \leq \frac{\theta q^{\alpha_2}(\theta) + \int_{\underline{\theta}}^{\overline{\theta}} q^{\alpha_2}(\tau)\,d\tau}{q^{\alpha_2\prime}(\theta)}, \quad (92)
$$

or

$$
\frac{\partial}{\partial\alpha}\left(\frac{\theta q^{\alpha}(\theta) + \int_{\underline{\theta}}^{\overline{\theta}} q^{\alpha}(\tau)\,d\tau}{q^{\alpha\prime}(\theta)}\right) \leq 0. \quad (93)
$$

■

We examine uniform distribution on $\Theta = [\underline{\theta}, \overline{\theta}]$ as an example. Under this example,

$$
\begin{aligned}
\frac{F(\theta)}{f(\theta)} &= \theta - \underline{\theta},\\
q^{\alpha}(\theta) &= \left[\theta + (1-\alpha)(\theta - \underline{\theta})\right]^{\frac{1}{\gamma-1}} = \left[(2-\alpha)\theta - (1-\alpha)\underline{\theta}\right]^{\frac{1}{\gamma-1}},\\
q^{\alpha\prime}(\theta) &= \frac{1}{\gamma-1}\left[(2-\alpha)\theta - (1-\alpha)\underline{\theta}\right]^{\frac{1}{\gamma-1}-1}(2-\alpha),\\
\int_{\underline{\theta}}^{\overline{\theta}} q^{\alpha}(\tau)\,d\tau &= \frac{1}{2-\alpha}\frac{\gamma-1}{\gamma}\left\{\left[(2-\alpha)\overline{\theta} - (1-\alpha)\underline{\theta}\right]^{\frac{\gamma}{\gamma-1}} - \left[(2-\alpha)\theta - (1-\alpha)\underline{\theta}\right]^{\frac{\gamma}{\gamma-1}}\right\}.
\end{aligned}
$$

The output ratio

$$
\frac{q^{\alpha_2}(\theta)}{q^{\alpha_1}(\theta)} = \frac{\left[\theta + (1-\alpha_2)(\theta - \underline{\theta})\right]^{\frac{1}{\gamma-1}}}{\left[\theta + (1-\alpha_1)(\theta - \underline{\theta})\right]^{\frac{1}{\gamma-1}}} = \left(\frac{(2-\alpha_2)\theta - (1-\alpha_2)\underline{\theta}}{(2-\alpha_1)\theta - (1-\alpha_1)\underline{\theta}}\right)^{\frac{1}{\gamma-1}}.
$$

$$
\begin{aligned}
& \frac{\partial}{\partial\theta}\left(\frac{(2-\alpha_2)\theta-(1-\alpha_2)\underline{\theta}}{(2-\alpha_1)\theta-(1-\alpha_1)\underline{\theta}}\right) \\
= & \frac{1}{\left[(2-\alpha_1)\theta-(1-\alpha_1)\underline{\theta}\right]^2}\left\{\left[(2-\alpha_1)(1-\alpha_2)-(2-\alpha_2)(1-\alpha_1)\right]\underline{\theta}\right\}, \\
= & \frac{1}{\left[(2-\alpha_1)\theta-(1-\alpha_1)\underline{\theta}\right]^2}\left[\left(\alpha_1-\alpha_2\right)\underline{\theta}\right] \geq 0,
\end{aligned}
$$

with equality when $\underline{\theta}=0$. Thus,

$$
\frac{\partial}{\partial\theta}\left(\frac{q^{\alpha_2}(\theta)}{q^{\alpha_1}(\theta)}\right) \leq 0, \tag{94}
$$

with equality when $\underline{\theta}=0$.

Then, denote $a \equiv \frac{1-\alpha}{2-\alpha} \geq 0$, $\frac{\partial a}{\partial\alpha} < 0$.

$$
\begin{aligned}
& \frac{\theta q^{\alpha}(\theta)+\int_{\theta}^{\overline{\theta}} q^{\alpha}(\tau)\,d\tau}{q^{\alpha\prime}(\theta)} \\
= & \frac{\theta\left[(2-\alpha)\theta-(1-\alpha)\underline{\theta}\right]^{\frac{1}{\gamma-1}}}{\frac{1}{\gamma-1}\left[(2-\alpha)\theta-(1-\alpha)\underline{\theta}\right]^{\frac{1}{\gamma-1}-1}(2-\alpha)} \\
& +\frac{\frac{1}{2-\alpha}\frac{\gamma-1}{\gamma}\left\{\left[(2-\alpha)\overline{\theta}-(1-\alpha)\underline{\theta}\right]^{\frac{\gamma}{\gamma-1}}-\left[(2-\alpha)\theta-(1-\alpha)\underline{\theta}\right]^{\frac{\gamma}{\gamma-1}}\right\}}{\frac{1}{\gamma-1}\left[(2-\alpha)\theta-(1-\alpha)\underline{\theta}\right]^{\frac{1}{\gamma-1}-1}(2-\alpha)} \\
= & \frac{\theta\left[(2-\alpha)\theta-(1-\alpha)\underline{\theta}\right]}{\frac{1}{\gamma-1}(2-\alpha)} \\
& +\frac{\frac{1}{2-\alpha}\frac{\gamma-1}{\gamma}}{\frac{1}{\gamma-1}(2-\alpha)}\left\{\frac{\left[(2-\alpha)\overline{\theta}-(1-\alpha)\underline{\theta}\right]^{\frac{\gamma}{\gamma-1}}}{\left[(2-\alpha)\theta-(1-\alpha)\underline{\theta}\right]^{\frac{1}{\gamma-1}-1}}-\left[(2-\alpha)\theta-(1-\alpha)\underline{\theta}\right]^2\right\}, \\
= & (\gamma-1)\left\{\theta\left(\theta-a\underline{\theta}\right)+\frac{1}{(2-\alpha)^2}\frac{\gamma-1}{\gamma}\left[\frac{(2-\alpha)^{\frac{\gamma}{\gamma-1}}\left(\overline{\theta}-a\underline{\theta}\right)^{\frac{\gamma}{\gamma-1}}}{(2-\alpha)^{\frac{1}{\gamma-1}-1}\left(\theta-a\underline{\theta}\right)^{\frac{1}{\gamma-1}-1}}-(2-\alpha)^2\left(\theta-a\underline{\theta}\right)^2\right]\right\} \\
= & (\gamma-1)\left\{\theta\left(\theta-a\underline{\theta}\right)+\frac{\gamma-1}{\gamma}\left[\frac{\left(\overline{\theta}-a\underline{\theta}\right)^{\frac{\gamma}{\gamma-1}}}{\left(\theta-a\underline{\theta}\right)^{\frac{1}{\gamma-1}-1}}-\left(\theta-a\underline{\theta}\right)^2\right]\right\}, \\
= & (\gamma-1)\,\theta\left(\theta-a\underline{\theta}\right)+\frac{(\gamma-1)^2}{\gamma}\left[\frac{\left(\overline{\theta}-a\underline{\theta}\right)^{\frac{\gamma}{\gamma-1}}-\left(\theta-a\underline{\theta}\right)^{\frac{\gamma}{\gamma-1}}}{\left(\theta-a\underline{\theta}\right)^{\frac{1}{\gamma-1}-1}}\right].
\end{aligned} \tag{95}
$$

Denote

$$\begin{aligned}
\delta &\equiv \frac{\partial}{\partial a}\left[\frac{\left(\overline{\theta}-a\underline{\theta}\right)^{\frac{\gamma}{\gamma-1}}-\left(\theta-a\underline{\theta}\right)^{\frac{\gamma}{\gamma-1}}}{\left(\theta-a\underline{\theta}\right)^{\frac{1}{\gamma-1}-1}}\right] \\
&= \frac{1}{\left[\left(\theta-a\underline{\theta}\right)^{\frac{1}{\gamma-1}-1}\right]^2}\frac{\underline{\theta}}{1-\gamma}\left(\theta-a\underline{\theta}\right)^{\frac{\gamma}{\gamma-1}+\frac{1}{\gamma-1}-2} \\
&\left\{\left(\frac{\overline{\theta}-a\underline{\theta}}{\theta-a\underline{\theta}}\right)^{\frac{\gamma}{\gamma-1}-1}\gamma-\left(\frac{\overline{\theta}-a\underline{\theta}}{\theta-a\underline{\theta}}\right)^{\frac{\gamma}{\gamma-1}}(2-\gamma)+2-2\gamma\right\}.
\end{aligned}$$

Denote $\eta\equiv\left\{\left(\frac{\overline{\theta}-a\underline{\theta}}{\theta-a\underline{\theta}}\right)^{\frac{\gamma}{\gamma-1}-1}\gamma-\left(\frac{\overline{\theta}-a\underline{\theta}}{\theta-a\underline{\theta}}\right)^{\frac{\gamma}{\gamma-1}}(2-\gamma)+2-2\gamma\right\}$.

