Income Distribution in a Dynamic Assignment Model*

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February 17, 2023

Abstract

This study investigates the income distribution in a dynamic assignment model with human-capital accumulation and endogenous firm size. Positive assortative matching between bosses and workers arises in labor market equilibrium. We solve the stationary equilibrium of two endogenous functions, the matching rule, and human-capital distribution; we also decompose two channels affecting the equilibrium matching rule, firm-size effect, and endogenous distribution effect. Finally, we perform a perturbation analysis of the equilibrium matching rule and the wage function, and examine the effects of technology improvement on income inequality.

JEL classifications: C78, D31, J24

Keywords: income distribution, matching, human-capital accumulation, firm size, perturbation method

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*We would like to thank Jan Eeckhout, Weicheng Lian, Yan Liang, Mehmet Ulubasoglu, Yong Wang, Chunzan Wu, and Shengxing Zhang for their comments and suggestions. We also thank seminar participants at Deakin University, Peking University, and the Australian Conference of Economists 2022 (Tasmania) for their helpful input.

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1 Introduction

Piketty (2014) finds that labor income inequality can be explained by the supply of and demand for different skills. Song et al. (2019) use administrative data for Social Security to investigate the sources of the increase in earnings inequality in the US during 1980–2013. They find that about one-third of the increase is attributable to the increase in the sorting of higher-paid workers into higher-paying firms.[1] Card et al. (2013) obtain similar results using German data. In the present study, based on Eeckhout and Kircher (2018), we investigate income inequality through a new channel using matching and sorting between bosses and workers in a dynamic assignment model.

Combining the matching between the worker quality and firm size, Eeckhout and Kircher (2018) find a mechanism under which the matching rule and equilibrium wage are determined simultaneously in the labor market.[2] However, Eeckhout and Kircher (2018) use a static assignment model. The driving force in Eeckhout and Kircher (2018) is the influence of firm size on the demand for workers, which connects the matching rule and equilibrium wage.[3] We extend the firm-size effect to a dynamic assignment model in which the boss' human capital accumulates over time. We view human-capital accumulation as a new mechanism, under which equilibrium wage and the matching rule are determined simultaneously, which is different from the mechanism in Eeckhout and Kircher (2018). We also investigate the situation in which the firm-size channel is shut down. With exogenous firm size, equilibrium wage in the labor market is determined by matching between bosses and workers with different human capital levels while human-capital accumulation is affected by equilibrium wage.

In the dynamic model, we investigate income distribution, which is the stationary distribution of its accumulation process. We show the existence of stationary human-capital distribution. Given this, the matching equilibrium is the same as in the static model of Eeckhout and Kircher (2018). Positive assortative matching between bosses and workers

[1] Song et al. (2019) use the variance of log earnings to measure inequality.

[2] Eeckhout and Kircher (2018) argue that firm size and matching are the two important cornerstones in macro, labor and industrial organization. More productive firms tend to hire more workers to produce while matching is an important factor for determining the firm's output.

[3] Without the firm-size channel, the matching rule influences equilibrium wage, while equilibrium wage does not affect the matching rule in the static assignment model.

arises in labor market equilibrium. Income distribution is determined by human-capital distribution and equilibrium wage. The human-capital accumulation process is determined by a random innate ability shock and parents' human capital and bequests. Equilibrium wage in the labor market is affected by matching between bosses and workers with different levels of human capital and by endogenous firm size. Further, the matching rule is determined by the human-capital distributions of bosses and workers. Thus, in stationary equilibrium, both the matching rule and human-capital distribution are endogenous and are determined simultaneously.

We use a perturbation method to decompose two channels affecting the equilibrium matching rule, the firm size effect and the endogenous distribution effect. We then numerically examine the magnitudes of these effects when the technology level is improved. The endogenous distribution channel has a much larger effect than firm-size on the matching rule. We also perform a perturbation analysis of wage function in labor market equilibrium. Moreover, to complement Eeckhout and Kircher (2018), we also conduct experiments while turning off the firm-size effect.

We numerically examine the effects of technology improvement on the matching rule, wage function, and income inequality. In our model, we find that Hicks-neutral technology improvement influences income inequality. This result contrasts with that of the static model of Eeckhout and Kircher (2018). In our dynamic model, technical change affects the distribution of the boss' human-capital distribution and thus on the matching rule. Therefore, Hicks-neutral technology improvement affects income distribution in our model.

In addition, we conduct experiments to examine the effects of technology improvement, the weight of the boss in output production, and the mean of employees' human capital on the matching rule, the wage function, and income inequality. These experiments are conducted using static and dynamic models. In the static model, the distribution of the boss' human capital is exogenously given, while in the dynamic model, its distribution is dynamically determined. The results of the dynamic model and the static model differ. This is because in the dynamic model, capital accumulation plays important roles in matching, labor, and firm size while the static model takes human capital as given and lowers the interactions among variables. The decomposition results obtained through perturbation for matching reveal that the firm-size channel and boss human-capital distribution channel deliver opposite effects in the dynamic model. Overall, the latter channel dominates the former.

1.1 Literature review

Our study links three streams of literature. The first follows the early work of Becker (1973) in which a standard frictionless matching model is used to explore sorting between firms and workers. Considering positive assortative matching (PAM) and negative assortative matching (NAM), Eeckhout and Kircher (2018) find that PAM helps explain why more productive firms tend to become larger (e.g., Google and Apple) and NAM helps explain why some large firms hire low-skilled workers (e.g., Walmart). We focus on PAM in our model. That is, highly productive firms hire high-skilled workers, and less productive firms hire low-skilled workers.[4]

Using assignment models, Gabaix and Landier (2008) and Tervio (2008) investigate income distribution, focusing on high-level CEO pay. Using an assignment model to generate income distribution, Scheuer and Werning (2017) study the taxation of superstars. To explain increases in inequality over the last four decades, mainly driven by the increase in the top income earners, Bao et al. (2022) focus on market power to explain managers' pay. They find that assortative matching between a firm and managers increases the gap in productivity across firms and eventually causes higher deadweight loss as productive firms do not pass productivity gains to the customer.

The second stream of literature is related to the inheritance effect on income inequality in dynamic models. Becker and Tomes (1979), Loury (1981), and Benhabib et al. (2011) investigate the inheritance effect on income and wealth inequality. In our study, we assume there is no credit market and employers cannot borrow but accumulate their human capital through education investment by using bequests from their parents.[5]

[4] Adhvaryu et al. (2020) also found NAM in some garment manufacturers in India because some suppliers are beholden to certain powerful global buyers and will allocate high-skilled managers to supervise low-skilled workers to achieve a minimum productivity level on each production line imposed by the buyers. Or, a more productive/skilled worker is matched with a less productive/skilled worker (e.g., pilots and copilots). We do not consider NAM in our study.

[5] For simplification, we assume that employees are hand-to-mouth and do not have human-capital accu-

The third stream of literature concerns the interaction between human-capital accumulation and matching. Teulings (2005) uses a static assignment model to investigate the effects of human-capital accumulation on wage distribution. That approach depends on the comparative statics. Our study, however, uses a dynamic model to investigate the effect of human-capital accumulation on income distribution. Using a dynamic matching model, Anderson and Smith (2010) and Anderson (2015) consider human-capital accumulation in frictionless assignment problems. Human-capital accumulation follows a stochastic process whose transition function depends on the human capital of the matched pair. Jovanovic (2014) uses a similar human capital accumulation function and considers the imperfect signal of the agent's human capital level. Herkenhoff et al. (2018) and Jarosch et al. (2021) estimate the parameters of the learning function. Those studies emphasize on-the-job learning and experiences while our study focuses on the role of schooling in human-capital accumulation.[6] Human-capital accumulation does not explicitly depend on wage function in these papers. Human-capital accumulation in our study, meanwhile, depends on education, and education input is determined by the boss' income, which is influenced by the wage function in the matching equilibrium. In Lise and Postel-Vinay (2020), the worker's human-capital accumulation depends on the worker' skills and the firm's technology in the matching. However, accumulation does not explicitly depend on wage function in their paper.[7]

The rest of this paper is organized as follows. We introduce the model in section 2 and present the perturbation results in section 3. Numerical experiments are conducted in section 4, and section 5 concludes the paper.

2 The model

There is a continuum of measure one of bosses in the economy. The measure of employees is 10. Bosses and employees match and produce outputs in an assignment frame-

mulation.

[6]Mincer (1974) considers both the role of schooling and the role of on-the-job learning in the human capital function.

[7]While our model is a frictionless matching model, Lise and Postel-Vinay (2020) uses a frictional search model.

work. Heterogeneity exists on both sides, that is, both bosses and employees are heterogeneous in their human capital. The human capital of bosses and employees, and the labor of employees are embedded and combined.[8] All bosses and employees live for one period. At the end of the period, each gives birth to one child so that the population stays constant. Bosses leave bequests to their children when they die. Following Becker and Tomes (1979), we assume that bosses, by leaving bequests, show care for not only their own consumption but also their children's income. They have a joy-of-giving bequest motive, and maximize their utility of constant relative risk aversion (CRRA) preference over consumption and bequests. Employees are hand-to-mouth. Their salary is only enough to cover their consumption.[9]

2.1 The boss problem

The boss in period t with human capital x_t chooses his or her consumption $c_{B,t}$ and bequest $b_{B,t+1}$ to maximize utility,

$$\max_{c_{B,t},b_{B,t+1}} \frac{c_{B,t}^{1-\gamma}}{1-\gamma} + \chi_B \frac{b_{B,t+1}^{1-\gamma}}{1-\gamma},$$

$$s.t.\ c_{B,t} + b_{B,t+1} = \pi(x_t),$$

where γ is the reciprocal of the intertemporal elasticity of substitution and χ_B represents the intensity of the bequest motive. The boss' income $\pi(x_t)$ is from the firm's profit, which is determined by bargaining and matching in the labor market. The optimal policy functions of the boss problem are

$$c_{B,t} = \frac{1}{1+\chi_B^{\frac{1}{\gamma}}}\pi(x_t), \tag{1}$$

[8]For an extensive review of sorting and matching in the labor market, see Eeckhout and Kircher (2018).

[9]We omit the labor supply channel in the benchmark model. In Appendix B.3 we introduce the employee's labor–leisure decision problem into the model.

and

$$b_{B,t+1} = \frac{\chi_B^{\frac{1}{\gamma}}}{1+\chi_B^{\frac{1}{\gamma}}}\pi(x_t). \tag{2}$$

Equation (2) implies that the higher the bequest motive intensity χ_B the higher the fraction of income the boss leaves to the child.

Bequests contribute to the formation of the child's human capital. Human-capital accumulation follows

$$x_{t+1} = \underline{x} + \kappa\theta_{t+1}x_t^{\epsilon}b_{B,t+1}^{\eta}, \tag{3}$$

where $\underline{x} > 0$ is the natural endowment of human capital, $\kappa > 0$ is a constant, and θ_{t+1} denotes the innate ability of generation $t+1$. We assume θ_{t+1} follows a uniform distribution, $\theta_{t+1} \sim U\left(0, \bar{\theta}\right)$. For simplification, we also assume that process $\{\theta_t\}_{t=0}^{\infty}$ is independent and identically distributed across generations. Further, ϵ and η denote the elasticity of the child's human capital with respect to the parent's human capital and bequests, with $0 \leq \epsilon, \eta < 1$.

The child inherits human capital from his or her parents through two channels. One is genetic inheritance x_t, and the other is the education channel through bequest inheritance $b_{B,t+1}$. The household cannot borrow money for the child's education. Substituting Equation (2) into Equation (3), we obtain

$$x_{t+1} = \underline{x} + \kappa\left(\frac{\chi_B^{\frac{1}{\gamma}}}{1+\chi_B^{\frac{1}{\gamma}}}\right)^{\eta}\theta_{t+1}x_t^{\epsilon}\pi(x_t)^{\eta}. \tag{4}$$

This equation implies the evolution of the boss' human-capital distribution. Thus, the boss' human-capital distribution is endogenous. Human-capital accumulation with an imperfect credit market can cause income inequality. This is the main difference between our model and models that assume the boss' distribution is exogenous.

2.2 The employee problem

To simplify the model, we assume the employee consumes all of his or her wage income. Thus, we have

$$c_{e,t} = W_t(y),$$

where $c_{e,t}$ is the worker's consumption in period t, and $W_t(y)$ is the worker's wage income in period t. We assume the employee's human capital is from public education, which is independent of the employee's income.[10] We assume that $y \in [\underline{y}, \bar{y}]$ is the human capital of the employee and is drawn from an exogenous distribution function $F_Y(y)$. Later, we will omit the subscript t in $W_t(y)$ since we only focus on the stationary equilibrium of the economy.

2.3 Matching

The labor market has no friction. The firm determines the quality and quantity of labor hiring. Following Eeckhout and Kircher (2018) and Grossman et al. (2017), a firm with a boss of human capital x hires L employees of human capital y to produce output according to the function,

$$F(x, y, L) = A\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta) y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi}, \tag{5}$$

where L represents firm size, $0 < \phi < 1$ is the impact coefficient of firm size, $A > 0$ is the firm's productivity, and $0 < \beta < 1$ denotes the weight of bosses in production. $\alpha > 0$ measures the elasticity of substitution between x and y.[11]

The boss hires workers from the labor market. The boss chooses not only the type of worker (i.e., the worker's human capital level) but also the number of workers, L, to maximize the firm' profits,

$$\pi(x) = \max_{y,L} A\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta) y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi} - W(y)L. \tag{6}$$

[10] If employees also have human-capital accumulation, the economy might have endogenous growth, which is beyond the scope of this study. To focus on the interaction between equilibrium wage and the matching rule, we assume that employees do not have human-capital accumulation.

[11] Our results do not depend on the function form of production. For example, we could adopt function $F(x, y, L) = A\left[\eta x^{\rho} + (1-\eta)(yL)^{\rho}\right]^{\frac{\gamma}{\rho}}$, as in Adamopoulos and Restuccia (2014).

From the firm's problem, we have the first-order conditions with respect to L and y,

$$W(y) = A\phi \left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta) y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi-1}, \tag{7}$$

and

$$W'(y) = A\,(1-\beta) \left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta) y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{1}{\alpha-1}} y^{-\frac{1}{\alpha}} L^{\phi-1}. \tag{8}$$

Equation (8) shows $W'(y) > 0$. This means workers with higher productivity receive a higher wage.

Parameter ϕ governs the decreasing marginal returns of firm size. Multiplying both sides of Equation (7) by L, we obtain

$$WL = \phi F(x, y, L). \tag{9}$$

Thus, payment to workers is a constant share of the total output of the firm. After we close the firm-size channel, the property in which the share of payment to workers is constant might not hold. We will explore this in section 4.

Each boss will choose workers with ability y to maximize the firm's profits. Following Eeckhout and Kircher (2018), we assume $0 < \alpha < 1$, such that we have positive assortative matching in labor market equilibrium.[12] The matching rule follows

$$x = m(y), \tag{10}$$

or

$$y = m^{-1}(x) \equiv \nu(x). \tag{11}$$

We know $m(\cdot)$ and $\nu(\cdot)$ are increasing functions. The matching rule is determined by the human capital distribution of bosses and that of workers.