Since $\overline{\theta}\in[\theta,+\infty)$, $\forall\theta$, when $\overline{\theta}=\theta$, $\eta=2\gamma-2+2-2\gamma=0$, and

$$\frac{\partial\eta}{\partial\overline{\theta}}=\left(\frac{\overline{\theta}-a\underline{\theta}}{\theta-a\underline{\theta}}\right)^{\frac{\gamma}{\gamma-1}-2}\frac{1}{\theta-a\underline{\theta}}\frac{\gamma}{\gamma-1}\left[1-(2-\gamma)\frac{\overline{\theta}-a\underline{\theta}}{\theta-a\underline{\theta}}\right]>0,$$

thus,

$$\eta\geq 0. \tag{96}$$

Moreover,

$$\delta|_{\underline{\theta}=0}=0. \tag{97}$$

Hence,

$$\delta\geq 0. \tag{98}$$

Moreover,

$$\frac{\partial\left[(\gamma-1)\,\theta\left(\theta-a\underline{\theta}\right)\right]}{\partial a}=\frac{\partial\left[(\gamma-1)\,\theta^2-(\gamma-1)\,\theta\underline{\theta}a\right]}{\partial a}=(1-\gamma)\,\theta\underline{\theta}\geq 0, \tag{99}$$

with equality when $\underline{\theta}=0$. Thus,

$$\frac{\partial}{\partial a}\left(\frac{\theta q^{\alpha}(\theta)+\int_{\theta}^{\overline{\theta}}q^{\alpha}(\tau)\,d\tau}{q^{\alpha\prime}(\theta)}\right)\geq 0. \tag{100}$$

With $\frac{\partial a}{\partial\alpha}<0$, we obtain

$$\frac{\partial}{\partial\alpha}\left(\frac{\theta q^{\alpha}(\theta)+\int_{\theta}^{\overline{\theta}}q^{\alpha}(\tau)\,d\tau}{q^{\alpha\prime}(\theta)}\right)\leq 0, \tag{101}$$

with equality when $\underline{\theta} = 0$. ■

Furthermore, we compare the case of $\alpha = 1$ with first best.

Corollary 10 *For two type* $\{\underline{\theta}, \overline{\theta}\}$*,* $\underline{t}^{\alpha=1} / \overline{t}^{\alpha=1} > \underline{t}^{FB} / \overline{t}^{FB}$*; for continous case,* $\frac{\partial}{\partial\theta}\left(\frac{t^{\alpha=1}(\theta)}{t^{FB}(\theta)}\right) \leq 0$.

Proof: For two type $\{\underline{\theta}, \overline{\theta}\}$,

$$\underline{q}^{\alpha=1} = \underline{q}^*, \tag{102}$$
$$\overline{q}^{\alpha=1} = \overline{q}^*, \tag{103}$$
$$\underline{t}^{\alpha=1} = \underline{\theta}\underline{q}^* + \triangle\theta\overline{q}^*, \tag{104}$$
$$\overline{t}^{\alpha=1} = \overline{\theta}\overline{q}^*. \tag{105}$$

$$\underline{t}^{FB} / \overline{t}^{FB} = \left(\underline{\theta} / \overline{\theta}\right)^{\frac{\gamma}{\gamma-1}}, \tag{106}$$
$$\underline{t}^{\alpha=1} / \overline{t}^{\alpha=1} = \frac{\underline{\theta}\underline{q}^* + \triangle\theta\overline{q}^*}{\overline{\theta}\overline{q}^*} = \left(\underline{\theta} / \overline{\theta}\right)^{\frac{\gamma}{\gamma-1}} + \triangle\theta / \overline{\theta} > \underline{t}^{FB} / \overline{t}^{FB}. \tag{107}$$

For continous type on $\Theta = [\underline{\theta}, \overline{\theta}]$,

$$q^{FB}(\theta) = \theta^{\frac{1}{\gamma-1}}, \tag{108}$$
$$t^{FB}(\theta) = \theta^{\frac{\gamma}{\gamma-1}}, \tag{109}$$
$$q^{\alpha=1}(\theta) = \theta^{\frac{1}{\gamma-1}}, \tag{110}$$
$$t^{\alpha=1}(\theta) = \theta^{\frac{\gamma}{\gamma-1}} + \frac{\gamma-1}{\gamma}\left(\overline{\theta}^{\frac{\gamma}{\gamma-1}} - \theta^{\frac{\gamma}{\gamma-1}}\right). \tag{111}$$

Therefore,

$$\frac{t^{\alpha=1}(\theta)}{t^{FB}(\theta)} = \frac{\theta^{\frac{\gamma}{\gamma-1}} + \frac{\gamma-1}{\gamma}\left(\overline{\theta}^{\frac{\gamma}{\gamma-1}} - \theta^{\frac{\gamma}{\gamma-1}}\right)}{\theta^{\frac{\gamma}{\gamma-1}}}, \tag{112}$$
$$= 1 + \frac{\gamma-1}{\gamma}\frac{\overline{\theta}^{\frac{\gamma}{\gamma-1}} - \theta^{\frac{\gamma}{\gamma-1}}}{\theta^{\frac{\gamma}{\gamma-1}}}, \tag{113}$$
$$= 1 + \frac{\gamma-1}{\gamma}\left(\frac{\overline{\theta}}{\theta}\right)^{\frac{\gamma}{\gamma-1}} - \frac{\gamma-1}{\gamma}, \tag{114}$$

which decreases with θ. ■

5 Change of information structure

5.1 More favorable distribution of types

Suppose that $0 < v < v_1 < 1$. We have

$$\Theta_1 = \begin{cases} \underline{\theta}, & \text{with probability } v_1 \\ \overline{\theta}, & \text{with probabiity } 1 - v_1 \end{cases} .$$

From Theorem **??** we have

$$\underline{q}_1^{SB} = (\underline{\theta})^{\frac{1}{\gamma-1}},$$

and

$$\overline{q}_1^{SB} = \left(\overline{\theta} + \frac{v_1}{1 - v_1}\Delta\theta\right)^{\frac{1}{\gamma-1}}.$$

Lemma 4 *Under Assumptions 1 and 2, if* $0 < v < v_1 < 1$*, we have*

$$\underline{q}_1^{SB}/\overline{q}_1^{SB} > \underline{q}^{SB}/\overline{q}^{SB}.$$

Since the output level of the efficient type is not affected by the probability v, Lemma 4 implies that the difference of the output levels between two types is larger when the distribution is more favorable. This result also holds for continuous-type distributions. If we use the first-best contract under complete information as a benchmark, which does not depends on the type distribution, we find that the output distortion relative to the first-best contract is larger when the distribution is more favorable.

By theorem 3, we obtain the following theorem.

Theorem 11 *Under Assumptions 1 and 2, if* $0 < v < v_1 < 1$*, we have*

$$\underline{t}_1^{SB}/\overline{t}_1^{SB} > \underline{t}^{SB}/\overline{t}^{SB}.$$

Even though income levels of both types change simultaneously, Theorem 11 implies that the wage profile is steeper when the distribution is more favorable.

For continuous-type distributions, according to the definition in Laffont and Tirole (1993, chap.1), the distribution on G on $\Theta = [\underline{\theta}, \overline{\theta}]$ is more favorable than the distribution F on the same interval if $G(\theta) \geq F(\theta)$ for all θ and $\frac{g(\theta)}{G(\theta)} \leq \frac{f(\theta)}{F(\theta)}$ for all θ. They are satisfied by any two cumulative distribution function G and F such that $G = M(F)$, where M is increasing and concave, for example, $G(\theta) = F(\theta)^{\beta}$ where $\beta \in (0, 1]$.

Assume $F(\theta) \sim U(\underline{\theta}, \overline{\theta})$, then,

$$\frac{F(\theta)}{f(\theta)} = \theta - \underline{\theta}, \tag{115}$$

$$\frac{G(\theta)}{g(\theta)} = \frac{1}{\beta}(\theta - \underline{\theta}). \tag{116}$$

Lemma 5 *Under assumption 3 , for $\beta_2 < \beta_1$,*

$$\frac{\partial}{\partial\theta}\left(\frac{q^{\beta_2}(\theta)}{q^{\beta_1}(\theta)}\right) \leq 0. \tag{117}$$

Proof:

$$\frac{q^{\beta_2}(\theta)}{q^{\beta_1}(\theta)} = \frac{\left(\theta + \frac{1}{\beta_2}(\theta - \underline{\theta})\right)^{\frac{1}{\gamma-1}}}{\left(\theta + \frac{1}{\beta_1}(\theta - \underline{\theta})\right)^{\frac{1}{\gamma-1}}} = \left(\frac{\left(1 + \frac{1}{\beta_2}\right)\theta - \frac{1}{\beta_2}\underline{\theta}}{\left(1 + \frac{1}{\beta_1}\right)\theta - \frac{1}{\beta_1}\underline{\theta}}\right)^{\frac{1}{\gamma-1}}. \tag{118}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\left(1 + \frac{1}{\beta_2}\right)\theta - \frac{1}{\beta_2}\underline{\theta}}{\left(1 + \frac{1}{\beta_1}\right)\theta - \frac{1}{\beta_1}\underline{\theta}}\right) = \frac{1}{\left(\left(1 + \frac{1}{\beta_1}\right)\theta - \frac{1}{\beta_1}\underline{\theta}\right)^2}\left(\frac{\beta_1 - \beta_2}{\beta_2\beta_1}\underline{\theta}\right) \geq 0, \tag{119}$$

with equality when $\underline{\theta} = 0$. Thus,

$$\frac{\partial}{\partial\theta}\left(\frac{q^{\beta_2}(\theta)}{q^{\beta_1}(\theta)}\right) \leq 0. \tag{120}$$

■

Then, we prove that

$$\frac{\partial}{\partial\beta}\left(\frac{\theta q^{\beta}(\theta) + \int_{\theta}^{\overline{\theta}} q^{\beta}(\tau)\, d\tau}{q^{\beta\prime}(\theta)}\right) \leq 0. \tag{121}$$

$$q^{\beta}(\theta) = \left[\left(1+\frac{1}{\beta}\right)\theta - \frac{1}{\beta}\underline{\theta}\right]^{\frac{1}{\gamma-1}}, \tag{122}$$

$$q^{\beta\prime}(\theta) = \frac{1}{\gamma-1}\left[\left(1+\frac{1}{\beta}\right)\theta - \frac{1}{\beta}\underline{\theta}\right]^{\frac{1}{\gamma-1}-1}\left(1+\frac{1}{\beta}\right), \tag{123}$$

$$\int_{\theta}^{\overline{\theta}} q^{\beta}(\tau)\,d\tau = \frac{1}{\left(1+\frac{1}{\beta}\right)}\frac{\gamma-1}{\gamma}\left\{\left[\left(1+\frac{1}{\beta}\right)\overline{\theta} - \frac{1}{\beta}\underline{\theta}\right]^{\frac{\gamma}{\gamma-1}} - \left[\left(1+\frac{1}{\beta}\right)\theta - \frac{1}{\beta}\underline{\theta}\right]^{\frac{\gamma}{\gamma-1}}\right\} \tag{124}$$

Thus, denote $b \equiv \frac{1/\beta}{1+1/\beta} \in (0,1]$, $\frac{\partial b}{\partial \beta} < 0$, we obtain

$$\frac{\theta q^{\beta}(\theta) + \int_{\theta}^{\overline{\theta}} q^{\beta}(\tau)\,d\tau}{q^{\beta\prime}(\theta)} \tag{125}$$

$$= (\gamma-1)\,\theta\,(\theta - b\underline{\theta}) + \frac{(\gamma-1)^2}{\gamma}\left[\frac{(\overline{\theta} - b\underline{\theta})^{\frac{\gamma}{\gamma-1}}}{(\theta - b\underline{\theta})^{\frac{1}{\gamma-1}-1}} - (\theta - b\underline{\theta})^2\right], \tag{126}$$

which is the same with equation (95) with a replaced by b. Thus, with the same logic, we obtain

$$\frac{\partial}{\partial \beta}\left(\frac{\theta q^{\beta}(\theta) + \int_{\theta}^{\overline{\theta}} q^{\beta}(\tau)\,d\tau}{q^{\beta\prime}(\theta)}\right) \leq 0. \tag{127}$$

with equality when $\underline{\theta} = 0$. ■

Thus, for continuous-type distributions, we show that $t_1(\theta')/t_1(\theta) > t(\theta')/t(\theta)$ for $\theta' < \theta$, when Θ_1 is more favorable than Θ.

The marginal cost of production does not change when the distribution is more favorable. The difference is the probability distribution of types. The change of the type distribution causes the principal to revise the contract and the wage offer accordingly. The principal offers a contract which displays a less equal wage profile. As in Costinot and Vogel (2010), the change of the wage profile refelct the changes in the return to skill. When the distribution is more favorable to the principal, the relative income difference between the efficient type agents and the inefficient type agents becomes larger.

6 Skill diversity

We investigate the imparcts of skill diversity on the contract. We show that a mean-preserving spread of the type distribution increases income inequality under the optimal contract. We prove it for continous type with uniform distribution.

The marginal cost $\theta \in \Theta = [a, b]$ is distributed according to the density function

$$f(\theta) = \phi \theta^{\beta},$$

and cumulative distribution function

$$F(\theta) = \int_a^{\theta} \phi \theta^{\beta} d\theta = \frac{\phi}{\beta + 1} \left(\theta^{\beta+1} - a^{\beta+1} \right),$$

where $\beta \geq 0$, $b > a > 0$ and $\phi = \frac{\beta+1}{b^{\beta+1} - a^{\beta+1}}$.

When $\beta = 0$, we have uniform distribution, and the second best outcomes are

$$\begin{aligned}
q^{SB}(\theta) &= 2^{\frac{1}{\gamma-1}} \left(\theta - \frac{a}{2} \right)^{\frac{1}{\gamma-1}}, \\
U^{SB}(\theta) &= 2^{\frac{1}{\gamma-1}} \int_{\theta}^{b} \left(\tau - \frac{a}{2} \right)^{\frac{1}{\gamma-1}} d\tau, \\
t^{SB}(\theta) &= 2^{\frac{1}{\gamma-1}} \int_{\theta}^{b} \left(\tau - \frac{a}{2} \right)^{\frac{1}{\gamma-1}} d\tau + \theta \cdot 2^{\frac{1}{\gamma-1}} \left(\theta - \frac{a}{2} \right)^{\frac{1}{\gamma-1}}.
\end{aligned}$$

By the change of variable,

$$\begin{aligned}
U^{SB}(\theta) &= 2^{\frac{1}{\gamma-1}} \int_{\theta}^{b} \left(\tau - \frac{a}{2} \right)^{\frac{1}{\gamma-1}} d\tau = 2^{\frac{1}{\gamma-1}} \int_{\theta-\frac{a}{2}}^{b-\frac{a}{2}} t^{\frac{1}{\gamma-1}} dt, \\
&= 2^{\frac{1}{\gamma-1}} \frac{\gamma-1}{\gamma} \left[t^{\frac{\gamma}{\gamma-1}} \right]_{\theta-\frac{a}{2}}^{b-\frac{a}{2}}, \\
&= 2^{\frac{1}{\gamma-1}} \frac{\gamma-1}{\gamma} \left[\left(b - \frac{a}{2} \right)^{\frac{\gamma}{\gamma-1}} - \left(\theta - \frac{a}{2} \right)^{\frac{\gamma}{\gamma-1}} \right].
\end{aligned}$$

Hence

$$\begin{aligned}
q^{SB}(\theta) &= 2^{\frac{1}{\gamma-1}}\left(\theta-\frac{a}{2}\right)^{\frac{1}{\gamma-1}}, \\
U^{SB}(\theta) &= 2^{\frac{1}{\gamma-1}}\frac{\gamma-1}{\gamma}\left[\left(b-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(\theta-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}\right], \\
t^{SB}(\theta) &= 2^{\frac{1}{\gamma-1}}\frac{\gamma-1}{\gamma}\left[\left(b-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(\theta-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]+\theta\cdot 2^{\frac{1}{\gamma-1}}\left(\theta-\frac{a}{2}\right)^{\frac{1}{\gamma-1}}.
\end{aligned}$$