Combining Equations (7) and (8), we have

$$\frac{W'(y)}{W(y)} = \frac{1-\beta}{\phi} \frac{y^{-\frac{1}{\alpha}}}{\beta m(y)^{\frac{\alpha-1}{\alpha}} + (1-\beta) y^{\frac{\alpha-1}{\alpha}}}, \tag{12}$$

[12]See Appendix A.1 for proof of the positive assortative matching with $0 < \alpha < 1$.

for $y \in [\underline{y}, \bar{y}]$, where $\underline{y}$ denotes the minimum human-capital level of employees. Thus, by Equation (12), we obtain the wage function,

$$W(y) = W(\underline{y}) \exp\left[\frac{1-\beta}{\phi}\int_{\underline{y}}^{y} \frac{z^{-\frac{1}{\alpha}}}{\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta)z^{\frac{\alpha-1}{\alpha}}} dz\right], \tag{13}$$

for $y \in [\underline{y}, \bar{y}]$.

From Equation (7), we derive labor demand as follows:

$$L(x) = \left\{\frac{A\phi\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)\nu(x)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}}{W(\nu(x))}\right\}^{\frac{1}{1-\phi}}, \tag{14}$$

which is referred to as the size of the firm with the boss' human capital x. Firm size connects the matching rule and equilibrium wage in the labor market. Firm size is the crucial force of labor market equilibrium in Eeckhout and Kircher (2018).

Equations (6) and (9) imply the boss' income:

$$\pi(x) = (1-\phi)F(x, y, L). \tag{15}$$

The boss' income is used for his or her own consumption and for leaving bequests, which are used for the child's education. Education expenditures build up the child's human capital. Given the distributions of X and Y, matching in the labor market is a standard assignment problem without friction. The novelty of our study is the endogenous distribution of X. The boss' income is determined by matching equilibrium in the labor market. This matching result influences the education expenditure of the next generation. Thus, human-capital distribution is determined by the matching rule.

2.4 Stationary distribution

In our study, human-capital accumulation is the new mechanism, which connects the matching rule and equilibrium wage in the labor market. The model can generate human-capital distribution through education investment. Substituting Equation (15) into Equation (4), we obtain the human-capital accumulation equation:

$$x_{t+1} = \underline{x} + \theta_{t+1} g(x_t), \tag{16}$$

where

$$g(x_t) = \rho x_t^{\epsilon} \left[\beta x_t^{\frac{\alpha-1}{\alpha}} + (1-\beta)\nu(x_t)^{\frac{\alpha-1}{\alpha}} \right]^{\frac{\alpha\eta}{\alpha-1}} L(x_t)^{\phi\eta},$$

with $\rho = \kappa \left[\frac{(1-\phi)A\chi_B^{\frac{1}{\gamma}}}{1+\chi_B^{\frac{1}{\gamma}}} \right]^{\eta}$.

The function $g(x)$ represents the deterministic part of human-capital accumulation, which includes genetic and bequest inheritance. From an initial human capital level x_0, Equation (16) leads to the human capital accumulation process $\{x_t\}_{t=0}^{\infty}$. We focus on stationary equilibrium in this study. Since $\theta_{t+1} \sim U\left(0, \bar{\theta}\right)$, the lower bound of the stationary distribution of $\{x_t\}_{t=0}^{\infty}$ is $\underline{x}$.

Theorem 1 *Suppose that there exists $\bar{x} > \underline{x}$, such that $\bar{x} = \underline{x} + \bar{\theta} g(\bar{x})$ and $x < \underline{x} + \bar{\theta} g(x)$ for $x \in [\underline{x}, \bar{x}]$. Assume $g(x)$ is continuously differentiable in $(\underline{x}, \bar{x})$ and $g'(x) > 0$ for $x \in (\underline{x}, \bar{x})$. The human-capital accumulation process $\{x_t\}_{t=0}^{\infty}$ is ergodic and hence has a unique stationary distribution.*

Given the matching rule and the wage function, process $\{x_t\}_{t=0}^{\infty}$ has a unique stationary distribution on $[\underline{x}, \bar{x}]$. Let X denote human capital with unique stationary distribution. The density function of X, $f_X(x)$, is determined by

$$f_X(x) = \int_{h(x)}^{\bar{x}} \frac{1}{\bar{\theta} g(u)} f_X(u) du, x \in [\underline{x}, \bar{x}], \tag{17}$$

where $h(x)$ is defined in Appendix A.3.[13] Human-capital distribution has stationary distribution in its accumulation process. We show the existence of stationary human-capital distribution. Given stationary human-capital distribution, matching equilibrium is the same as in the static model of Eeckhout and Kircher (2018). Positive assortative matching between bosses and workers arises in labor market equilibrium.

[13] See the detailed derivations in Appendix A.3.

2.5 Stationary equilibrium

We can define stationary equilibrium as follows:

Definition 1 *Stationary equilibrium consists of*

(i) the solutions to the utility maximization of boss and employee: boss' bequest b_B and consumption c_B, employee's consumption c_e;

(ii) the solutions to production: wage $W(y)$ and firm size $L(x)$ determined by Equations (13) and (14);

(iii) the density function of the stationary distributions of X, $f_X(x)$ given by Equation (17);

(iv) the matching rule: $m(x)$ determined by the labor market clearing condition,

$$\int_{\underline{x}}^{m(y)} L(z) f_X(z) dz = \int_{\underline{y}}^{y} f_Y(z) dz, \tag{18}$$

for $y \in [\underline{y}, \bar{y}]$, where $f_Y(z)$, the density function of Y, is exogenously given.

Stationary equilibrium is determined by Equations (17) and (18).[14] The two endogenous functions, the matching rule and human-capital distribution, are determined by these equations. Equilibrium wage and the matching rule are determined simultaneously. Thus, we can investigate the interaction between human-capital accumulation and the matching mechanism in the labor market. Equation (18) implies that

$$m'(y) = \frac{f_Y(y)}{L\left(m(y)\right) f_X\left(m(y)\right)} > 0, \tag{19}$$

for $y \in [\underline{y}, \bar{y}]$.

Without the firm size effect, Equation (18) is replaced by $\int_{\underline{x}}^{m(y)} f_X(z)dz = \int_{\underline{y}}^{y} f_Y(z)dz$. In a static model, the human capital distribution of bosses X is exogenous. Thus, the matching rule is independent of the wage rate in the labor market. In a dynamic model,

[14] Even though we have shown that process $\{x_t\}_{t=0}^{\infty}$ has unique stationary distribution, it does not imply that stationary equilibrium is unique. Different stationary equilibria might have different matching rules and wage functions.

human-capital distribution X is determined by Equation (17), which is influenced by the matching rule in the labor market. Thus, we find that human-capital accumulation is a new mechanism, under which equilibrium wage and the matching rule are determined simultaneously, which is different from the mechanism in Eeckhout and Kircher (2018).

Intergenerational transfers influence income inequality through human-capital accumulation. Assortative matching contributes to income inequality in a static model. We combine these two channels to study income inequality in a dynamic model with endogenous human-capital distribution and investigate the interaction of these two forces. The sorting equilibrium in the labor market generates the cross sectional income inequality, and intergerational transmission exaggerates distributional effects through inheritance.

Our benchmark model includes both the inheritance effect and the firm size effect. On the one hand, we extend the firm size effect of Eeckhout and Kircher (2018) to a dynamic assignment model in which the boss' human capital accumulates over time. On the other hand, we can investigate the interactions between the firm size effect and the inheritance effect on income inequality in one model. Our parsimonious model focuses on the interactions between the matching rule and the wage function. We omit frictions and institutional factors in the labor market. Our model does not take government policy into account. Sachs et al. (2020) investigate the effect of taxes on the assignment between tasks and workers.

3 Perturbation

Following Costinot and Vogel (2010), we apply comparative static analysis to the matching rule and wage function in the equilibrium. Since Equations (17) and (18) are both nonlinear, it is impossible to analytically examine the effect of A, β, α, and ϕ on the wage rate and matching rule in the equilibrium. Therefore, we use the perturbation method to investigate the effects of these parameters.[15]

Here we present the analysis of the perturbation effect of A and obtain the channels through which A influences equilibrium. We use hat to represent the derivative and the

[15]Due to space limitations, we show the perturbation analysis of β, α, and ϕ in Appendix A.4.

subscript to represent the variable with respect to which we calculate the derivative. For example, $\hat{L}_A(x)$ means the derivative of $L(x)$ with respect to A.[16]

3.1 Perturbation on the matching rule

Rosen (1981) emphasize sorting and matching as the reason for the superstar phenomenon, which implies that relatively small numbers of people have very high earnings. Matching plays an important role in income inequality. We can derive the matching rule from Equation (18). Even though the equation is nonlinear, we can use the perturbation method to investigate effects of A on the matching rule. Differentiating both sides of Equation (18) with respect to A, we obtain,

$$\hat{m}_A(y) = -\frac{\int_{\underline{x}}^{m(y)} \hat{L}_A(z) f_X(z) dz}{L(m(y)) f_X(m(y))} - \frac{\int_{\underline{x}}^{m(y)} L(z) \hat{f}_{(X;A)}(z) dz}{L(m(y)) f_X(m(y))}, \tag{20}$$

for all $y \in [\underline{y}, \bar{y}]$. Equation (20) shows two channels through which A affects matching: the firm-size channel and human-capital accumulation channel. The first term on the right-hand side of Equation (20) represents the firm-size channel, and the second term on the right-hand side represents the human-capital distribution channel. Moreover, through Equation (14), we find that wage rate links these channels through the interactions between firm size and matching.

Human capital distribution is exogenous in the static model. The only channel through which the wage function can influence the matching rule is firm size. If we further close the firm-size effect, the matching rule is determined only by the exogenous distributions of X and Y. In the dynamic model, the wage function influences human-capital distribution X and thus affects on the matching rule. The matching rule and equilibrium wage are determined simultaneously in the labor market. The advantage of the perturbation method

[16] Haanwinckel (2020) decomposes the changes in wage and sorting into contributions from education, technology, minimum wage, and other shocks, and finds that inequality is mainly driven by the minimum wage. Bhandari et al (2021) decompose welfare into three components using the perturbation method and discuss how policy changes can affect aggregate efficiency, redistribution, and insurance. Their method can be applied to both static and dynamic stochastic economies. We do not implement perturbation on social welfare here.

is that different channels are expressed by the linear relationship after we differentiate the nonlinear equations that determine general equilibrium.

3.2 Perturbation of the wage function

Sattinger (1993) summarizes the mechanisms through which sorting determines earnings distribution in the economy. The wage function is determined in the sorting equilibrium. Differentiating both sides of Equation (13) with respect to A, we have

$$\frac{\hat{W}_A(y)}{W(y)} = \frac{\hat{W}_A(\underline{y})}{W(\underline{y})} - \frac{1-\beta}{\phi}\int_{\underline{y}}^{y} \frac{z^{-\frac{1}{\alpha}}\beta m(z)^{\frac{\alpha-1}{\alpha}}}{\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta)z^{\frac{\alpha-1}{\alpha}}\right]^2}\frac{\alpha-1}{\alpha}\frac{\hat{m}_A(z)}{m(z)}dz, \tag{21}$$

for all $y \in [\underline{y}, \bar{y}]$. The first term represents the lower bound wage channel, and the second term is the matching rule channel. The first term in the perturbation equation does not change with y while the second term shows that the change in the wage rate at level y depends on the change in the matching rule at all levels below y. If the matching rule does not change, then the second term becomes zero, and the wage rates only change proportionally.

Derivatives and gradients are normally used in comparative statics. We extend the perturbation analysis to general equilibrium, which consists of two endogenous functions. We find the derivative of each point in the endogenous function with respect to the change in the exogenous variable, such as technology level A. Thus, we can find the heterogeneous effects of the exogenous variable on workers and bosses with different human capital levels. These heterogeneous effects are important for research on the labor market and income inequality.

The perturbation analysis indicates that the matching rule is affected by the wage function through two channels, firm size and X distribution, while Eeckhout and Kircher (2018) only emphasize firm size in the static model. Both firm size and human-capital accumulation have effects on the matching rule and the wage function in equilibrium. We will implement more numerical exercises to separate these different channels.

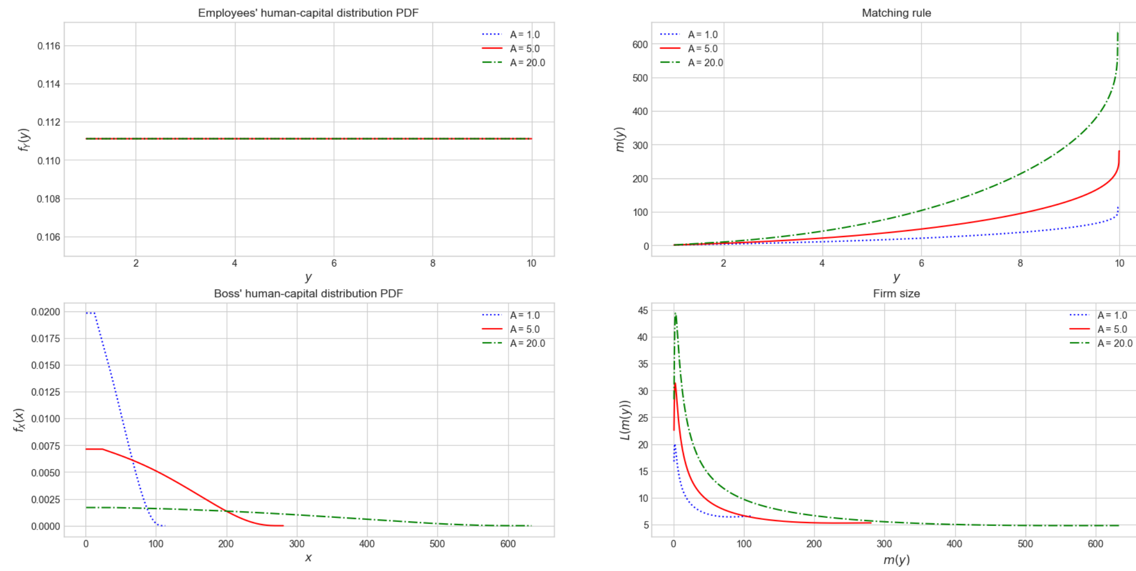
4 Experiments

To examine how inequality changes through interactions between the matching rule and the wage function in the model, in this section, we implement some numerical experiments. We first assign values to the parameters of our model. Following Eeckhout and Kircher (2018), we assume that Y follows a uniform distribution $U(1, 10)$ and $f_Y(y) = 1/9$, and θ follows a uniform distribution $U(0, 10)$ and $f_\Theta(\theta) = 0.1$. We also select $\alpha = 0.5$ and $\beta = 0.3$, as in Eeckhout and Kircher (2018). We use $\phi = 0.66$, representing the employee's wage accounting for 66% in the firm's total income. We choose $\kappa = 1$, $\epsilon = 0.35$, and $\eta = 0.4$, as in Bénabou (2002). We set $\underline{x} = 1$. In the dynamic model, the distribution of X is endogenous, and thus the upper bound $\bar{x}$ changes as the parameter values change.