After mean preserving spread, $\widetilde{\theta}\in\widetilde{\Theta}=[e,d]=[a-c,b+c]$. With uniform distribution, we have

$$\widetilde{\theta}=h+p\theta.$$

Thus

$$\begin{aligned}
h+pa &= e, \\
h+pb &= d.
\end{aligned}$$

Hence,

$$\begin{aligned}
p &= \frac{d-e}{b-a}, \\
h &= e-\frac{d-e}{b-a}a.
\end{aligned}$$

Thus,

$$\begin{aligned}
\widetilde{\theta} &= e-\frac{d-e}{b-a}a+\frac{d-e}{b-a}\theta, \\
&= a-c-\frac{b+c-(a-c)}{b-a}a+\frac{b+c-(a-c)}{b-a}\theta, \\
&= a-c-\frac{b-a+2c}{b-a}a+\frac{b-a+2c}{b-a}\theta.
\end{aligned}$$

Denote $A\equiv\frac{b-a+2c}{b-a}$, $E=a-c-Aa=-\frac{a+b}{b-a}c$, thus $\widetilde{\theta}=E+A\theta=a-c-Aa+A\theta=A(\theta-a)+a-c$.

Proposition 5 *i)* $\frac{q^{SB}(\theta)}{\widetilde{q}^{SB}(\widetilde{\theta})}$ *and* $\frac{U^{SB}(\theta)}{\widetilde{U}^{SB}(\widetilde{\theta})}$ *increase with* θ*;*

ii) if $\gamma>\widehat{\gamma}$, $\frac{\theta q^{SB}(\theta)}{\widetilde{\theta}\widetilde{q}^{SB}(\widetilde{\theta})}$ *and* $\frac{t^{SB}(\theta)}{\widetilde{t}^{SB}(\widetilde{\theta})}$ *increase with* θ*, where* $\widehat{\gamma}=\frac{(3b-2a)(b-a+2c)+(a-c)(b-a)}{(4b-2a)(b-a+2c)+(a-c)b}$.

Proof:

$$\frac{q^{SB}(\theta)}{\widetilde{q}^{SB}\left(\widetilde{\theta}\right)} = \frac{2^{\frac{1}{\gamma-1}}\left(\theta - \frac{a}{2}\right)^{\frac{1}{\gamma-1}}}{2^{\frac{1}{\gamma-1}}\left(E + A\theta - \frac{a-c}{2}\right)^{\frac{1}{\gamma-1}}} = \left(\frac{\theta - \frac{a}{2}}{E + A\theta - \frac{a-c}{2}}\right)^{\frac{1}{\gamma-1}}.$$

$$\frac{\partial}{\partial\theta}\left(\frac{\theta - \frac{a}{2}}{E + A\theta - \frac{a-c}{2}}\right) = \frac{1}{\left(E + A\theta - \frac{a-c}{2}\right)^2}\left[\left(E + A\theta - \frac{a-c}{2}\right) - A\left(\theta - \frac{a}{2}\right)\right],$$

where

$$\begin{aligned}
& \left(E + A\theta - \frac{a-c}{2}\right) - A\left(\theta - \frac{a}{2}\right), \\
= \quad & E + A\theta - \frac{a-c}{2} - A\theta + A\frac{a}{2}, \\
= \quad & E - \frac{a-c}{2} + A\frac{a}{2}, \\
= \quad & -\frac{a+b}{b-a}c - \frac{a-c}{2} + \frac{b-a+2c}{b-a}\frac{a}{2}, \\
= \quad & -\frac{2(a+b)c + (a-c)(b-a)}{2(b-a)} + \frac{(b-a+2c)a}{2(b-a)}, \\
= \quad & \frac{ba - a^2 + 2ca - 2(a+b)c - (ab - a^2 - cb + ca)}{2(b-a)}, \\
= \quad & \frac{ba - a^2 + 2ca - 2ac - 2bc - ab + a^2 + cb - ca}{2(b-a)}, \\
= \quad & \frac{-bc - ca}{2(b-a)}, \\
= \quad & -\frac{(b+a)c}{2(b-a)} < 0.
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial\theta}\left(\frac{\theta - \frac{a}{2}}{B + A\theta - \frac{a-c}{2}}\right) < 0,$$

and

$$\frac{\partial}{\partial\theta}\left(\frac{q^{SB}(\theta)}{\widetilde{q}^{SB}\left(\widetilde{\theta}\right)}\right) > 0.$$

Then,

$$\begin{aligned}
\frac{\theta q^{SB}(\theta)}{\widetilde{\theta}\widetilde{q}^{SB}\left(\widetilde{\theta}\right)} &= \frac{\theta \cdot 2^{\frac{1}{\gamma-1}}\left(\theta - \frac{a}{2}\right)^{\frac{1}{\gamma-1}}}{\widetilde{\theta} \cdot 2^{\frac{1}{\gamma-1}}\left(\widetilde{\theta} - \frac{a-c}{2}\right)^{\frac{1}{\gamma-1}}}, \\
&= \left(\frac{\theta^{\gamma-1}\left(\theta - \frac{a}{2}\right)}{\widetilde{\theta}^{\gamma-1}\left(\widetilde{\theta} - \frac{a-c}{2}\right)}\right)^{\frac{1}{\gamma-1}}, \\
&= \left(\frac{\theta^{\gamma-1}\left(\theta - \frac{a}{2}\right)}{(A(\theta-a)+a-c)^{\gamma-1}\left(A(\theta-a)+\frac{a-c}{2}\right)}\right)^{\frac{1}{\gamma-1}}, \\
&= \left(\frac{(Z+a)^{\gamma-1}\left(Z+\frac{a}{2}\right)}{(AZ+a-c)^{\gamma-1}\left(AZ+\frac{a-c}{2}\right)}\right)^{\frac{1}{\gamma-1}},
\end{aligned}$$

where $Z = (\theta - a) \in [0, a-b]$.

$$\begin{aligned}
&\frac{\partial}{\partial Z}\left[\frac{(Z+a)^{\gamma-1}\left(Z+\frac{a}{2}\right)}{(AZ+a-c)^{\gamma-1}\left(AZ+\frac{a-c}{2}\right)}\right] \\
= \; & \frac{1}{\left[(AZ+a-c)^{\gamma-1}\left(AZ+\frac{a-c}{2}\right)\right]^2} \cdot (Z+a)^{\gamma-2}(AZ+a-c)^{\gamma-2} E \cdot \\
& \left[(\gamma-1)\left(Z+\frac{a}{2}\right)\left(AZ+\frac{a-c}{2}\right)+\frac{1}{2}(Z+a)(AZ+a-c)\right].
\end{aligned}$$

Since $E = -\frac{a+b}{b-a}c < 0$, we look at whether $(\gamma-1)\left(Z+\frac{a}{2}\right)\left(AZ+\frac{a-c}{2}\right) + \frac{1}{2}(Z+a)(AZ+a-c) > 0$. This is the case if

$$\frac{1}{2}\left[(\gamma-1)(2Z+a)(2AZ+a-c)+(Z+a)(AZ+a-c)\right] > 0.$$

Since $Z = \theta - a$, the above inequality is equivalent to

$$(4\gamma-3)AZ^2 + (2\gamma-1)(a-c+aA)Z + \gamma a(a-c) > 0.$$

$$\triangle = (2\gamma-1)^2(a-c+aA)^2 - 4(4\gamma-3)A\gamma a(a-c).$$

Since $(2\gamma-1)^2 > (4\gamma-3)\gamma > 0$, $(a-c+aA)^2 > 4Aa(a-c) > 0$, we have $\triangle > 0$. Hence, for $\gamma > \frac{3}{4} > \frac{1}{2}$, the left hand side is a parabola towards the upside with negative axis of symmetry and two negative roots. The intersept with the vertical axis is $\gamma a(a-c) > 0$. Thus, for $Z \in [0, b-a]$, the parabola is positive.

For $\gamma = \frac{3}{4}$,

$$\begin{aligned} & (4\gamma - 3)AZ^2 + (2\gamma - 1)(a - c + aA)Z + \gamma a(a - c), \\ = & \frac{1}{2}(a - c + aA)Z + \frac{3}{4}a(a - c). \end{aligned}$$

It is a straight line with negative slope and positive intercept with the vertical axis. Hence for it to be positive on the interval, it is sufficient that the line is positive at $b - a$.

$$\begin{aligned} & \frac{1}{2}(a - c + aA)(b - a) + \frac{3}{4}a(a - c) \\ = & \frac{1}{4}[2(a - c + aA)(b - a) + 3a(a - c)], \\ = & \frac{1}{4}\left[2\left(a - c + a\frac{b - a + 2c}{b - a}\right)(b - a) + 3a(a - c)\right], \\ = & \frac{1}{4}[2(a - c)(b - a) + a(b - a + 2c) + 3a(a - c)] > 0. \end{aligned}$$

Thus, $\frac{1}{2}(a - c + aA)Z + \frac{3}{4}a(a - c)$ is positive on $Z \in [0, b - a]$.

For $\gamma < \frac{3}{4}$, it is a parabola towards the downside with positive intercept. For the parabola to be positive on the interval $Z \in [0, b - a]$, it is sufficient that it is positive at $b - a$. That is, it is sufficient that

$$(4\gamma - 3)A(b - a)^2 + (2\gamma - 1)(a - c + aA)(b - a) + \gamma a(a - c) > 0,$$

or we can solve for

$$\gamma > \widehat{\gamma} \equiv \frac{(3b - 2a)(b - a + 2c) + (a - c)(b - a)}{(4b - 2a)(b - a + 2c) + (a - c)b}.$$

where $0 < \frac{(3b-2a)(b-a+2c)+(a-c)(b-a)}{(4b-2a)(b-a+2c)+(a-c)b} < \frac{3}{4}$.

Thus, if $\gamma > \widehat{\gamma}$, $(4\gamma-3)AZ^2+(2\gamma - 1)(a - c + aA)Z+\gamma a(a - c) > 0$ for all Z. Hence, if $\gamma > \widehat{\gamma}$, $\frac{\partial}{\partial Z}\left[\frac{(Z+a)^{\gamma-1}\left(Z+\frac{a}{2}\right)}{(AZ+a-c)^{\gamma-1}\left(AZ+\frac{a-c}{2}\right)}\right] < 0$, and $\frac{\partial}{\partial \theta}\left(\frac{\theta q^{SB}(\theta)}{\widetilde{\theta}\widetilde{q}^{SB}\left(\widetilde{\theta}\right)}\right) > 0$.

Next, we examine the ratio of informational rent.

$$
\begin{aligned}
\frac{U^{SB}(\theta)}{\widetilde{U}^{SB}\left(\widetilde{\theta}\right)} &= \frac{2^{\frac{1}{\gamma-1}}\frac{\gamma-1}{\gamma}\left[\left(b-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(\theta-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]}{2^{\frac{1}{\gamma-1}}\frac{\gamma-1}{\gamma}\left[\left(b+c-\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(\widetilde{\theta}-\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]}, \\
&= \frac{\left(b-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(\theta-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}}{\left(b+c-\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(A(\theta-a)+\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}}, \\
&= \frac{1}{A^{\frac{\gamma}{\gamma-1}}}\frac{\left(b-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(\theta-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{b+c-\frac{a-c}{2}}{A}\right)^{\frac{\gamma}{\gamma-1}}-\left(\theta-a+\frac{a-c}{2A}\right)^{\frac{\gamma}{\gamma-1}}}.
\end{aligned}
$$

Denote $X \equiv \theta - a + \frac{a-c}{2A}$, $B \equiv \frac{a}{2} - \frac{a-c}{2A}$, $N \equiv \left(\frac{b+c-\frac{a-c}{2}}{A}\right)^{\mu} = [Q(b-a)]^{\mu}$, where $Q = \frac{b-\frac{a}{2}+\frac{3}{2}c}{b-a+2c}$, $M \equiv \left(b-\frac{a}{2}\right)^{\mu}$, $\mu = \frac{\gamma}{\gamma-1}$. M is the maximum of $(X+B)^{\mu}$, and N is the maximum of X^{μ}. We can also prove that $M < N$.

$$
\frac{U^{SB}(\theta)}{\widetilde{U}^{SB}\left(\widetilde{\theta}\right)} = \frac{1}{A^{\mu}}\frac{M-(X+B)^{\mu}}{N-X^{\mu}} = \frac{1}{A^{\mu}}\frac{(X+B)^{\mu}-M}{X^{\mu}-N},
$$

where $X \in \left[\frac{a-c}{2A}, Q(b-a)\right]$, $M > 0$, $N > 0$, $\mu \in (-\infty, 0)$. Denote

$$
\frac{(X+B)^{\mu}-M}{X^{\mu}-N} \equiv f(X).
$$

$$
\begin{aligned}
f'(X) &= \frac{1}{(X^{\mu}-N)^2}\left[\mu(X+B)^{\mu-1}(X^{\mu}-N)-((X+B)^{\mu}-M)\mu X^{\mu-1}\right], \\
&= \frac{1}{(X^{\mu}-N)^2}\mu(X+B)^{\mu-1}X^{\mu-1}\left[M(X+B)^{1-\mu}-NX^{1-\mu}-B\right].
\end{aligned}
$$

Since $\mu < 0$, $f'(X) > 0$ if and only if

$$
F(X) \equiv M(X+B)^{1-\mu} - NX^{1-\mu} - B < 0.
$$

When $\theta = b$, $\theta - \frac{a}{2} = b - \frac{a}{2}$, $\theta - a + \frac{a-c}{2A} = \frac{b+c-\frac{a-c}{2}}{A} = Q\left(b-a\right)$, thus

$$\begin{aligned} F(X) &= M\left(X+B\right)^{1-\mu} - NX^{1-\mu} - B, \\ &= \left(b-\frac{a}{2}\right)^{\mu}\left(\theta-\frac{a}{2}\right)^{1-\mu} - \left[Q\left(b-a\right)\right]^{\mu}\left(\theta-a+\frac{a-c}{2A}\right)^{1-\mu} - \left(\frac{a}{2}-\frac{a-c}{2A}\right), \\ &= \left(b-\frac{a}{2}\right) - \left(\theta-a+\frac{a-c}{2A}\right) - \left(\frac{a}{2}-\frac{a-c}{2A}\right), \\ &= 0. \end{aligned} \tag{128}$$

Moreover,

$$F'(X) = (1-\mu)\left[M\left(X+B\right)^{-\mu} - NX^{-\mu}\right] < 0$$

if and only if

$$M\left(X+B\right)^{-\mu} - NX^{-\mu} < 0,$$

or

$$X > \frac{BQ(b-a)}{b-\frac{a}{2}-Q(b-a)} = Q(b-a).$$

since we can prove that $\frac{B}{b-\frac{a}{2}-Q(b-a)} = 1$. Thus, if and only if $X > Q(b-a)$, which is impossible, since $X \in \left[\frac{a-c}{2A}, Q(b-a)\right]$. Hence,

$$\begin{aligned} F'(X) &> 0 \text{ for } X \in [\frac{a-c}{2A}, Q(b-a)); \\ F'(X) &= 0 \text{ for } X = Q(b-a). \end{aligned}$$

Since $F\left(X\right)|_{\theta=b} = 0$, and $F'(X) \geq 0$ on $X \in [\frac{a-c}{2A}, Q(b-a)]$, we obtain $F\left(X\right) \leq 0$ on $X \in \left[\frac{a-c}{2A}, Q(b-a)\right]$, that is, $F\left(X\right) < 0$ on the interval $X \in [\frac{a-c}{2A}, Q(b-a))$ and $F\left(X\right)|_{\theta=b} = 0$. Thus, $f'\left(X\right) > 0$ on $X \in [\frac{a-c}{2A}, Q(b-a))$, and $f'\left(X\right) = 0$ at $Q(b-a)$. In another word, $\frac{\partial}{\partial\theta}\left(\frac{U^{SB}(\theta)}{\widetilde{U}^{SB}\left(\widetilde{\theta}\right)}\right) > 0$ on $\theta \in [a, b)$ and $\frac{\partial}{\partial\theta}\left(\frac{U^{SB}(\theta)}{\widetilde{U}^{SB}\left(\widetilde{\theta}\right)}\right) = 0$ at $\theta = b$.