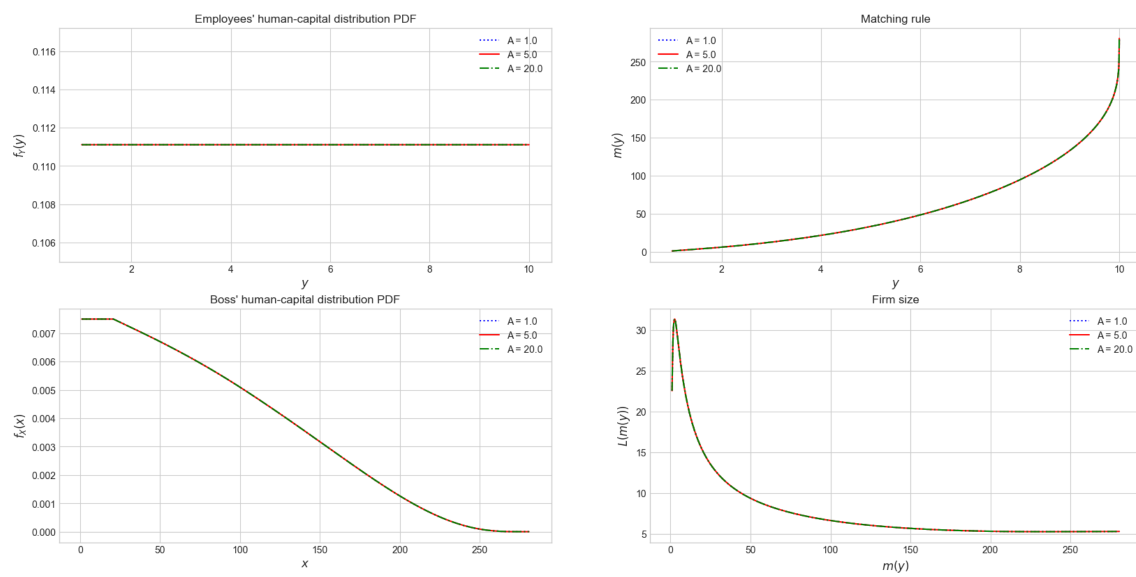
4.1 Technology improvement

Human-capital accumulation with an imperfect credit market can cause a skewed distribution of human capital. The household cannot borrow money for the children's education. The inheritance channel accumulates the effects of luck experienced by each generation. Thus, inheritance is one factor causing income inequality. This mechanism is emphasized by Bénabou (2000). Heterogeneous types of bosses and workers and type complementarity in the production function lead to sorting in labor market equilibrium. This is another factor causing income inequality. This mechanism is emphasized by Eeckhout and Kircher (2018). We combine these two channels to study income inequality. By investigating the effect of technology improvement on income inequality, we find interesting interactions between these two forces.

We investigate the dynamic and static models for different productivity levels, $A = 1, 5, 20$. Other parameters are mentioned at the beginning of this section. For the convenience of comparison, we set the support of distribution X in the static model the same as that in the dynamic model when $A = 5$; that is, $X \sim U(1, 280.77)$ in the static model.



(a) The dynamic model



(b) The static model

Figure 1: Changes in matching, firm size, and human-capital distribution for different values of A.

Technology improvement influences the matching rule and firm size in the dynamic model. Technology improvement influences X distribution through bequests as shown by Equation (3) and Figure 1(a). Figure 1(a) shows that the same level y matches with higher $x = m(y)$ as A increases. Furthermore, a worker with a higher level of y matches a boss with much higher human capital, x, as A increases.

We first show that technology improvement does not influence the matching rule and firm size (see Figure 1(b)) or income distribution (see Figure 3) in the static model. Suppose that firm-size function $L(x)$ does not change when technology level increases from A to A'. From Equation (18), we know that the matching rule does not change. Equations (7) and (8) imply that wage function $W(y)$ changes proportionally and $L(x)$ does not change. From Equation (15), we know that the boss' income also increases proportionally. It is demonstrated, therefore, that income inequality does not change with A in the static model. Technology improvement increases incomes proportionally at all human capital levels. Hicks-neutral technology improvement does not influence income distribution in the static model.[17] These conclusions are illustrated by numerical examples in Figure 3.

In Eeckhout and Kircher (2018), $f_Y(y)$ and $f_X(x)$ are exogenously given while firm size is endogenous. In our dynamic model, both firm size and X distribution are endogenous. Thus, technology improvement has different effects in the static and dynamic models. Technology improvement affects income inequality through the boss' human-capital distribution and the matching rule in the dynamic model. From Equation (9), we know that the income share of workers in each firm is ϕ, when the firm size is endogenous. This constant share does not change along with human capital level x. Thus, technology improvement has no effect on income share in the economy. This constant share property holds both in the dynamic and the static model. Even though technology improvement has no effect on the income shares of bosses and workers in the economy, it does influence income inequality within the boss group and that within the worker group.

As shown by Equation (7), employees' wage $W(y)$ increases with y when technology A is given. To investigate wage inequality, we calculate the wage ratio of different human

[17] Minimum wage $W(\underline{y})$ increases with technology proportionately in the labor market, even though it is determined by law in the real world.

capital levels. From Equation (13), we have

$$\frac{W(y',A)}{W(y,A)} = \exp\left[\frac{1-\beta}{\phi}\int_y^{y'} \frac{z^{-\frac{1}{\alpha}}}{\beta m(z,A)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}} dz\right],$$

where $y' \geq y$. The ratio depends on A through matching rule in the labor market equilibrium. If technology improvement promotes matching, the wage increases with y at a faster rate. In the static model, the matching rule does not change with the technology level. Thus, the wage ratio in the static model does not change.

From Figure 1(a) we can see that the matching rule varies with A since the endogenous human capital distribution of bosses changes in the dynamic model. Thus, the ratio $\frac{W(y',A)}{W(y,A)}$ changes with A in the dynamic model. Technology improvement has effects on wage inequality among workers as shown in Figure 3. From Equation (15), we know that the income inequality of bosses is identical to that of the output of firms. Figure 3 shows that the Gini coefficient of the boss' income decreases with A. This implies that technology improvement lowers the variance of firms' output in the whole economy.

Equation (20) holds for all $y \in [\underline{y}, \bar{y}]$. Thus, we can draw perturbation curves in Figure 2 to show the perturbation of the matching rule when A changes from 2 to 2.01. The blue solid line shows the overall positive effects of A on matching through the firm-size channel and X distribution channel. All workers find bosses with higher human capital levels when A increases. The total effect can be decomposed into two channels through which A affects matching: the firm-size channel and X distribution channel (i.e., the boss' endogenous human-capital accumulation channel). The X distribution channel and firm size have the opposite effects. The X distribution effects dominate the firm-size effects. Thus, the matching perturbation changes are positive for all $y \in [\underline{y}, \bar{y}]$. Employees, who have higher human capital y can enjoy much easier matching with high-skilled bosses. These results confirm the matching rule curves in Figure 1(a).

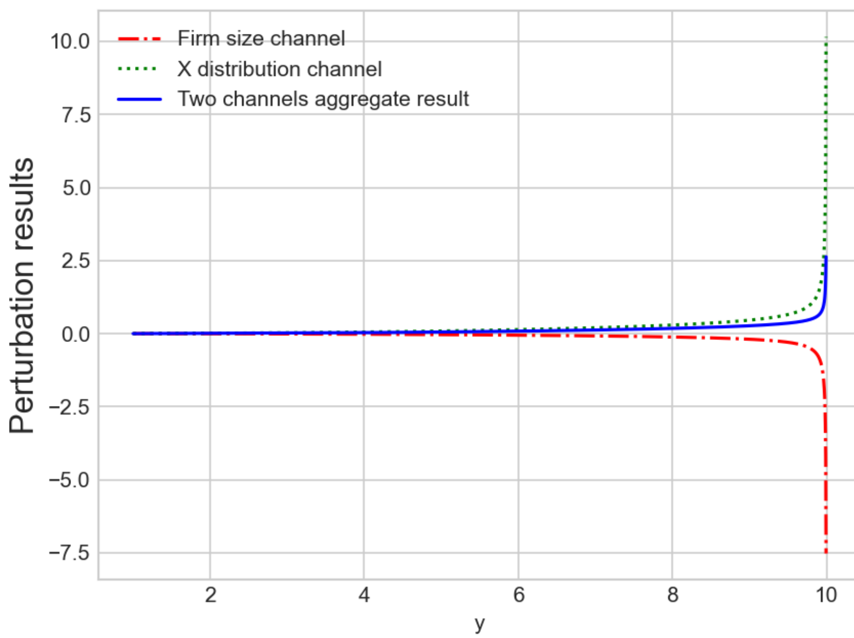


Figure 2: Matching perturbation in the dynamic model when A changes from 2 to 2.01.

Since the income shares of bosses and employees are constant, we investigate the Gini coefficient of the income distribution within each group. Then we plot the Gini coefficient of the income distribution for the whole economy. Figure 3 shows that the Gini coefficients of the employee's wage distribution increase slightly with technology improvement in the dynamic model but do not change in the static model, as we theoretically explained above. The Gini coefficients of the boss' income distribution decrease as A increases. Figure 3 shows that the boss' income distribution becomes more equalized because of more equalized X distribution as shown in Figure 1(a). Inequality among bosses decreases with technology improvement. The Gini coefficient of the entire economy increases slightly.

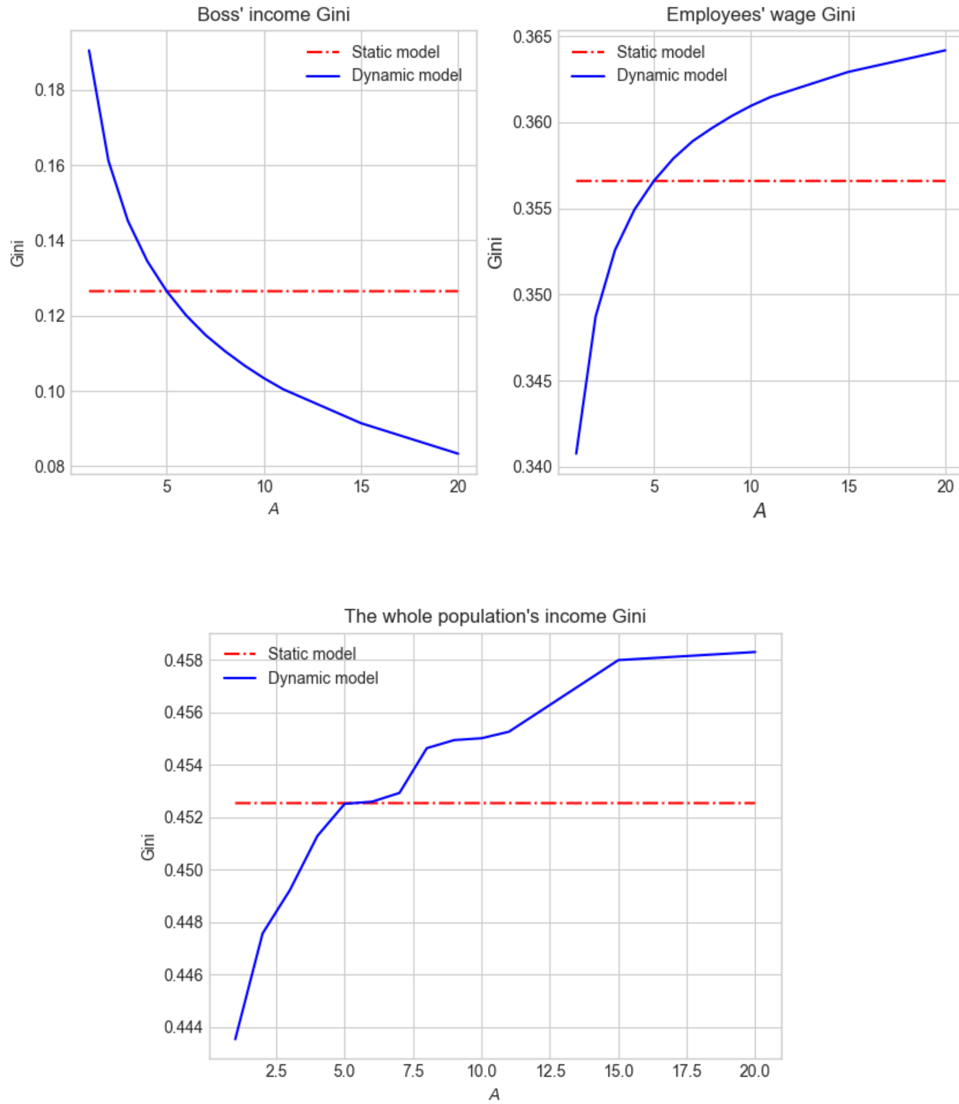


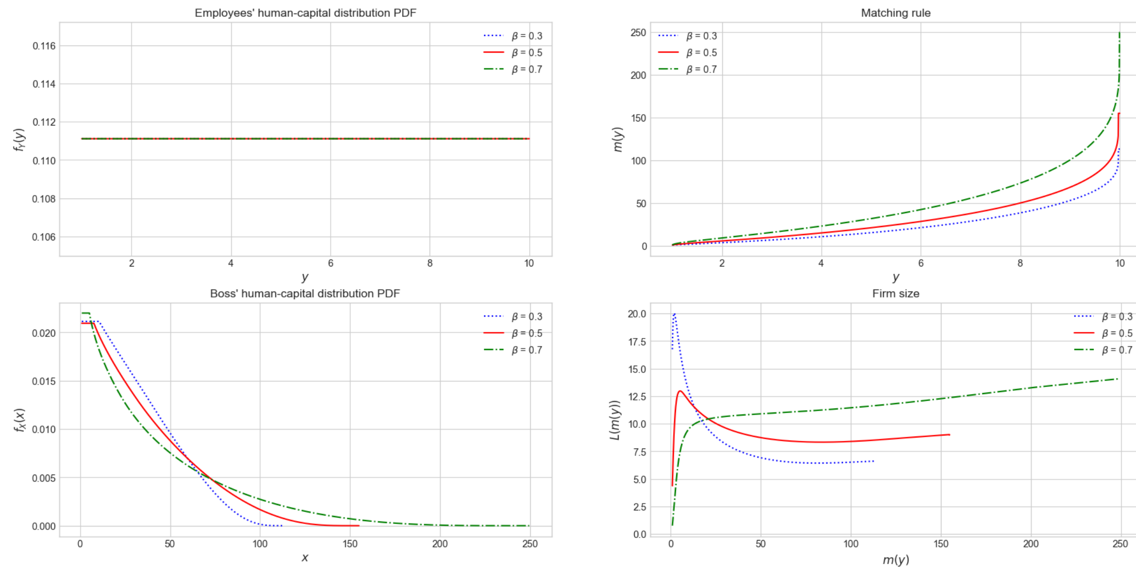
Figure 3: Gini coefficient of the boss' income, employees' wage, and the whole population's income for different values of A.

The most striking finding is that technology improvement has effects on income inequality in the dynamic model but not in the static model. Autor et al. (1998), Krusell et al. (2000), and Acemoglu (2003) emphasize that skill-biased technological change can explain the increase in earnings inequality in the US since 1970s. Different from those studies, we use Hicks-neutral technical change in our study. We find that technical change has effects on the distribution of X and thus on the matching rule; therefore, technical change influences income inequality in the dynamic model. Our findings are consistent with the literature. For example, using microdata from 1984 to 1989, Krueger (1993) finds that workers who use computers for their jobs could have more wage income. Similar findings are seen in Autor et al. (1998), Caselli (1999), Galor and Moav (2000), and Acemoglu (2002). Ales et al. (2015) investigate the implications of technical change for the structure of wages and employment, using a model that assigns talent to tasks. However, the effects in that study are from skill-biased technical change.