Next, we look at the income ratio. Denote $x \equiv \frac{\widetilde{U}}{\widetilde{U}+\widetilde{\theta}\widetilde{q}}$,

$$\begin{aligned}
\frac{t^{SB}(\theta)}{\widetilde{t}^{SB}\left(\widetilde{\theta}\right)} &= \frac{U+\theta q}{\widetilde{U}+\widetilde{\theta}\widetilde{q}}, \\
&= \frac{U}{\widetilde{U}+\widetilde{\theta}\widetilde{q}} + \frac{\theta q}{\widetilde{U}+\widetilde{\theta}\widetilde{q}}, \\
&= \frac{\widetilde{U}}{\widetilde{U}+\widetilde{\theta}\widetilde{q}}\frac{U}{\widetilde{U}} + \frac{\widetilde{\theta}\widetilde{q}}{\widetilde{U}+\widetilde{\theta}\widetilde{q}}\frac{\theta q}{\widetilde{\theta}\widetilde{q}}, \\
&= \frac{\widetilde{U}}{\widetilde{U}+\widetilde{\theta}\widetilde{q}}\frac{U}{\widetilde{U}} + \left(1-\frac{\widetilde{U}}{\widetilde{U}+\widetilde{\theta}\widetilde{q}}\right)\frac{\theta q}{\widetilde{\theta}\widetilde{q}}, \\
&= x\frac{U}{\widetilde{U}} + (1-x)\frac{\theta q}{\widetilde{\theta}\widetilde{q}}, \\
&= x\frac{U}{\widetilde{U}} + \frac{\theta q}{\widetilde{\theta}\widetilde{q}} - x\frac{\theta q}{\widetilde{\theta}\widetilde{q}}, \\
&= \frac{\theta q}{\widetilde{\theta}\widetilde{q}} + x\left(\frac{U}{\widetilde{U}} - \frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\frac{t^{SB}(\theta)}{\widetilde{t}^{SB}\left(\widetilde{\theta}\right)}\right) &= \frac{\partial}{\partial\theta}\left(\frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right) + \frac{\partial x}{\partial\theta}\left(\frac{U}{\widetilde{U}} - \frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right) + x\left(\frac{\partial}{\partial\theta}\left(\frac{U}{\widetilde{U}}\right) - \frac{\partial}{\partial\theta}\left(\frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right)\right), \\
&= \frac{\partial}{\partial\theta}\left(\frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right) + \frac{\partial x}{\partial\theta}\left(\frac{U}{\widetilde{U}} - \frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right) + x\frac{\partial}{\partial\theta}\left(\frac{U}{\widetilde{U}}\right) - x\frac{\partial}{\partial\theta}\left(\frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right), \\
&= (1-x)\frac{\partial}{\partial\theta}\left(\frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right) + \frac{\partial x}{\partial\theta}\left(\frac{U}{\widetilde{U}} - \frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right) + x\frac{\partial}{\partial\theta}\left(\frac{U}{\widetilde{U}}\right).
\end{aligned}$$

We have known that $x > 0$, $1-x > 0$, $\frac{\partial}{\partial\theta}\left(\frac{\theta q}{\widetilde{\theta}\widetilde{q}}\right) > 0$ if $\gamma > \widetilde{\gamma}$, and $\frac{\partial}{\partial\theta}\left(\frac{U}{\widetilde{U}}\right) \geq 0$.

$$x = \frac{\widetilde{U}}{\widetilde{U}+\widetilde{\theta}\widetilde{q}} = \frac{1}{1+\frac{\widetilde{\theta}\widetilde{q}}{\widetilde{U}}}.$$

$$
\begin{aligned}
\frac{\widetilde{\theta}\widetilde{q}}{\widetilde{U}} &= \frac{\left(A(\theta-a)+a-c\right)2^{\frac{1}{\gamma-1}}\left(A(\theta-a)+\frac{a-c}{2}\right)^{\frac{1}{\gamma-1}}}{2^{\frac{1}{\gamma-1}}\frac{\gamma-1}{\gamma}\left[\left(b+c-\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(A(\theta-a)+\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]}, \\
&= \frac{\left(A(\theta-a)+a-c\right)\left(A(\theta-a)+\frac{a-c}{2}\right)^{\frac{1}{\gamma-1}}}{\frac{\gamma-1}{\gamma}\left[\left(b+c-\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}-\left(A(\theta-a)+\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]}, \\
&= \frac{\left(X+D\right)X^{\mu-1}}{\frac{1}{\mu}\left(N-X^{\mu}\right)} \equiv g\left(X\right).
\end{aligned}
$$

$$
\begin{aligned}
g'\left(X\right) &= \frac{1}{\left(\frac{1}{\mu}\left(N-X^{\mu}\right)\right)^{2}}\cdot \\
&\quad \left\{\left[X^{\mu-1}+\left(X+D\right)\left(\mu-1\right)X^{\mu-2}\right]\frac{1}{\mu}\left(N-X^{\mu}\right)-\frac{1}{\mu}\left(-\mu X^{\mu-1}\right)\left(X+D\right)X^{\mu-1}\right\}, \\
&= \frac{1}{\left(\frac{1}{\mu}\left(N-X^{\mu}\right)\right)^{2}}\cdot \\
&\quad \left\{\frac{1}{\mu}X^{\mu-1}\left(N-X^{\mu}\right)+\frac{\mu-1}{\mu}\left(X+D\right)X^{\mu-2}\left(N-X^{\mu}\right)+X^{\mu-1}\left(X+D\right)X^{\mu-1}\right\}, \\
&= \frac{1}{\left(\frac{1}{\mu}\left(N-X^{\mu}\right)\right)^{2}}\cdot \\
&\quad \left\{\frac{1}{\mu}XX^{\mu-2}\left(N-X^{\mu}\right)+\frac{\mu-1}{\mu}\left(X+D\right)X^{\mu-2}\left(N-X^{\mu}\right)+X^{\mu-1}\left(X+D\right)X^{\mu-1}\right\}, \\
&= \frac{1}{\left(\frac{1}{\mu}\left(N-X^{\mu}\right)\right)^{2}}\cdot \\
&\quad \left\{\left[\frac{1}{\mu}X+\frac{\mu-1}{\mu}\left(X+D\right)\right]X^{\mu-2}\left(N-X^{\mu}\right)+X^{\mu-1}\left(X+D\right)X^{\mu-1}\right\}, \\
&= \frac{1}{\left(\frac{1}{\mu}\left(N-X^{\mu}\right)\right)^{2}}\cdot \\
&\quad \left\{\left[\frac{1}{\mu}X+\left(1-\frac{1}{\mu}\right)X+\left(1-\frac{1}{\mu}\right)D\right]X^{\mu-2}\left(N-X^{\mu}\right)+X^{\mu-1}\left(X+D\right)X^{\mu-1}\right\}, \\
&= \frac{1}{\left(\frac{1}{\mu}\left(N-X^{\mu}\right)\right)^{2}}\cdot \\
&\quad \left\{\left[X+\left(1-\frac{1}{\mu}\right)D\right]X^{\mu-2}\left(N-X^{\mu}\right)+X^{\mu-1}\left(X+D\right)X^{\mu-1}\right\}>0. \qquad (129)
\end{aligned}
$$