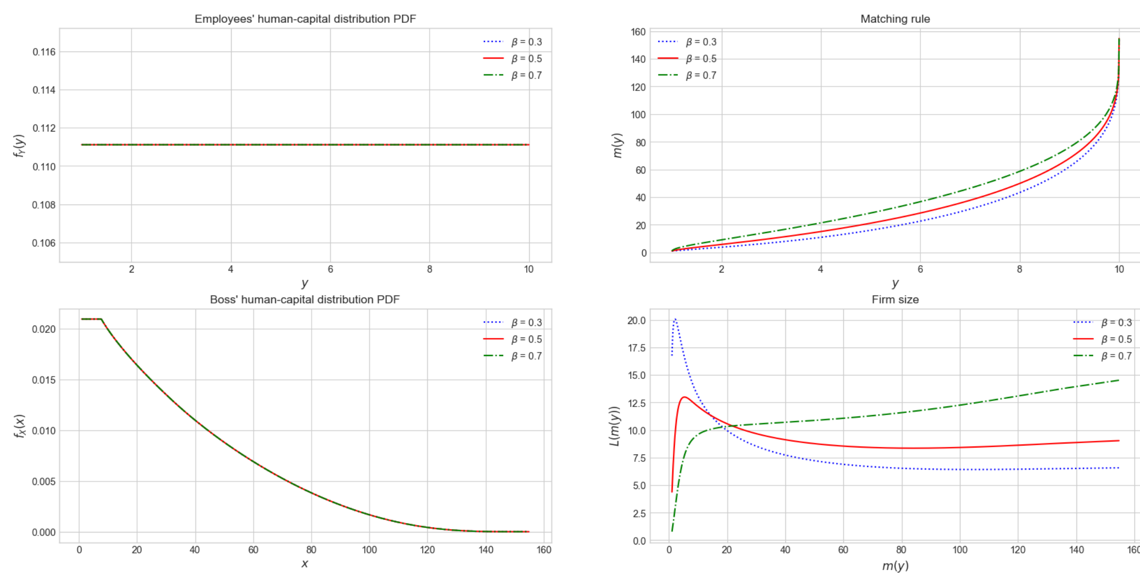
4.2 Weights of bosses and employees in production

Parameter β in Equation (5) measures the weight of the boss in producing output. The higher β is, the more important the boss' role is. To investigate the effects of the boss' role, we implement experiments with different values of β. In the experiments, we assume the employee's human capital distribution Y follows $U(1, 10)$ in both static and dynamic models. The boss' human capital distribution X in the static model follows the stationary distribution of the dynamic model when $\beta = 0.5$. In this subsection, we choose $A = 1$ and other parameters as mentioned at the beginning of this section.

Unlike the change of A, which causes proportional changes in wages and profits in the static model, the change of β alters the relative importance between bosses and workers in production. The matching rule varies with β, even in the static model. The change of β affects the income distribution. Firm size will also change correspondingly. We again use numerical experiments to investigate the effects of β on labor market equilibrium. Figure 4 includes these results.



(a) The dynamic model



(b) The static model

Figure 4: Changes in matching, firm size, and human-capital distribution for different values of β.

Figure 4(a) shows that a worker with human capital level y matches a boss with a much higher human capital level at $\beta = 0.7$ than at $\beta = 0.3$. The change in the matching rule in the dynamic model reflects two channels: X distribution and firm size. The change in the matching rule in Figure 4(b) is only through the firm size channel. Since X distribution does not change in the static model, neither $m(\underline{y})$ nor $m(\bar{y})$ changes with β. Thus, the change in the matching rule has larger magnitudes in the middle of Y distribution than the two ends of $\underline{y}$ and $\bar{y}$.

Figure 4 shows that firm size has a nonmonotone relationship with $m(y)$ at $\beta = 0.3$, and firm size is increasing with $m(y)$ at $\beta = 0.7$. This is because, from the firm's problem, we have[18]

$$L'(m(y)) = \frac{\frac{F_y}{L} - F_{xL}m'(y) - F_{Ly}}{F_{LL}m'(y)}. \tag{22}$$

Equation (22) shows that firm size could be an increasing or decreasing function of $m(y)$ depending on the relative forces of complementarity between bosses and employees, and the span of control complementarity between firm type and the number of workers.

To investigate how wage changes with β, we conduct a perturbation exercise implemented by increasing β from 0.4 to 0.41.[19] Figure 5 shows the perturbation results in the dynamic model. The horizontal red dashed line in Figure 5 represents the lower bound effect of the wage curve. The wage function $W(y)$, where $y \in [\underline{y}, \bar{y}]$, is determined by labor market equilibrium. Thus, the wage rate at the lower bound $W(\underline{y})$ is endogenous, and the lower-bound effect is identical for $y \in [\underline{y}, \bar{y}]$. The numerical result shows that this effect is positive. Figure 5 shows that the β direct channel is negative while the matching channel is positive. In terms of the trend, the direct channel dominates the matching channel. Combining these three components, the total effects on the wage rate are positive. In terms of wage level, the employee earns more as the weight of the boss in production increases. The advantage of perturbation analysis is that we can use curves to represent different effects at different levels of y.

[18] To derive Equation (22), we combine Equations (24) and (26) in Appendix A.1.

[19] See the perturbation decomposition of Equation (28) in Appendix A.4.2.

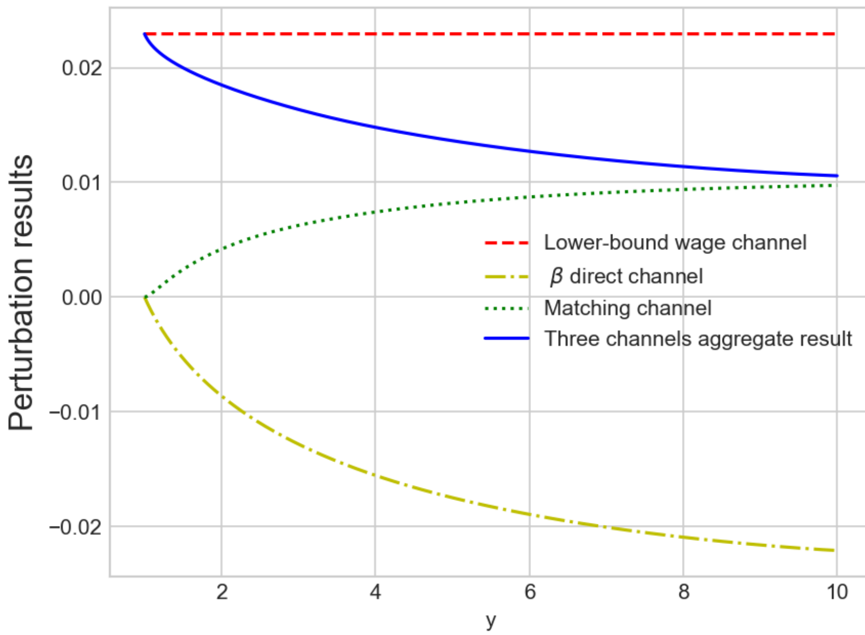


Figure 5: Wage perturbation in the dynamic model.

Figure 6 shows the Gini coefficient of income distributions in the dynamic and static models for different values of β. The static model adopts the stationary distribution of the dynamic model when $\beta = 0.5$. If β is larger than 0.5, the value of the Gini coefficient in the dynamic model is larger than that of the static model. In addition, the boss' Gini coefficient trends upward in both the dynamic and static models while the employee's Gini coefficient trends downward. The whole populations' Gini lines trend slightly downward in both the dynamic and static models when β is small while they trend upward when β becomes large. When β is small, the decrease in the employee's Gini coefficient dominates the increase in the boss' Gini coefficient, and then the whole population's Gini cofficient decreases. As β is greater than a certain value, however, the increase in the boss' Gini coefficient dominates the decrease in the employee's Gini coefficient. Therefore, the whole population's Gini coefficient increases with β. The trend patterns in the dynamic model are similar to those in the static model.

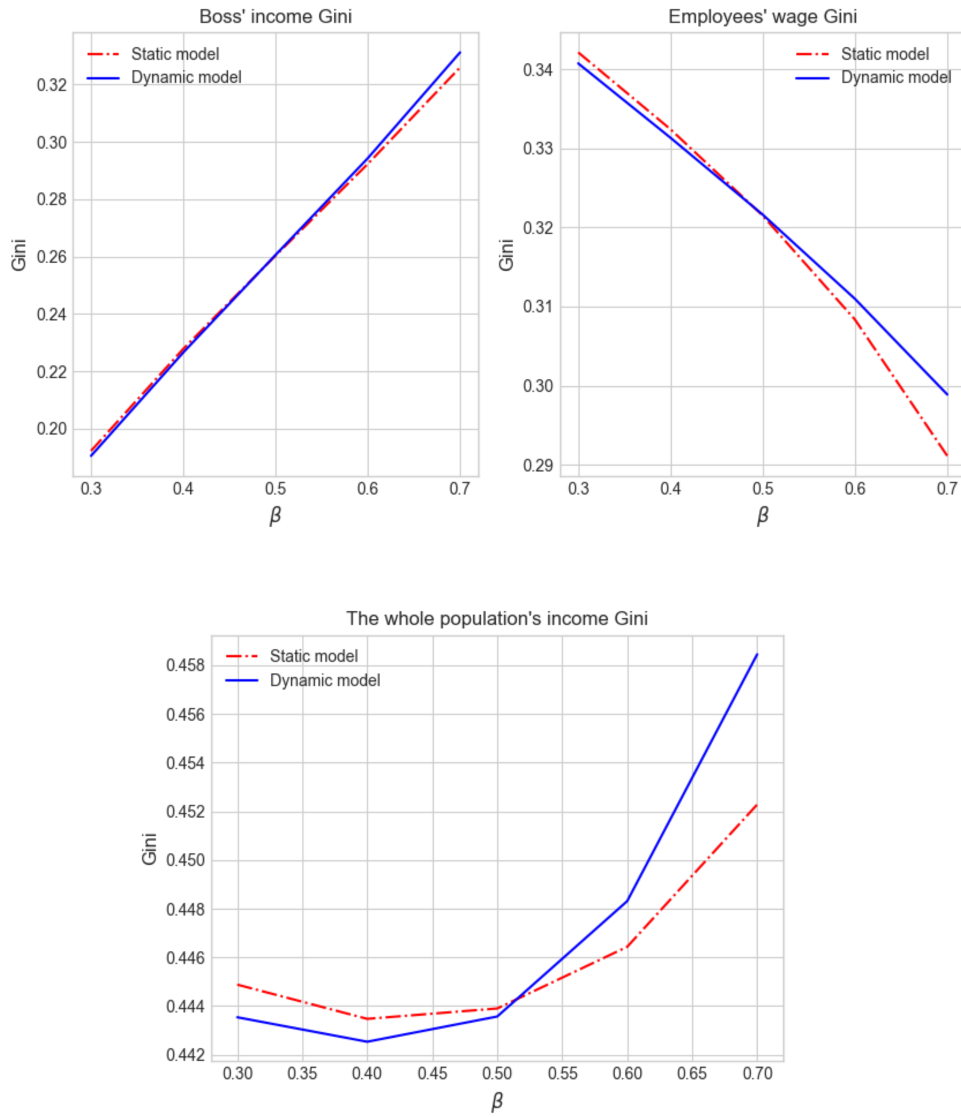


Figure 6: Gini coefficient of the boss' income, employees' wage, and the whole population's income for different values of β.

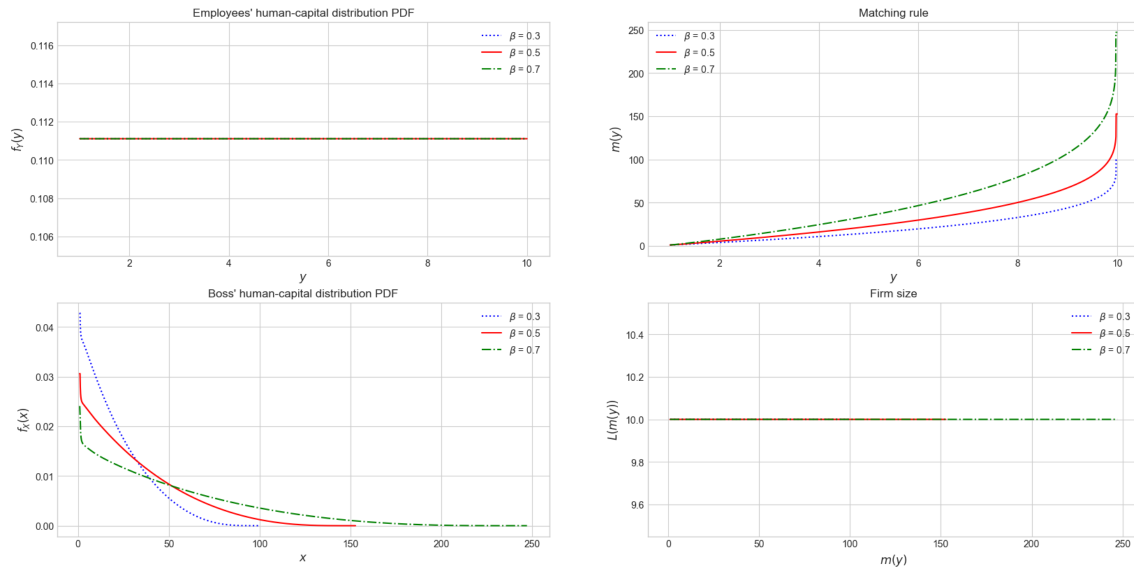
From Equation (9), we know that the income share of workers in the economy is ϕ when firm size is endogenous. The endogenous firm size implies the constant bargaining power between bosses and workers. Thus, weight β has no effect on income share in the economy. This constant share property holds in the dynamic and static models.[20]

To examine the effect of β on labor share, we shut down the firm size channel to implement some exercises. We set firm size $L(x) = 10$. Figure 7 shows the equilibrium results of the dynamic and static models with exogenous firm size.[21] If firm size is exogenously given, we can see via Equation (18) that in the static model, the matching rule does not change with β , since $f_X(x)$ and $f_Y(y)$ are independent of β. In this case β has no effect on the matching rule. Figure 7(b) confirms this conclusion.

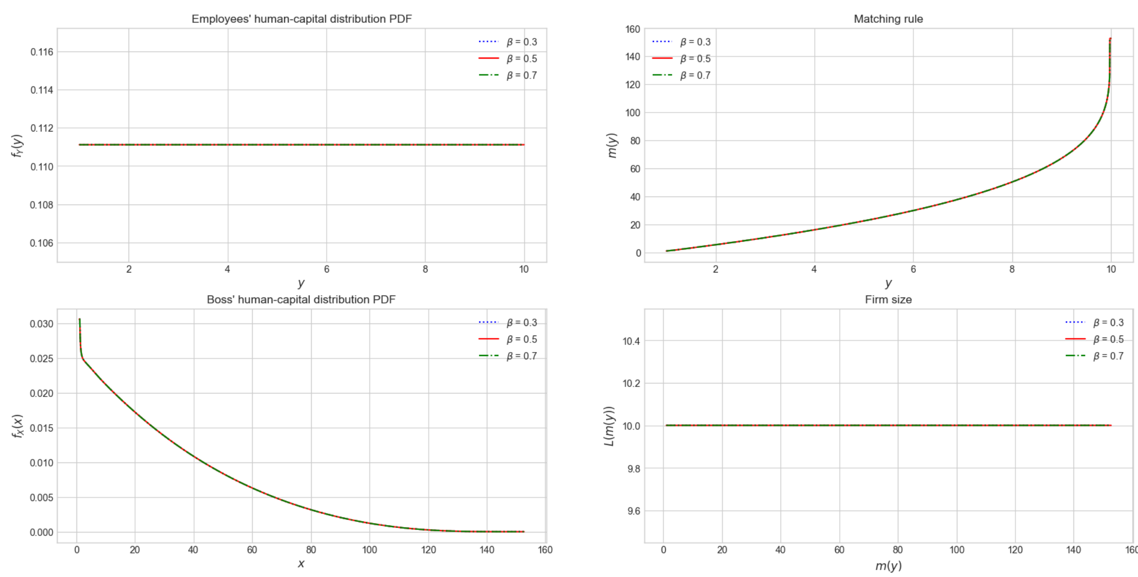
Figure 7(a) shows that X distribution changes with β owing to human-capital accumulation in the dynamic model. Thus, the matching rule also varies with β. With exogenous firm size, equilibrium wage in the labor market is determined by matching between bosses and workers with different human capital levels while human-capital accumulation is affected by equilibrium wage. Therefore, we view human-capital accumulation as a new mechanism, under which equilibrium wage and the matching rule are determined simultaneously, which is different from the mechanism in Eeckhout and Kircher (2018).

[20]Using a sorting model, Gabaix and Landier (2008) find that increases in firm size can explain the rapid increases in CEO compensation since the 1970s in the US. Eeckhout and Kircher (2018) emphasize the firm-size channel in an assignment model between firms and workers with experiments of varying β.

[21]See the derivation details of shutting down the firm size channel in Appendix B.2.



(a) The dynamic model



(b) The static model

Figure 7: Changes in matching, firm size, and human-capital distribution in the models with exogenous firm size for different values of β.

In the matching model without the firm size effect (i.e., the firm size is constant) income share is determined by bargaining power between bosses and workers, and bargaining power is endogenously determined in labor market equilibrium. Analyzing the employee's labor income share $\frac{WL}{F(x,y,L)}$ with different β in the dynamic and static models with exogenous firm size can help us analyze the inequality between bosses and employees.

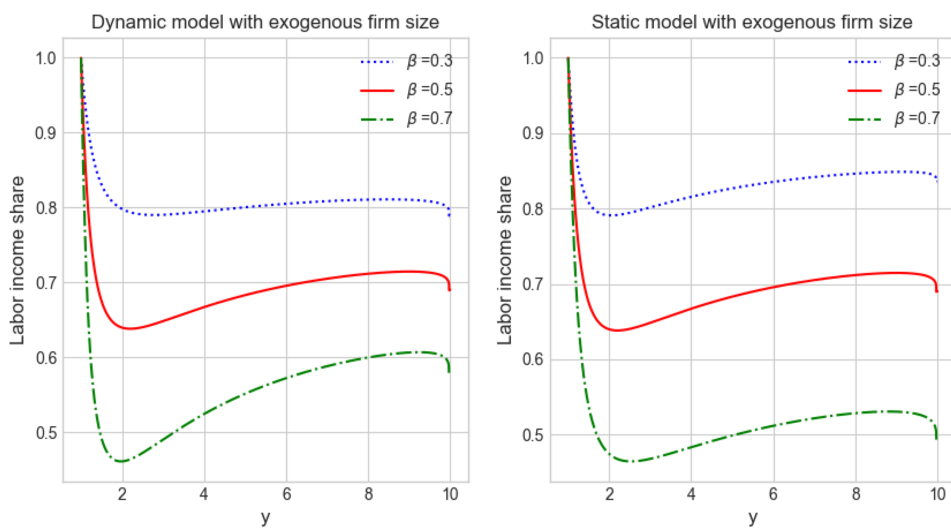


Figure 8: Labor income share comparison.

To specify equilibrium wage in the model with exogenous firm size, we assume employees have dominant bargaining power over bosses at the lower bound of $\underline{y}$.[22] Thus, the boss' profits are zero at $\underline{x}$. However, even under this assumption regarding the lower end of Y distribution, employees with higher human capital have endogenous bargaining power. This is shown in Figure 8 for labor income share with a fixed β.

Furthermore, we find that the labor income share curve shifts when β changes. For a fixed level of y, β and labor share are negatively correlated. The boss weight in the production function influences endogenous bargaining power. A large β implies high importance

[22]We need this assumption to pin down the boundary condition of the wage function, $W(\underline{y})$.

in production and thus the large bargaining power of the boss. Therefore, a boss with large β takes a large share of output. By contrast, an employee's labor share declines with β as he or she becomes relatively less important.

De Loecker et al. (2020) find that markups by firms increased rapidly in the US after 1980. The increase in both markups and profitability shows that market power has increased. Eeckhout (2021) suggests that market power could have caused the declining labor income share in the US. Here, we find that the labor income share is endogenous in the matching equilibrium of the labor market and could be affected by the relative importance of the boss in production. Empirical investigations along these lines can be left to future research.

4.3 Improvement of employee's ability

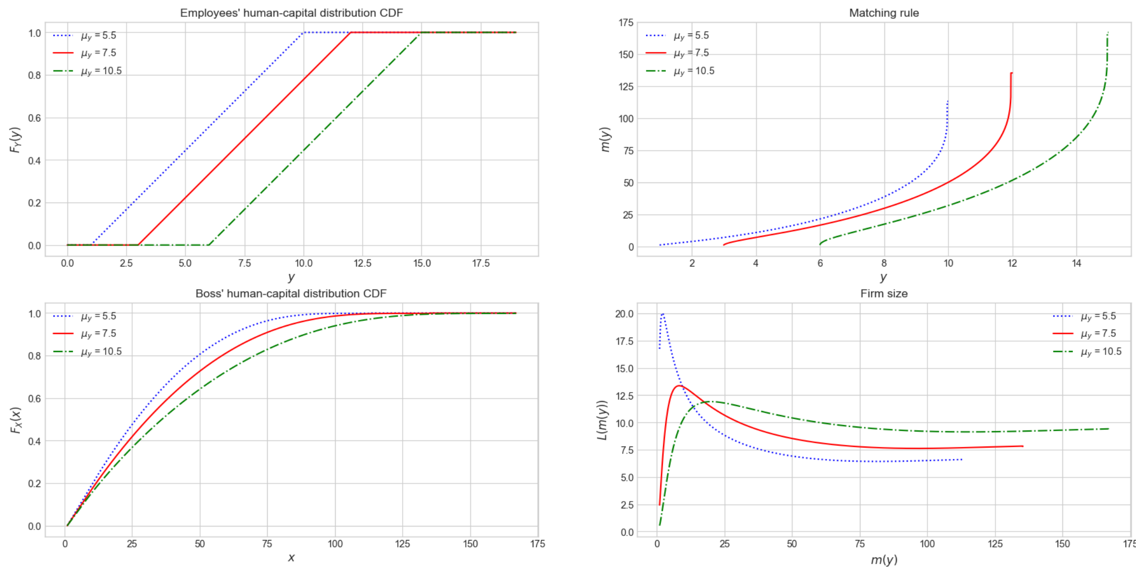
We can also use our model to investigate the effects of distribution Y on stationary distribution X, matching, and income distribution. Assume that u follows a uniform distribution on $(1, 10)$, and $Y = u + d$, where $d \geq 0$ is a constant. Thus, we have the mean of Y, $\mu_y = 5.5 + d$. Similar to skill abundance in Costinot and Vogel (2010), the increase of d represents the improvement of the employee's ability. The mean of Y distribution μ_y changes.[23] In this experiment, we adopt $A = 1$ and other parameters as mentioned at the beginning of this section.[24]

When Y distribution changes,[25] output and the boss' income increase accordingly. Thus, the boss can leave a larger amount of bequests. A change in Y distribution finally affects X distribution and thus the matching rule, through the inheritance effect on education. Figure 9(a) shows the cumulative distribution function (CDF) of X distribution. It is shown that X distribution shifts to the right when μ_y increases.

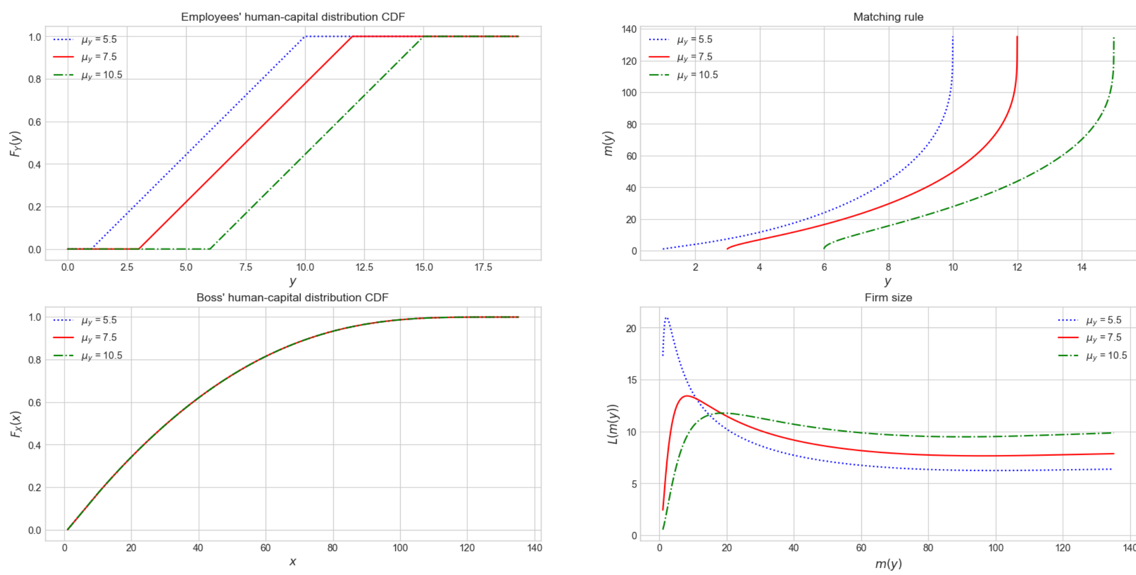
[23] We also conduct experiments by varying the dispersion of the employee's ability distribution while keeping the mean constant (see Appendix B.1). We find that income inequality increases with the employee's skill diversity.

[24] The boss' human-capital distribution X in the static model follows the stationary distribution of the dynamic model when $d = 2$.

[25] The employee's average human capital improves for certain reasons, such as the improvement of public education or the immigration of foreign workers.



(a) The dynamic model



(b) The static model

Figure 9: Changes in matching, firm size, and human-capital distribution for different values of μ_y.

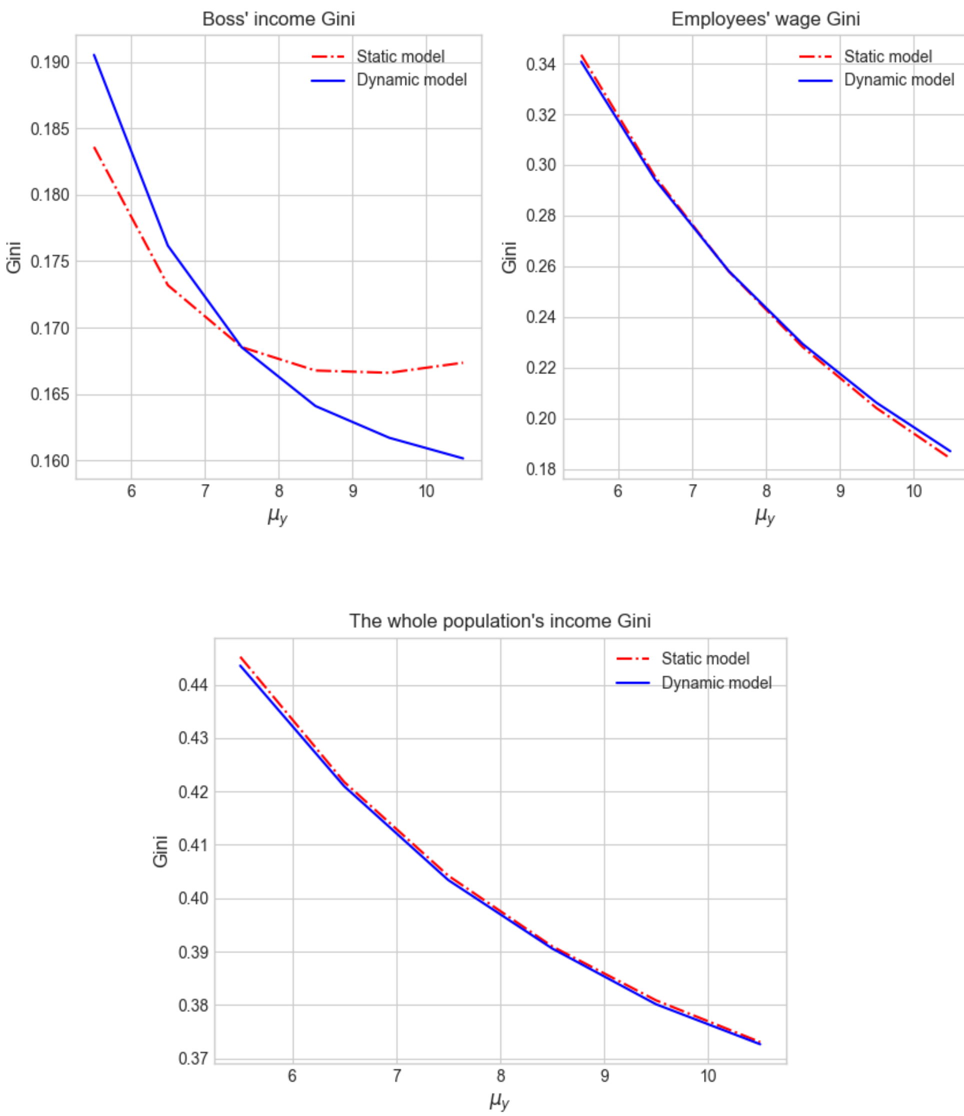


Figure 10: Gini coefficient of the boss' income, employees' wage, and the whole population's income for different values of μ_y.

Figure 9 shows that firm size displays similar patterns in the static and dynamic models. Firm size has an inverted U-shaped relationship with x even though the support of X distribution changes. It decreases with μ_y when $m(y)$ (i.e., x) is small. This is because low-skilled bosses need to hire more low-skilled employees to substitute quality for quantity. However, when $m(y)$ is large enough (i.e., x is not small) firm size increases with μ_y because high-skilled bosses will hire more high-skilled employees owing to the complementarity between bosses and employees in output production.

Figure 10 shows that the Gini coefficient of the boss' income, employees' wage and the whole population's income in both the dynamic and static models trends downward. The shifts in the support of Y distribution to the right cause the Lorenz curve to be flatter. The improvement of employees' human capital promotes equality in not only the employees' wage distribution but also the boss' income distribution. Thus, the Gini coefficient of income distribution in the entire economy decreases.

5 Conclusion

In this study, we extend Eeckhout and Kircher (2018) to a dynamic assignment model in which human-capital distribution of the boss is determined by the human capital accumulation. We solve the stationary equilibrium of the dynamic model with two endogenous functions: the matching rule and human-capital distribution. Aside from considering endogenous firm size, as in Eeckhout and Kircher (2018), we also investigate the situation in which the firm size channel is shut down. With exogenous firm size, the equilibrium wage in the labor market is determined by matching between bosses and workers with different human capital levels, while human-capital accumulation is affected by the equilibrium wage. Therefore, we view human-capital accumulation as a new mechanism, under which equilibrium wage and the matching rule are determined simultaneously, which is different from the mechanism in Eeckhout and Kircher (2018).

We combine human-capital accumulation and assortative matching into a dynamic assignment model to study income inequality in a new framework and investigate the interaction of these two forces. We find that technology improvement promotes matching mainly through human-capital distribution channel, instead of the firm-size channel, which negatively affects matching. The whole population's income inequality, mainly driven by employees' income inequality, increases with technology improvement in the dynamic model but not in the static model. The whole population's income inequality presents a

U-shaped relationship with the importance of the boss' role in output production because of the offsetting effects between the boss income distribution and employees' income distribution. Bargaining power between the boss and the employee is also examined after turning off the firm-size effect by checking the change in labor share. We find that labor share declines as the role of the boss in output production becomes more important. Keeping the boss role constant, the labor share exhibits a U shape against the employees human capital. Firm size and employees ability exhibit a non-linear relationship owing to the complementarity between the boss and the employee in output production. An improvement in the employees ability or an increase in the mean of the employee's human capital helps lower both the boss and employee's income inequality.