Thus, $\frac{\partial}{\partial\theta}\left(\frac{\widetilde{\theta}\widetilde{q}}{\widetilde{U}}\right) > 0$, and we have

$$\frac{\partial x}{\partial \theta} < 0.$$

$$\begin{aligned}
& \frac{U}{\widetilde{U}} - \frac{\theta q}{\widetilde{\theta}\widetilde{q}} \\
= \; & \frac{2^{\frac{1}{\gamma-1}}\frac{\gamma-1}{\gamma}\left[\left(b-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}} - \left(\theta-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]}{2^{\frac{1}{\gamma-1}}\frac{\gamma-1}{\gamma}\left[\left(b+c-\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}} - \left(\widetilde{\theta}-\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]} - \frac{\theta\cdot 2^{\frac{1}{\gamma-1}}\left(\theta-\frac{a}{2}\right)^{\frac{1}{\gamma-1}}}{\widetilde{\theta}\cdot 2^{\frac{1}{\gamma-1}}\left(\widetilde{\theta}-\frac{a-c}{2}\right)^{\frac{1}{\gamma-1}}}, \\
= \; & \frac{\left[\left(b-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}} - \left(\theta-\frac{a}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]}{\left[\left(b+c-\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}} - \left(A(\theta-a)+\frac{a-c}{2}\right)^{\frac{\gamma}{\gamma-1}}\right]} - \frac{\theta\cdot\left(\theta-\frac{a}{2}\right)^{\frac{1}{\gamma-1}}}{(A(\theta-a)+a-c)\cdot\left(A(\theta-a)+\frac{a-c}{2}\right)^{\frac{1}{\gamma-1}}}, \\
= \; & \frac{1}{A^{\mu}}\left\{\frac{M-(X+B)^{\mu}}{N-X^{m}} - \frac{(X+C)\,(X+B)^{\mu-1}}{(X+D)\,X^{\mu-1}}\right\}, \\
= \; & \frac{1}{A^{\mu}}\left\{\frac{\left[M-(X+B)^{\mu}\right](X+D)\,X^{\mu-1} - (N-X^{m})\,(X+C)\,(X+B)^{\mu-1}}{(N-X^{m})\,(X+D)\,X^{\mu-1}}\right\}.
\end{aligned}$$

$D < C$ implies that

$$0 < (X+D) < (X+C)\,.$$

Furthermore,

$$\begin{aligned}
& \left[M-(X+B)^{\mu}\right]X^{\mu-1} - (N-X^{\mu})\,(X+B)^{\mu-1}\,, \\
= \; & (X+B)^{\mu-1}\,X^{\mu-1}\left[M\,(X+B)^{1-\mu} - NX^{1-\mu} - B\right] < 0.
\end{aligned}$$

Thus,

$$0 < \left[M-(X+B)^{\mu}\right]X^{\mu-1} < (N-X^{\mu})\,(X+B)^{\mu-1}\,.$$

Hence,

$$\left[M-(X+B)^{\mu}\right](X+D)\,X^{\mu-1} < (N-X^{\mu})\,(X+C)\,(X+B)^{\mu-1}\,,$$

and thus

$$\left[M-(X+B)^{\mu}\right](X+D)\,X^{\mu-1} - (N-X^{\mu})\,(X+C)\,(X+B)^{\mu-1} < 0.$$

Hence,

$$\frac{U}{\widetilde{\widetilde{U}}} - \frac{\theta q}{\widetilde{\widetilde{\theta}}\widetilde{q}} < 0.$$

Hence,

$$\frac{\partial x}{\partial \theta}\left(\frac{U}{\widetilde{\widetilde{U}}} - \frac{\theta q}{\widetilde{\widetilde{\theta}}\widetilde{q}}\right) > 0.$$

Thus, if $\gamma > \widetilde{\gamma}$ (sufficient condition),

$$\frac{\partial}{\partial \theta}\left(\frac{t^{SB}(\theta)}{\widetilde{t}^{SB}\left(\widetilde{\theta}\right)}\right) = (1-x)\frac{\partial}{\partial \theta}\left(\frac{\theta q}{\widetilde{\widetilde{\theta}}\widetilde{q}}\right) + \frac{\partial x}{\partial \theta}\left(\frac{U}{\widetilde{\widetilde{U}}} - \frac{\theta q}{\widetilde{\widetilde{\theta}}\widetilde{q}}\right) + x\frac{\partial}{\partial \theta}\left(\frac{U}{\widetilde{\widetilde{U}}}\right) > 0.$$

7 Conclusion

We investigate how information frictions influence income distributions when there is a trade-off between rent extraction and efficiency. We study the adverse selection problem in this paper and show that the information rent increases income inequality.

We find that information frictions cause inequality. And the optimal contract incurs a less equal output distribution.

We also find that the change of social norms influences inequality under the optimal contract.

We then study the impacts of information structures on income inequality.

8 Appendix

8.1 Proof of Lemma 4

Proof: We have

$$\begin{aligned}
\underline{q}_1^{SB}/\overline{q}_1^{SB} &= (\underline{\theta})^{\frac{1}{\gamma-1}} / \left(\overline{\theta} + \frac{v_1}{1-v_1}\Delta\theta\right)^{\frac{1}{\gamma-1}} \\
&= \left(\frac{\overline{\theta}}{\underline{\theta}} + \frac{v_1}{1-v_1}\Delta\theta\right)^{\frac{1}{1-\gamma}} \\
&> \left(\frac{\overline{\theta}}{\underline{\theta}} + \frac{v}{1-v}\Delta\theta\right)^{\frac{1}{1-\gamma}} \\
&= \underline{q}^{SB}/\overline{q}^{SB},
\end{aligned}$$

since $0 < v < v_1 < 1$. ■

8.2 Proof of Theorem 11

Proof: We have

$$\begin{aligned}
\underline{t}_1^{SB}/\overline{t}_1^{SB} &= \left(\underline{\theta}\underline{q}_1^{SB} + \Delta\theta\overline{q}_1^{SB}\right)/\overline{\theta}\overline{q}_1^{SB} \\
&= \left(\underline{\theta}/\overline{\theta}\right)\left(\underline{q}_1^{SB}/\overline{q}_1^{SB}\right) + \Delta\theta/\overline{\theta} \\
&> \left(\underline{\theta}/\overline{\theta}\right)\left(\underline{q}^{SB}/\overline{q}^{SB}\right) + \Delta\theta/\overline{\theta} \\
&= \underline{t}^{SB}/\overline{t}^{SB},
\end{aligned}$$

since we have $\underline{q}_1^{SB}/\overline{q}_1^{SB} > \underline{q}^{SB}/\overline{q}^{SB}$ from Lemma 4. ■

8.3 Tools

To compare the income distributions under different contracts, we need more tools.

To establish the Lorenz ordering between two non-negative random variables, we can find the connection between the Lorenz ordering and the second order stochastic dominance. Following Ok (2020) we define the second order stochastic dominance as fololows.

Definition 3 *Let $F_X(\cdot)$ and $F_Y(\cdot)$ be the distribution functions of random* *variables X and Y, respectively. X second order stochastic dominates Y, denoted as $X \succeq_{SSD} Y$, if, and only if,*

$$\int_{-\infty}^{z} F_X(\tau)d\tau \leq \int_{-\infty}^{z} F_Y(\tau)d\tau,$$

for all $z \in \mathbb{R}$, provided that the integrals exist.

Following Shaked and Shanthikumar (2010) we define the convex order of two random variables as follows.

Definition 4 *For two random variables X and Y, X is smaller than Y in the convex order, denoted as $X \preceq_{cx} Y$, if, and only if,*

$$E\left[\phi\left(X\right)\right] \leq E\left[\phi\left(Y\right)\right],$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided that the expectations exist.

Proposition 6 *Let X and Y be two random variables such that $E\left(X\right) = E\left(Y\right)$. Then $X \preceq_{cx} Y$ if, and only if, $X \succeq_{SSD} Y$, i.e.*

$$X \preceq_{cx} Y \Longleftrightarrow X \succeq_{SSD} Y.$$

Proposition 7 *Let X and Y be two non-negative random variables such that $E\left(X\right) = E\left(Y\right)$. Then $X \preceq_{cx} Y$ if, and only if, $X \succeq_{L} Y$, i.e.*

$$X \preceq_{cx} Y \Longleftrightarrow X \succeq_{L} Y.$$

Propositions 6 and 7 imply that $X \succeq_L Y$, $X \preceq_{cx} Y$, and $X \succeq_{SSD} Y$ are equivalent if X and Y are two non-negative random variables with equal means. For two non-negative random variables X and Y such that $E(X) > 0$, $E(Y) > 0$, and $E(X) \neq E(Y)$, we cannot use Propositions 6 and 7 directly. However, for any non-negative random variable X with $E(X) > 0$, we know that X and $\frac{X}{E(X)}$ have the same Lorenz curve. Thus, $X \succeq_L Y$ is equivalent to $\frac{X}{E(X)} \succeq_L \frac{Y}{E(Y)}$. In order to compare Lorenz curves of random variables X and Y, we can investigate random variables $\frac{X}{E(X)}$ and $\frac{Y}{E(Y)}$, since $E\left(\frac{X}{E(X)}\right) = E\left(\frac{Y}{E(Y)}\right) = 1$.