Regarding policy implications, our findings suggest that while encouraging technology improvement, which could increase inequality, policy makers should pay more attention to improving average human capital, which in turn could not only help improve technology but also reduce inequality.

Each agent lives for one period in our model. Thus, the agent does not need to know the probability distribution of the matching rule and the wage function in future periods. Future extension could extend the model into an infinite horizon problem. Thus, we would have to solve a functional rational expectation equilibrium since both the matching rule and the wage rate are functions. This could pose a difficult problem when using the traditional computation methods. However, a new method of machine learning, as in Azinovic et al. (2022), could solve this problem. This framework could also be used to measure the effects of immigrants or government interventions, such as education subsidies, on income inequality. We leave this to future work.

Appendix A Derivations and proofs

A.1 Derivation of positive assortative matching

The firm's profit-maximization problem is

$$\pi(x) = \max_{y,L} F(x, y, L) - W(y)L.$$

The first-order conditions of the firm's profit-maximization problem are

$$W(y) = F_L(x, y, L), \tag{23}$$

and

$$W'(y)L = F_y(x, y, L). \tag{24}$$

The second-order necessary condition for maximization requires the Hessian matrix $\mathbf{H}$ to be negative definite, where

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 \pi}{\partial L^2} & \frac{\partial^2 \pi}{\partial L \partial y} \\ \frac{\partial^2 \pi}{\partial y \partial L} & \frac{\partial^2 \pi}{\partial y^2} \end{pmatrix}.$$

We know that

$$\frac{\partial^2 \pi}{\partial L^2} = F_{LL},$$

$$\frac{\partial^2 \pi}{\partial L \partial y} = \frac{\partial^2 \pi}{\partial y \partial L} = F_{Ly} - W'(y),$$

and

$$\frac{\partial^2 \pi}{\partial y^2} = F_{yy} - W''(y)L.$$

Since $0 < \phi < 1$, we have $F_{LL} < 0$. The Hessian matrix $\mathbf{H}$ is negative definite. Thus, the determinant $|\mathbf{H}|$ is positive. We have

$$\begin{aligned} |\mathbf{H}| &= \frac{\partial^2 \pi}{\partial L^2} \frac{\partial^2 \pi}{\partial y^2} - \left(\frac{\partial^2 \pi}{\partial L \partial y} \right)^2 \\ &= F_{LL}(F_{yy} - W''(y)L) - (F_{Ly} - W'(y))^2 \geq 0. \end{aligned} \tag{25}$$

Plugging $x = m(y)$ into Equation (23), we then differentiate $W(y)$ with respect to y and obtain

$$W'(y) = F_{xL}m'(y) + F_{Ly} + F_{LL}L'(m(y))m'(y). \tag{26}$$

Plugging $x = m(y)$ into Equation (24), we then differentiate both sides with respect to y and obtain

$$F_{yy} - W''(y)L = -F_{xy}m'(y) + (W'(y) - F_{Ly})L'(m(y))m'(y). \tag{27}$$

Combining Equations (25) and (27), we have

$$\begin{aligned}
|\mathbf{H}| &= F_{LL}[-F_{xy}m'(y) + (W'(y) - F_{Ly})L'(m(y))m'(y)] - (W'(y) - F_{Ly})^2 \\
&= -F_{LL}F_{xy}m'(y) + [F_{LL}L'(m(y))m'(y) - (W'(y) - F_{Ly})](W'(y) - F_{Ly}) \\
&= [F_{xL}(F_{Ly} - W'(y)) - F_{LL}F_{xy}]m'(y),
\end{aligned}$$

where the third line uses $W'(y) - F_{Ly} = F_{xL}m'(y) + F_{LL}L'(m(y))m'(y)$ from Equation (26).

From Equation (24), we have $W'(y) = \frac{F_y}{L}$. Thus, we have

$$|\mathbf{H}| = \left[F_{xL}\left(F_{Ly} - \frac{F_y}{L}\right) - F_{LL}F_{xy}\right]m'(y).$$

Using the production function $F(x, y, L) = A\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi}$, we have

$$|\mathbf{H}| = \left(\frac{1}{\alpha} - 1\right) A^2\phi(1-\phi)\beta(1-\beta)[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)y^{\frac{\alpha-1}{\alpha}}]^{\frac{2}{\alpha-1}} x^{-\frac{1}{\alpha}} y^{-\frac{1}{\alpha}} L^{2\phi-2} m'(y) \geq 0.$$

Since $0 < \alpha < 1$, $0 < \beta < 1$, and $0 < \phi < 1$, we have $m'(y) \geq 0$. Thus, we have positive assortative matching in labor market equilibrium.

A.2 Ergodicity of the human-capital accumulation process

Definition 2 *A Markovian process $\{s_t\}_{t=0}^{\infty}$ is monotone if for every increasing function $r(\cdot)$, $E[r(s_{t+1})|s_t]$ is an increasing function of s_t.*

Lemma 1 *The human-capital accumulation process $\{x_t\}_{t=0}^{\infty}$ is monotone.*

Proof of Lemma 1: Suppose $r(\cdot)$ is an increasing function. For $x_t \leq \tilde{x}_t$, we have

$$\begin{aligned} E[r(x_{t+1})|x_t] &= \int_0^{\bar{\theta}} r(\underline{x} + \theta' g(x_t)) \frac{1}{\bar{\theta}} d\theta' \\ &\leq \int_0^{\bar{\theta}} r(\underline{x} + \theta' g(\tilde{x}_t)) \frac{1}{\bar{\theta}} d\theta' \\ &= E[r(x_{t+1})|\tilde{x}_t]. \end{aligned}$$

Thus, process $\{x_t\}_{t=0}^{\infty}$ is monotone. ■

Definition 3 *A Markov process $\{s_t\}_{t=0}^{\infty}$ has the Feller property if for any bounded continuous function $r(\cdot)$, $E[r(s_{t+1})|s_t]$ is a bounded continuous function of s_t.*

Lemma 2 *The human-capital accumulation process $\{x_t\}_{t=0}^{\infty}$ has the Feller property.*

Proof of Lemma 2: Suppose $r(\cdot)$ is a bounded continuous function. For sequence $\{x_t^n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_t^n = x_t$, we have

$$\begin{aligned} \lim_{n\to\infty} E[r(x_{t+1})|x_t^n] &= \lim_{n\to\infty} \int_0^{\bar{\theta}} r(\underline{x} + \theta' g(x_t^n)) \frac{1}{\bar{\theta}} d\theta' \\ &= \int_0^{\bar{\theta}} \lim_{n\to\infty} r(\underline{x} + \theta' g(x_t^n)) \frac{1}{\bar{\theta}} d\theta' \\ &= \int_0^{\bar{\theta}} r(\underline{x} + \theta' g(x_t)) \frac{1}{\bar{\theta}} d\theta' \\ &= E[r(x_{t+1})|x_t], \end{aligned}$$

where the second line uses the bounded convergence theorem. Thus, process $\{x_t\}_{t=0}^{\infty}$ has the Feller property. ■

Definition 4 *A Markov process $\{s_t\}_{t=0}^{\infty}$ on $[a, b]$ satisfies the "mixing" condition if there exists $s^* \in [a, b]$, $\varepsilon > 0$, and $N \geq 1$ such that $P^N(a, [s^*, b]) \geq \varepsilon$ and $P^N(b, [a, s^*]) \geq \varepsilon$.*[26]

Lemma 3 *The stochastic process $\{x_t\}_{t=0}^{\infty}$ satisfies the "mixing" condition.*

Proof of Lemma 3: Let

$$x^* = \underline{x} + \frac{\bar{\theta}}{2} g\left(\underline{x}\right).$$

If $\theta_{t+1} = 0$, then $x_{t+1} = \underline{x}$ for $x_t = \bar{x}$. There exists $0 < \delta < \bar{\theta}$ such that $x_{t+1} < x^*$ for $x_t = \bar{x}$ and $\theta_{t+1} \in [0, \delta]$. Thus, we have

$$Pr(\bar{x}, [\underline{x}, x^*]) \geq Pr(\theta_{t+1} \in [0, \delta]) = \delta/\bar{\theta},$$

and

$$Pr(\underline{x}, [x^*, \bar{x}]) \geq Pr\left(\theta_{t+1} \in [\frac{\bar{\theta}}{2}, \bar{\theta}]\right) = \frac{1}{2}.$$

Pick $N = 1$ and let

$$\varepsilon = \min\left\{\frac{1}{2}, \delta/\bar{\theta}\right\}.$$

We have $Pr(\underline{x}, [x^*, \bar{x}]) \geq \varepsilon$ and $Pr(\bar{x}, [\underline{x}, x^*]) \geq \varepsilon$. Thus, process $\{x_t\}_{t=0}^{\infty}$ satisfies the "mixing" condition. ■

Proof of Theorem 1: Lemma 1 shows that process $\{x_t\}_{t=0}^{\infty}$ is monotone. From Lemmas 2 and 3, we know that process $\{x_t\}_{t=0}^{\infty}$ has the Feller property and satisfies the "mixing" condition. Based on Theorem 12.12 in Stokey and Lucas (1989), process $\{x_t\}_{t=0}^{\infty}$ is ergodic. ■

A.3 Stationary distribution as a fixed point

We have $\bar{x} = \underline{x} + \bar{\theta} g(\bar{x})$. Let $\tilde{x} = \underline{x} + \bar{\theta} g(\underline{x})$. We have $\underline{x} \leq \tilde{x} \leq \bar{x}$, since $g'(x) > 0$ for $x \in (\underline{x}, \bar{x})$. From the implicit function theorem, we know that there exists a differentiable function $p(x)$ such that

$$x = \underline{x} + \bar{\theta} g(p(x)),$$

[26] $P^n(\cdot, \cdot)$ denotes the n-step transition probability of $\{s_t\}_{t=0}^{\infty}$.

for $x \in [\tilde{x}, \bar{x}]$. We know that $\underline{x} \le p(x) \le \bar{x}$ since $\tilde{x} = \underline{x} + \bar{\theta} g(\underline{x})$ and $\bar{x} = \underline{x} + \bar{\theta} g(\bar{x})$. Let

$$h(x) = \begin{cases} \underline{x} & , \quad \text{if } x \in [\underline{x}, \tilde{x}] \\ p(x) & , \quad \text{if } x \in [\tilde{x}, \bar{x}] \end{cases}$$

We use $f_{X_t}(x)$ to denote the probability density function of x_t. From $x_{t+1} = \underline{x} + \theta_{t+1} g(x_t)$, we have

$$\begin{aligned} Pr\left(x_{t+1} \le x\right) &= \int_{\underline{x}}^{\bar{x}} Pr\left(\underline{x} + \theta_{t+1} g(u) \le x\right) f_{X_t}(u) du \\ &= \int_{\underline{x}}^{\bar{x}} Pr\left(\theta_{t+1} \le \frac{x - \underline{x}}{g(u)}\right) f_{X_t}(u) du \\ &= \int_{\underline{x}}^{h(x)} f_{X_t}(u) du + \int_{h(x)}^{\bar{x}} \frac{1}{\bar{\theta}} \frac{x - \underline{x}}{g(u)} f_{X_t}(u) du, \end{aligned}$$

since θ_{t+1} follows a uniform distribution on $[0, \bar{\theta}]$.

For $x \in [\underline{x}, \tilde{x}]$, the cumulative distribution function of x_{t+1}, $F_{X_{t+1}}(x)$, is

$$F_{X_{t+1}}(x) = \int_{\underline{x}}^{\bar{x}} \frac{1}{\bar{\theta}} \frac{x - \underline{x}}{g(u)} f_{X_t}(u) du.$$

Thus, we have

$$f_{X_{t+1}}(x) = \int_{\underline{x}}^{\bar{x}} \frac{1}{\bar{\theta} g(u)} f_{X_t}(u) du.$$

For $x \in [\tilde{x}, \bar{x}]$, the cumulative distribution function $F_{X_{t+1}}(x)$ is

$$F_{X_{t+1}}(x) = \int_{\underline{x}}^{p(x)} f_{X_t}(u) du + \int_{p(x)}^{\bar{x}} \frac{1}{\bar{\theta}} \frac{x - \underline{x}}{g(u)} f_{X_t}(u) du.$$

Thus, we obtain

$$\begin{aligned} f_{X_{t+1}}(x) &= f_{X_t}(p(x)) p'(x) - \frac{1}{\bar{\theta}} \frac{x - \underline{x}}{g(p(x))} f_{X_t}(p(x)) p'(x) + \int_{p(x)}^{\bar{x}} \frac{1}{\bar{\theta}} \frac{1}{g(u)} f_{X_t}(u) du \\ &= \int_{p(x)}^{\bar{x}} \frac{1}{\bar{\theta} g(u)} f_{X_t}(u) du, \end{aligned}$$

since $x = \underline{x} + \bar{\theta}g(p(x))$ for $x \in [\tilde{x}, \bar{x}]$.

Thus, for all $x \in [\underline{x}, \bar{x}]$, we have

$$f_{X_{t+1}}(x) = \int_{h(x)}^{\bar{x}} \frac{1}{\bar{\theta}g(u)} f_{X_t}(u)du.$$

The probability density function of stationary distribution X, $f_X(x)$, satisfies

$$f_X(x) = \int_{h(x)}^{\bar{x}} \frac{1}{\bar{\theta}g(u)} f_X(u)du,$$

for all $x \in [\underline{x}, \bar{x}]$.

A.4 Perturbation

A.4.1 Technology improvement

Differentiating both sides of Equation (13) with respect to A, we have

$$\hat{W}_A(y) = W(y) \left[\frac{\hat{W}_A(\underline{y})}{W(\underline{y})} - \frac{1-\beta}{\phi} \int_{\underline{y}}^{y} \frac{z^{-\frac{1}{\alpha}} \beta m(z)^{\frac{\alpha-1}{\alpha}}}{\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right]^2} \frac{\alpha-1}{\alpha} \frac{\hat{m}_A(z)}{m(z)} dz \right].$$

Thus, we have Equation (21).

A.4.2 Changing the weight of bosses

Differentiating both sides of Equation (18) with respect to β, we obtain

$$\hat{m}_\beta(y) = -\frac{\int_{\underline{x}}^{m(y)} \hat{L}_\beta(z) f_X(z)dz}{L(m(y))f_X(m(y))} - \frac{\int_{\underline{x}}^{m(y)} L(z)\hat{f}_{(X;\beta)}(z)dz}{L(m(y))f_X(m(y))}.$$

Differentiating Equation (13), we obtain the employee's wage $W(y)$ perturbation with respect to β, as below:

$$\frac{\hat{W}_\beta(y)}{W(y)} = \frac{\hat{W}_\beta(\underline{y})}{W(\underline{y})} - \int_{\underline{y}}^{y} \frac{1-\beta}{\phi} \frac{z^{-\frac{1}{\alpha}}\beta m(z)^{\frac{\alpha-1}{\alpha}}}{\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right]^2} \frac{\alpha-1}{\alpha} \frac{\hat{m}_\beta(z)}{m(z)} dz$$
$$- \int_{\underline{y}}^{y} \left\{ \frac{1}{\phi} \frac{z^{-\frac{1}{\alpha}}}{\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}} + \frac{1-\beta}{\phi} \frac{z^{-\frac{1}{\alpha}}\left[m(z)^{\frac{\alpha-1}{\alpha}} - z^{\frac{\alpha-1}{\alpha}}\right]}{\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right]^2} \right\} dz, \tag{28}$$

where the first term is the lower-bound wage channel effect, the second term is the matching channel effect, and the last one is the direct channel effect of β.

A.4.3 Changing the substitution coefficient of bosses and employees

Differentiating both sides of Equation (18) with respect to α, we obtain

$$\hat{m}_\alpha(y) = -\frac{\int_{\underline{x}}^{m(y)} \hat{L}_\alpha(z) f_X(z) dz}{L(m(y)) f_X(m(y))} - \frac{\int_{\underline{x}}^{m(y)} L(z) \hat{f}_{(X;\alpha)}(z) dz}{L(m(y)) f_X(m(y))}.$$

Differentiating Equation (13), we obtain the employee's wage $W(y)$ perturbation with respect to α, as below:

$$\frac{\hat{W}_\alpha(y)}{W(y)} = \frac{\hat{W}_\alpha(\underline{y})}{W(\underline{y})} - \frac{1-\beta}{\phi} \int_{\underline{y}}^{y} \frac{z^{-\frac{1}{\alpha}}\beta m(z)^{\frac{\alpha-1}{\alpha}}}{\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right]^2} \frac{\hat{m}_\alpha(z)}{m(z)} dz$$
$$+ \frac{1-\beta}{\phi} \int_{\underline{y}}^{y} z^{-\frac{1}{\alpha}} \left\{ \frac{\ln z \left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right] - \beta m(z)^{\frac{\alpha-1}{\alpha}} \ln m(z)}{\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right]^2} - \frac{(1-\beta)\, z^{\frac{\alpha-1}{\alpha}} \ln z}{\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right]^2} \right\} \frac{1}{\alpha^2} dz,$$

where the first term is the lower-bound wage channel effect, the second term is the matching channel effect, and the last one is the direct channel of α.

A.4.4 Changing the impact coefficient of firm size

Differentiating both sides of Equation (18) with respect to ϕ, we obtain

$$\hat{m}_\phi(y) = -\frac{\int_{\underline{x}}^{m(y)} \hat{L}_\phi(z) f_X(z) dz}{L(m(y)) f_X(m(y))} - \frac{\int_{\underline{x}}^{m(y)} L(z) \hat{f}_{(X;\phi)}(z) dz}{L(m(y)) f_X(m(y))}.$$

Differentiating Equation (13), we obtain the employee's wage $W(y)$ perturbation with respect to ϕ, as below:

$$\begin{aligned}\frac{\hat{W}_\phi(y)}{W(y)} &= \frac{\hat{W}_\phi(\underline{y})}{W(\underline{y})} - \frac{1-\beta}{\phi} \int_{\underline{y}}^{y} \frac{z^{-\frac{1}{\alpha}} \beta m(z)^{\frac{\alpha-1}{\alpha}}}{\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right]^2} \frac{\alpha-1}{\alpha} \frac{\hat{m}_\phi(z)}{m(z)} dz \\ &\quad - \frac{1-\beta}{\phi} \int_{\underline{y}}^{y} \frac{1}{\phi} \frac{z^{-\frac{1}{\alpha}}}{\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}} dz,\end{aligned}$$

where the first term is the lower-bound wage channel effect, the second term is the matching channel effect, and the last one is the direct channel of ϕ.

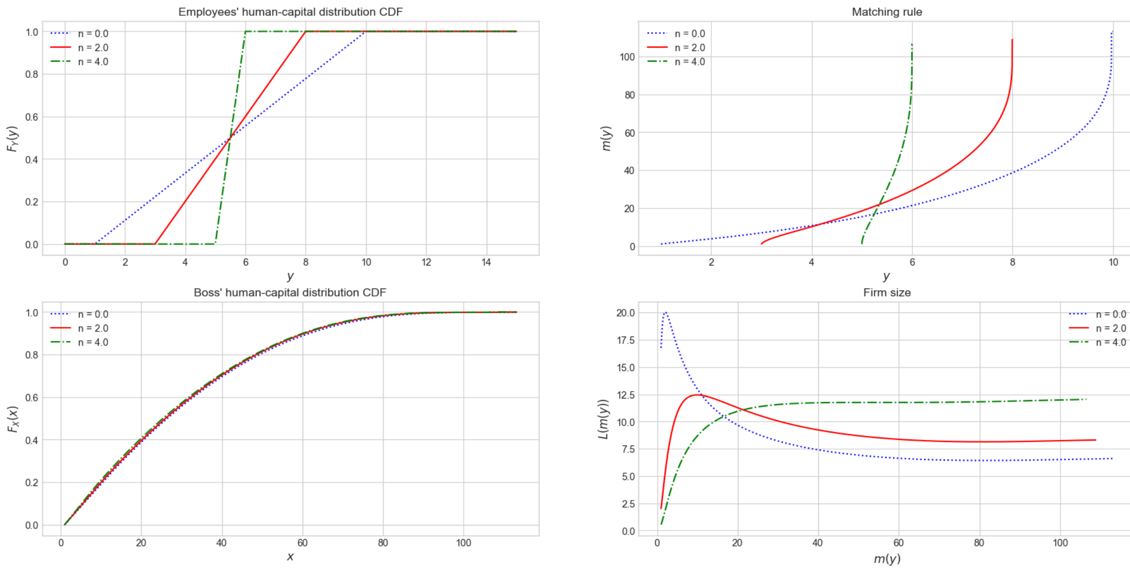
Appendix B Model extensions

B.1 Change in employees' ability dispersion

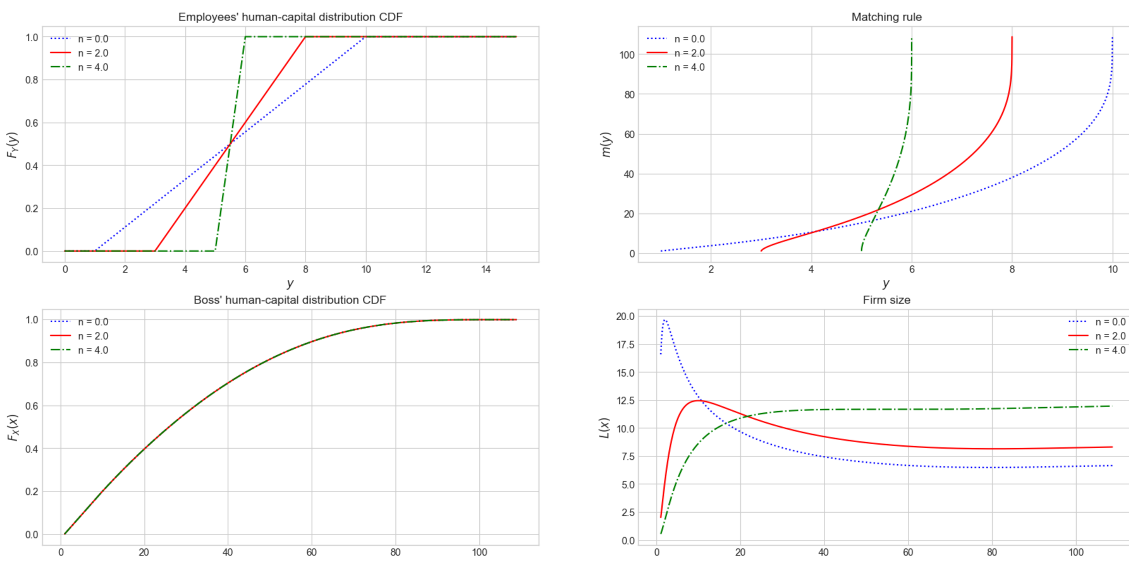
In this subsection, we present the changes in the results of the dynamic and static models when the dispersion of employee's ability distribution varies. The change of n shows the ability dispersion. Assume that Y follows a uniform distribution on $(1+n, 10-n)$, where $n \geq 0$ is a constant. When we increase n, the mean of Y distribution, $\mu_y = 5.5$, does not change while the dispersion of Y distribution decreases. In this experiment, we use the same parameters as in subsection 4.3.[27]

By extending the analysis of Costinot and Vogel (2010) to a dynamic assignment model while keeping the mean-preserving spread of Y, we find that a change in n leads to a change in stationary distribution X, since human-capital accumulation depends on the matching rule and the wage function in the labor market. The numerical experiments show that firm size has a nonmonotone relationship of x when the dispersion of distribution Y is low. Firm size is decreasing in x when the dispersion of distribution Y is high. Figure 11 shows different cases of $n = 0, 2, 4$. The X distributions in the dynamic model respond slightly to the change.

[27] The boss' human-capital distribution X in the static model below follows the stationary distribution of the dynamic model when $n = 2$.



(a) The dynamic model



(b) The static model

Figure 11: Changes in matching, firm size, and human-capital distribution for different values of n.

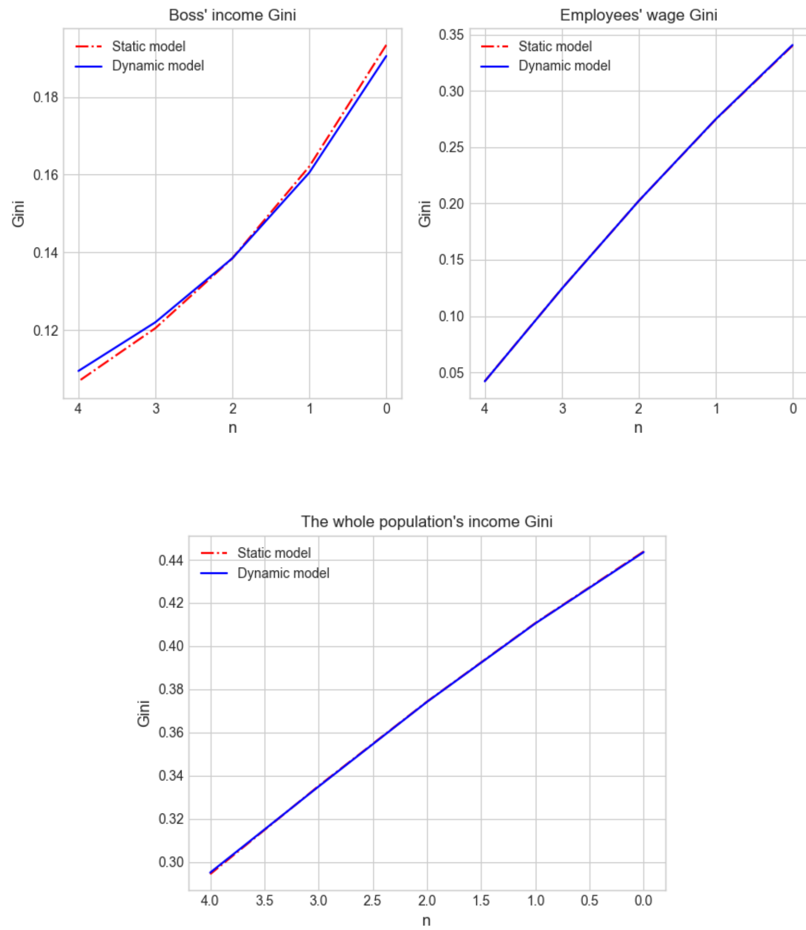


Figure 12: Gini coefficient of the boss' income, employees' wage and the whole population's income for different values of n.

Dispersion magnitude is negatively related to n. Thus, inequality decreasing with n means that income inequality increases with the employees skill diversity. As shown in Figure 12, the boss' income Gini in the dynamic model is slightly different from the static model while employees are the same in both the dynamic and static models. However, all trends in Figure 12 indicate that income inequality increases with employees' skill diversity.

B.2 Model with exogenous firm size

In this model, we assume firm size L is exogenous and $L = 10$. The boss' income is

$$\pi(x) = \max_y A \left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta) y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi} - W(y)L.$$

Thus, wage is determined by

$$W'(y) = A\,(1-\beta)\left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta) y^{\frac{\alpha-1}{\alpha}}\right]^{\frac{1}{\alpha-1}} y^{-\frac{1}{\alpha}} L^{\phi-1}.$$

Therefore, the wage function is

$$W(y) = W(\underline{y}) + A\,(1-\beta)\,L^{\phi-1}\int_{\underline{y}}^{y}\left[\beta m(z)^{\frac{\alpha-1}{\alpha}} + (1-\beta) z^{\frac{\alpha-1}{\alpha}}\right]^{\frac{1}{\alpha-1}} z^{-\frac{1}{\alpha}} dz.$$

The boss' income is

$$\pi(x) = A \left[\beta x^{\frac{\alpha-1}{\alpha}} + (1-\beta)\nu(x)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi} - W(\nu(x))L.$$

Then, we obtain the human capital accumulation equation,

$$x_{t+1} = \theta_{t+1} g(x_t),$$

where $g(x_t) = \rho x_t^{\epsilon}\left\{A\left[\beta x_t^{\frac{\alpha-1}{\alpha}} + (1-\beta)\nu(x_t)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} L^{\phi} - W(\nu(x_t))L\right\}^{\eta}$, with $\rho = \kappa\left(\frac{\chi_B^{1/\gamma}}{1+\chi_B^{1/\gamma}}\right)^{\eta}$. The labor market clearing condition is

$$L\int_{\underline{x}}^{m(y)} f_X(z)dz = \int_{\underline{y}}^{y} f_Y(z)dz.$$

B.3 Model with endogenous labor supply

In this subsection, we extend our benchmark model to a model with employee labor supply, and we only show the differences here. The employee chooses his or her consumption $c_{e,t}$ and labor ℓ_t to maximize his or her utility, subject to income:

$$\max_{c_{e,t},l_t} \frac{c_{e,t}^{1-\gamma}}{1-\gamma} - \chi_e \frac{\ell_t^{1+\delta}}{1+\delta},$$

s.t.

$$c_{e,t} = W_t(y)\ell_t,$$

where $\frac{1}{\delta}$ denotes the Frisch elasticity of labor supply. Solving the employee's problem we have

$$c_{e,t} = \chi_e^{-\frac{1}{\delta+\gamma}} W_t(y)^{\frac{\delta+1}{\delta+\gamma}},$$

and

$$\ell_t = \chi_e^{-\frac{1}{\delta+\gamma}} W_t(y)^{\frac{1-\gamma}{\delta+\gamma}}.$$

The labor market clearing condition now becomes

$$\int_{\underline{x}}^{m(y)} L(z) f_X(z) dz = \int_{\underline{y}}^{y} \ell(z) f_Y(z) dz. \tag{29}$$