The coefficient of variation (CV) of a random variable X is definded as

$$CV(X) = \sqrt{\frac{E\,(X - EX)^2}{(EX)^2}}.$$

From Proposition 7 we know that $\frac{X}{E(X)} \succeq_L \frac{Y}{E(Y)}$ implies that $\frac{X}{E(X)} \preceq_{cx} \frac{Y}{E(Y)}$. By the definition of the convex order we know that

$$E\left(\frac{X}{E(X)} - 1\right)^2 \leq E\left(\frac{Y}{E(Y)} - 1\right)^2,$$

since $\phi(x) = (x-1)^2$ is a convex function. Therefore, $X \succeq_L Y$ implies

$$CV(X) = \sqrt{E\left(\frac{X}{E(X)} - 1\right)^2} \leq \sqrt{E\left(\frac{Y}{E(Y)} - 1\right)^2} = CV(Y).$$

Proposition 8 *For two non-negative random variables X and Y with $E\,(X) = E\,(Y)$,*

$$X = \begin{cases} x, & \text{with probability } v \\ x', & \text{with probabiity } 1 - v \end{cases},$$

where $x' < x$. And

$$Y = \begin{cases} y, & \text{with probability } v \\ y', & \text{with probabiity } 1 - v \end{cases},$$

where $y' < y$. If $x' \geq y'$, then we have $X \succeq_{SSD} Y$.

Proof of Propostion 8: The distribution function of X, $F_X(\tau)$, is[2]

$$F_X(\tau) = (1-v)I_{[x',x)}(\tau) + I_{[x,\infty)}(\tau),\ \tau \in [0,\infty).$$

and the distribution function of Y, $F_Y(\tau)$, is

$$F_Y(\tau) = (1-v)I_{[y',y)}(\tau) + I_{[y,\infty)}(\tau),\ \tau \in [0,\infty).$$

[2]The indicator function $I_A(\tau)$ is defined as

$$I_A(\tau) = \begin{cases} 1, & \text{if } \tau \in A \\ 0, & \text{if } \tau \notin A \end{cases}.$$

Since $E(X) = E(Y)$, $x' \geq y'$ implies $x \leq y$. Thus, for $z \in [0, y')$, we have $\int_0^z [F_X(\tau) - F_Y(\tau)]\, d\tau = 0$, and for $z \in [y', x')$ we have

$$\int_0^z [F_X(\tau) - F_Y(\tau)]\, ds = -\int_{y'}^z (1-v) ds = -(1-v)(z - y') \leq 0.$$

For $z \in [x', x)$ we have

$$\int_0^z [F_X(\tau) - F_Y(\tau)]\, d\tau = -\int_{y'}^{x'} (1-v) ds = -(1-v)(x' - y') \leq 0.$$

For $z \in [x, y)$ we have

$$\begin{aligned}
\int_0^z [F_X(\tau) - F_Y(\tau)]\, d\tau &= -(1-v)(x' - y') + \int_x^z v d\tau \\
&\leq -(1-v)(x' - y') + v(y - x) \\
&= E(Y) - E(X) \\
&= 0.
\end{aligned}$$

For $z \in [y, \infty)$ we have

$$\begin{aligned}
\int_0^z [F_X(\tau) - F_Y(\tau)]\, d\tau &= -(1-v)(x' - y') + v(y - x) \\
&= E(Y) - E(X) \\
&= 0.
\end{aligned}$$

Thus we have

$$\int_0^z [F_X(\tau) - F_Y(\tau)]\, d\tau \leq 0, \text{ for } \forall z \in [0, \infty),$$

which implies $\int_0^z F_X(\tau) d\tau \leq \int_0^z F_Y(\tau) d\tau$ for all $z \in [0, \infty)$. Therefore, we have $X \succeq_{SSD} Y$. ■

We use Proposition 8 to investigate the impact of information frictions and technological changes on the income distribution.

To draw the conclusion on inequality comparison, we need the following proposition.

Proposition 9 *Let X be a non-negative non-degenerate random variable on $[\underline{x}, \bar{x}]$, and let $g(x)$ and $h(x)$ be non-negative decreasing fuctions of $x \in$*

$[\underline{x}, \bar{x}]$ *such that* $g(x) > 0$ *and* $h(x) > 0$ *for* $x < \bar{x}$. *Then we have*

$$g(X) \succeq_L h(X),$$

if $\frac{g(x)}{h(x)}$ *is increasing in* $x \in [\underline{x}, \bar{x})$.

References

[1] Baron, David and Roger Myerson (1982): "Rgulating a monopolist with unknow costs," *Econometrica*, 50, 911-930.

[2] Che, Yeon-Koo and Ian Gale (1998): "Standard auctions with financially constrained bidders," *Review of Economic Studies*, 65, 1-21.

[3] Che, Yeon-Koo, Ian Gale, and Jinwoo Kim (2013): "Assigning resources to budget-constrained agents," *Review of Economic Studies*, 80, 73-107.

[4] Condorelli, Daniele (2013): "Market and non-market mechanisms for the optimal allocation of scarce resources," *Games and Economic Behabior*, 82, 582-591.

[5] Dworczak, Piotr, Scott Kominers, and Mohammad Akbarpour (2021): "Redistribution through markets," *Econometrica*, 89, 1665-1698.

[6] Fernandez, Raquel and Jordi Gali (1999): "To each according to...? Markets, tournaments, and the matching problem with borrowing constraints," *Review of Economic Studies*, 66, 799-824.

[7] Gastwirth, Joseph (1971): "a general definition of the Lorenz curve," *Econometrica*, 39, 1037-1039.

[8] Geerolf, Francois (2017): "A theory of Pareto distributions," mimeo, University of California, Los Angeles.

[9] Green, Jerry and Nancy Stokey (1983): "A comparison of tournaments and contracts," *Journal of Political Economy*, 91, 349-364.

[10] Heckman, James and Bo Honoré (1990): "The empirical content of the Roy model," *Econometrica*, 58, 1121-1149.

[11] Laffont, Jean-Jacques, and David Martimort (2002): *The Theory of Incentives: The Principal-agent Model*, Princeton University Press, Princeton, NJ.

[12] Lazear, Edward and Sherwin Rosen (1981): "Rank-order Tournaments as optimal labor contracts," *Journal of Political Economy*, 89, 841-864.

[13] Mirrlees, James (1971): "An exploration in the theory of optimum income taxation," *Review of Economic Studies*, 38, 175-208.

[14] Nalebuff, Barry and Joseph Stiglitz (1983): "Prizes and incentives: Towards a general theory of compensation and competition," *Bell Journal of Economics*, 14, 21-43.

[15] Ok, Efe (2020): *Measure and Probability Theory with Economic Applications*, mimeo, New York University.

[16] Piketty, Thomas (2020): *Capital and Ideology*, Harvard University Press, Cambridge, MA.

[17] Piketty, Thomas and Emmanuel Saez (2003): "Income inequality in the United States, 1913-1998," *Quarterly Journal of Economics*, 118, 1-39.

[18] Saez, Emmanuel (2001): "Using elasticities to derive optimal income tax rates," *Review of Economic Studies*, 68, 205-229.

[19] Saez, Emmanuel and Gabriel Zucman (2016): "Wealth inequality in the United States since 1913: Evidence from capitalized income tax data," *Quarterly Journal of Economics*, 131, 519-578.

[20] Sattinger, Michael (1975): "Comparative advantage and the distributions of earnings and abilities," *Econometrica*, 43, 455-468.

[21] Shaked, Moshe and George Shanthikumar (2010): *Stochastic Orders*, Springer, New York, NY.

[22] Terviö, Marko (2008): "The difference that CEOs make: An assignment model approach," *American Economic Review*, 98, 642-668.

[23] Terviö, Marko (2009): "Superstars and mediocrities: Market failure in the discovery of talent," *Review of Economic Studies*, 76, 829-850.

[24] Teulings, Coen (1995): "The wage distribution in a model of the assignment of skills to jobs," *Journal of Political Economy*, 103, 280-315.

[25] Wu, Yaping and Shenghao Zhu (2021): "Nonlinear income tax and inequality," mimeo, University of International Business and Economics.

[26] Zhu, Shenghao (2013): "Comparisons of stationary distributions of linear models," *Economics Letters*, 119, 221-223.