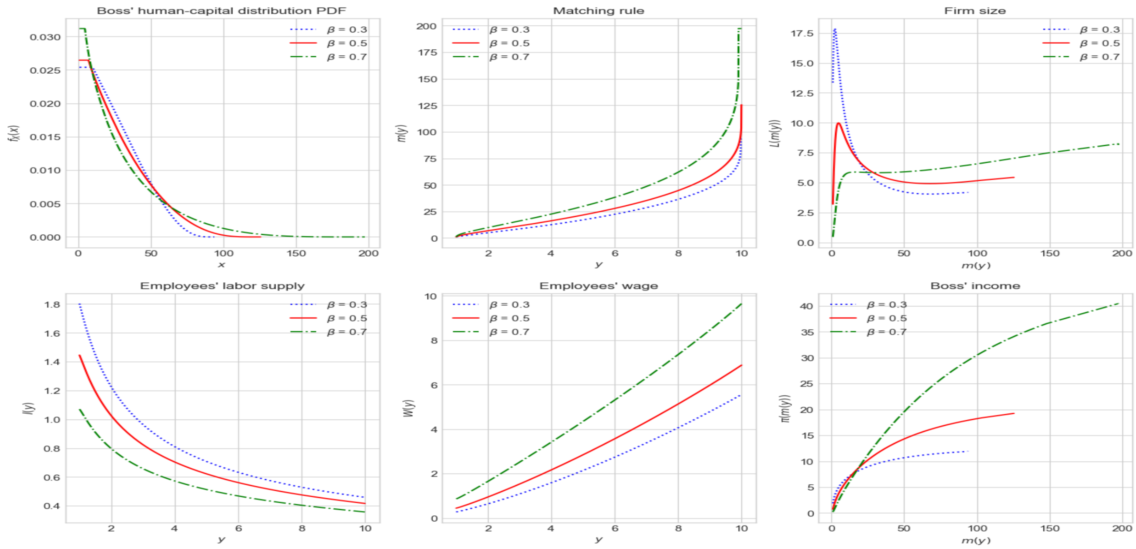
Differentiating Equation (29) with respect to y, we obtain

$$m'(y) = \frac{\ell(y) f_Y(y)}{L(m(y)) f_X(m(y))},$$

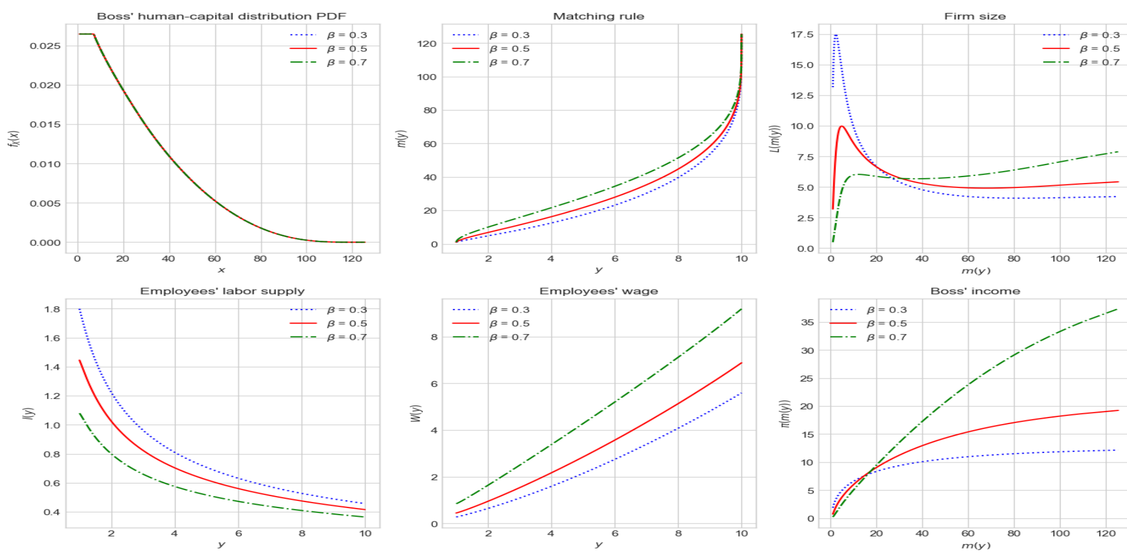
and the matching rule now is

$$m(y) = \underline{x} + \int_{\underline{y}}^{y} \frac{\ell(z) f_Y(z)}{L(m(z)) f_X(m(z))} dz.$$

We implement numerical experiments of different β in the model with endogenous labor supply. We adopt $\gamma = 2$, $\delta = 0.2$ and $\chi_e = 1$; the other parameters are the same as in subsection 4.2. Figure 13 shows the results. Comparing Figure 13 and 4, we find that the results with endogenous labor supply are consistent with the results with exogenous labor supply.



(a) The dynamic model



(b) The static model

Figure 13: Changes in matching, firm size, labor supply, employee's wage, and boss' income, and boss' human-capital distribution for different values of β.

Appendix C Numerical methods

C.1 Matching and market clearing

The two important parts for solving the model are the boss' human capital density function $f_X(x)$ and the matching. In subsection C.2, we obtain $f_X(x)$, and in this subsection, we obtain the matching.

Based on Equation (19) , we obtain the matching rule $m(y)$:

$$m(y) = \underline{x} + \int_{\underline{y}}^{y} \frac{f_Y(z)}{L\left(m(z)\right) f_X\left(m(z)\right)} dz. \tag{30}$$

To solve the matching, we follow the following steps:

Step 1: Set the $\boldsymbol{x}$ grids and $\boldsymbol{y}$ distribution In the model, we assume y follows an exogenous uniform distribution from 1 to 10 and initialize x to range from 1 to 5000. We set the number of the x and y grids to be n_b and n_e; thus, x and y are $n_b \times 1$ and $n_e \times 1$ vectors, $\{x_1, x_2, \cdots, x_{n_b}\}^T$ and $\{y_1, y_2, \cdots, y_{n_e}\}^T$, respectively. Since the support of x is endogenously determined, in step 1, we use 5000 as the initial value of the upper limit of x. When we calculate the x stationary distribution in subsection C.2, we will get the new upper limit of x.

Step 2: Initialize the matching We set the initial matching $m(y)$ ranges from 1 to $x_{n_b}/20$. If the upper limit of the initial value of $m(y)$ is much larger than the upper limit of x, it might make the next matching $m(y)$ increase by too much, making it unable to converge. Thus, we suggest that the upper limit of the initial value should be less than the upper limit of x.

Step 3: Obtain $\boldsymbol{W(y)}$, $\boldsymbol{L(m(y))}$, $\boldsymbol{\pi(m(y))}$, $\boldsymbol{g(m(y))}$, and $\boldsymbol{f_X(x)}$ according to the model Following the model in this paper, we need to fix $W(\underline{y})$ first and then use $m(y)$ and y to obtain these variables and functions. Thus, in the code, we set WL and WH to represent the initial minimum and maximum value of $W(y_1)$, respectively, and the initial value of $W(y_1) = \frac{WL+WH}{2}$. Then, we obtain $W(y)$, $L(m(y))$, $\pi(m(y))$, and $g(m(y))$.

Step 4: Obtain the matching $\boldsymbol{m(y)}$ Following subsection C.2, we obtain $f_X(m(y))$

and then use interpolation to get $f_X(x)$. Since the elements of $m(y)$ might be larger than $\bar{x}$, the extrapolation might lead to some element of $f_X(m(y)) \leq 0$; then, we need to set the ones whose values are nonpositive and equal to the minimum positive values of $f_X(m(y))$. Following the method in Eeckhout and Kircher (2018), multiply $f_Y(y)$ by 100—the population ratio of employees to bosses. According to the Occupational Employment and Wage Statistics provided by the US Bureau of Labor Statistics, managers, first-line supervisors, superintendents, and administrators accounted for about 10.4% of the workforce in 2018; thus, we assume the population ratio of employees to bosses is 10. Then, we obtain the matching $m(y)$ following Equation (30). The element of $m(y)$ might be larger than $\bar{x}$, especially when the elements of $f_X(m(y))$ approximate to 0. Thus, we set the elements whose values are larger than $\bar{x}$ equal to $\bar{x}$. Then, we obtain the final matching in this step.

Step 5: Check if $\boldsymbol{x}$ is exhausted Following Eeckhout and Kircher (2018), we set diff_my_xmax $= \frac{max(m(y))-\bar{x}}{\bar{x}}$. When the absolute value of diff_my_xmax is larger than 10^{-3}, it means the boss' human capital x is not exhausted. Thus, if diff_my_xmax is larger than 10^{-3}, we need to decrease the wage to increase $L(m(y))$; thus, we set $WH^{new} = W(y_1)$ and $W(y_1)^{new} = \frac{WL+WH^{new}}{2}$. Similar to this, when diff_my_xmax is less than -10^{-3}, we set $WL^{new} = W(y_1)$ and $W(y_1)^{new} = \frac{WL^{new}+WH}{2}$.

Step 6: Check if matching converges In step 4, we obtain the matching $m(y)_1$, but it might not converge. Thus, we run the code again with this matching $m(y)_1$ to get a new matching, $m(y)_2$, which might not be the same as $m(y)_1$. We therefore need to check if $m(y)_2$ is the same as $m(y)_1$. First, we define a vector equal to the difference between $m(y)_2$ and $m(y)_1$. Second, we define a variable dm equal to the norm of the vector, divided by the square root of n_e; n_e is the number of elements in $m(y)$. If dm is less than 10^{-2}, we consider that the matching converges. Repeat steps 3–6 until the matching converges.

Step 7: Check if the market clears We set a vector equal to the difference between the RHS and LHS of Equation (18). This vector is $n_e \times 1$. Then, we define a variable Labor_market_clear equal to the norm of this vector, divided by the square root of n_e. When Labor_market_clear is less than 10^{-3}, the market is clear. If the LHS is larger than the RHS, it means that labor demand is larger than the labor supply; thus, we decrease the wage. Let $WH^{new} = W(y_1)$ and $W(y_1)^{new} = \frac{WL+WH^{new}}{2}$. Otherwise, let $WL^{new} = W(y_1)$ and $W(y_1)^{new} = \frac{WL^{new}+WH}{2}$. Repeat steps 2–7 until the market clears.

Dynamic model with exogenous firm size The part that is different from the benchmark model is the lowest wage $W(y_1)$ setting. In step 3, we set the smallest value of $W(y_1)$, which satisfies $\pi(m(\underline{y})) > 0$ as the initial value. Assume $\tilde{W}$ makes $\pi(m(\underline{y})) = 0$; then, we set the initial value of $W(y_1)$ equal to $\tilde{W} - 10^{-5}$. The other steps are the same as in the benchmark model.

Static model with endogenous firm size Since the x distribution is exogenously given, we use one of the x stationary distributions of the benchmark model as the x distribution. We adjust $W(y_1)$ to make the market consume all x and y. This means we make the minimum and maximum of $m(y)$ equal to those of x. Since we use interpolation to obtain matching, the minimum of $m(y)$ is always equal to the minimum of x. We just increase or decrease $W(\underline{y})$ to make the maximum of $m(y)$ equal to the maximum of x when the former is smaller or larger than the latter. Other steps are the same as in the benchmark model.

Static model with exogenous firm size We use the same method as in the dynamic model with exogenous firm size. The differences from the dynamic model with exogenous firm size are as follows: First, we use that of the x stationary distributions of the latter as the x distribution in the former. Second, we use that of the stationary matching rules of the latter as the initial matching rule in the former.

The experiment is implemented using different β, A, μ_y, and σ_y to repeat steps 1–7. Then, we plot the figures with these results.

C.2 Density function $f_X(x)$

To calculate the density function numerically, we follow the steps below:

Step 1: Calculate $\bar{x}$ After obtaining $m(y)$ and $g(m(y))$ in step 3 of section C.1, based on the equation $\bar{x} = \underline{x} + \bar{\theta} g(\bar{x})$, we can get the new $\bar{x}$. First, we use interpolation between $m(y)$ and $g(m(y))$ to obtain $g(\cdot)$. Second, the x lower limit $x_1 = \underline{x}$ is fixed, and therefore we assume $x_u = 10\bar{x}$ and $x_l = \underline{x}$, where $\bar{x} = 5000$ in the first run of the code because we initialize the maximum of $x = 5000$. Let $b(x) = \bar{\theta} g(x) + \underline{x} - x$; we need to calculate $b(x_l)$ and $b(x_u)$. If $b(x_l)b(x_u) < 0$, let $\bar{x}_1 = \frac{x_l + x_u}{2}$; otherwise, there must not exist a new $\bar{x}$, the initial wage support might not be right, and we need to adjust the initial wage support and rerun the code. If the absolute value of $b(\bar{x}_1) \leq 10^{-3}$, then the new $\bar{x} = \bar{x}_1$. Third, if the absolute value of $b(\bar{x}_1) > 10^{-3}$, we need to check the sign of $b(\bar{x}_1)b(x_l)$ and $b(\bar{x}_1)b(x_u)$. If $b(\bar{x}_1)b(x_l) < 0$, let $\bar{x}_2 = \frac{x_l+\bar{x}_1}{2}$ and $x_u = \bar{x}_1$. Otherwise, there must be $b(\bar{x}_1)b(x_u) < 0$; let $\bar{x}_2 = \frac{x_u+\bar{x}_1}{2}$ and $x_l = \bar{x}_1$. If the absolute value of $b(\bar{x}_2) \leq 10^{-3}$, then the new $\bar{x} = \bar{x}_2$. Fourth, repeat the third step until the absolute value of $b(\bar{x}_j) \leq 10^{-3}$, where j is the number of times the third step is repeated.

Step 2: Calculate $\boldsymbol{h(x)}$ Given the new $\bar{x}$, the x support should be $[\underline{x}, \bar{x}]$; then, use the discretized grids of x, $(x_1, x_2, ..., x_{n_b})^T$ to represent x. First, according to $\tilde{x} = \underline{x} + \bar{\theta}g(\underline{x})$, we obtain $\tilde{x}$. Second, when $x_i < \tilde{x}$, $h(x_i) = \underline{x}$; otherwise, $h(x_i) = g^{-1}(\frac{x_i - \underline{x}}{\bar{\theta}}) \leq \bar{x}$.

Step 4: Obtain g(u) In steps 2 and 3, we get $h(x)$ and the new $\bar{x}$; then, the support of u is $[h(x_i), \bar{x}]$. For different x_i, we get different g(u) with $g(\cdot)$ in step 1. In the first run of the code, we assume the initial x distribution $f_X(x)_0$ is a uniform distribution, and the density is $\{\underbrace{1, 1, \cdots, 1}_{n_b}\}/n_b$. Use interpolation between x and $f_X(x)$ to get $f_X(\cdot)$; then, we can get $f_X(u)$. Then, according to Equation (17), we use the trapezoidal rule to obtain $f_X(x_i)$ and normalize $f_X(x_i)$ for all x_i to get the final $f_X(x)_1$.

Step 5: Obtain the stationary distribution $\boldsymbol{f_X(x)}$ From steps 1–4, we obtain $f_X(x)_1$, but it might not be stationary distribution. Thus, we rerun the code with $f_X(x)_1$ as the initial distribution to get $f_X(x)_2$. Then, define a vector equal to $f_X(x)_2 - f_X(x)_1$ and divide the norm of this vector by the root of n_b—the number of elements in this vector. Finally, use this result as dis_fx. If dis_fx is less than $< 10^{-3}$, the distribution is stationary. Otherwise, run the code with $f_X(x)_2, f_X(x)_3, \cdots$ until dis_fx is less than 10^{-3}.

C.3 Numerical methods for the perturbation in section 3

In the main text, we use hat to represent the derivative and the subscript to represent the variable with respect to which we do the derivative. For example, $\hat{L}_A(x)$ means the differentiation of $L(x)$ with respect to A. In the code, $\hat{L}_A(x) = L_{A_2}(x) - L_{A_1}(x)$.

The following steps are employed to accomplish perturbation of the matching rule.

Step 1: Follow section C.1 and solve the model with $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ In this step, we just run the model and solve it following section C.1 with a different A.

Step 2: Obtain the perturbations of the related functions by subtracting the corresponding results In step 1, we obtain all functions with two results (e.g., $L_{A_2}(x)$ and $L_{A_1}(x)$). Then, we can obtain the perturbation of $L(x)$, $\hat{L}_A(x) = L_{A_2}(x) - L_{A_1}(x)$.

Step 3: Calculate the matching perturbation Based on Equation (20) in the paper,

$$\hat{m}_A(y) = -\frac{\int_{\underline{x}}^{m(y)} \hat{L}_A(z) f_X(z) dz}{L(m(y)) f_X(m(y))} - \frac{\int_{\underline{x}}^{m(y)} L(z) \hat{f}_{(X;A)}(z) dz}{L(m(y)) f_X(m(y))}.$$

The LHS can be easily obtained using step 1 and step 2; RHS should use integration. Here, we use the trapezoidal rule. Since the integral domain is from $\underline{x}$ to $m(y)$, the first value of this integration is 0. Consequently, when we use the trapezoidal rule, we need to set $initial = 0$.

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