# Online Appendix of "Existence of the Stationary Equilibrium in an Incomplete-market Model with Endogenous Labor Supply"

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This online appendix contains all proofs of the paper "Existence of the stationary equilibrium in an incomplete-market model with endogenous labor supply."

## 1 Appendix A

### 1.1 Proof of Proposition 1

Proof: 1) $J(y, q)$ is bounded since $u(c, h)$ is bounded.

2) Suppose that $0 < y_1 < y_2$. $c^s(y_1, q)$ and $h^s(y_1, q)$ are the optimal choices

for the intratemporal problem. We have

$$\begin{aligned} J(y_1, q) &= u\left[c^s(y_1, q), h^s(y_1, q)\right] \\ &< u\left[c^s(y_1, q) + y_2 - y_1, h^s(y_1, q)\right] \\ &\leq J(y_2, q). \end{aligned}$$

Thus $J(y, q)$ is strictly increasing in $y$.

For any $y_1, y_2 > 0$ and $y_1 \neq y_2$, we have $(c^s(y_1, q), h^s(y_1, q)) \neq (c^s(y_2, q), h^s(y_2, q))$, since $u(c, h)$ is strictly increasing in $c$ and $h$. Since $u(c, h)$ is strictly concave in $c$ and $h$, we have

$$\begin{aligned} & J\left[\lambda y_1 + (1-\lambda) y_2, q\right] \\ \geq\ & u\left[\lambda c^s(y_1, q) + (1-\lambda) c^s(y_2, q), \lambda h^s(y_1, q) + (1-\lambda) h^s(y_2, q)\right] \\ >\ & \lambda u\left[c^s(y_1, q), h^s(y_1, q)\right] + (1-\lambda) u\left[c^s(y_2, q), h^s(y_2, q)\right] \\ =\ & \lambda J(y_1, q) + (1-\lambda) J(y_2, q), \end{aligned}$$

for $\lambda \in (0, 1)$. Thus, $J(y, q)$ is strictly concave in $y$.

3) By Theorem 3.6 (Theorem of the Maximum) posited by Stokey and Lucas (1989), $c^s(y, q)$ and $h^s(y, q)$ are continuous in $y \in (0, \infty)$.

If Case ii) of Assumption 2 holds, we have $h^s(y, q) = 0$ and $c^s(y, q) = y$. Thus, $c^s(y, q)$ and $h^s(y, q)$ are increasing in $y$.

Next I will concentrate on Case i) of Assumption 2. In this case, we have $h^s(y, q) > 0$ and

$$\frac{u_2\left[c^s(y, q), h^s(y, q)\right]}{u_1\left[c^s(y, q), h^s(y, q)\right]} \geq q,$$

for $y \in (0, \infty)$. For $0 < y_1 < y_2$, $h^s(y_1, q) = 1$ implies that $h^s(y_2, q) = 1$. Suppose that $h^s(y_2, q) < 1$. Then $c^s(y_1, q) < c^s(y_2, q)$. $u_{21}u_1 - u_{11}u_2 > 0$ implies that $\frac{\partial\left(\frac{u_2}{u_1}\right)}{\partial c} > 0$. Additionally, $u_{12}u_2 - u_{22}u_1 > 0$ implies that $\frac{\partial\left(\frac{u_2}{u_1}\right)}{\partial h} < 0$. Therefore, we have

$$\frac{u_2\left[c^s(y_1, q), h^s(y_1, q)\right]}{u_1\left[c^s(y_1, q), h^s(y_1, q)\right]} < \frac{u_2\left[c^s(y_2, q), h^s(y_2, q)\right]}{u_1\left[c^s(y_2, q), h^s(y_2, q)\right]} = q.$$

We have a contradiction. Thus we have $h^s(y_2, q) = 1$. $c^s(y_2, q) = y_2 - q > y_1 - q = c^s(y_1, q)$.

Suppose that $h^s(y, q) \in (0, 1)$ for some $y > 0$. We have

$$u_2\left[c^s(y, q), h^s(y, q)\right] = u_1\left[c^s(y, q), h^s(y, q)\right] q,$$

and

$$c^s(y, q) + h^s(y, q)q = y.$$

Thus, using the Implicit Function Theorem, we have

$$\frac{\partial c^s(y, q)}{\partial y} = \frac{(u_{12}u_2 - u_{22}u_1)u_1}{(u_{12}u_2 - u_{22}u_1)u_1 + (u_{21}u_1 - u_{11}u_2)u_2} > 0,$$

and

$$\frac{\partial h^s(y, q)}{\partial y} = \frac{(u_{21}u_1 - u_{11}u_2)u_1}{(u_{12}u_2 - u_{22}u_1)u_1 + (u_{21}u_1 - u_{11}u_2)u_2} > 0,$$

since $u_{21}u_1 - u_{11}u_2 > 0$ and $u_{12}u_2 - u_{22}u_1 > 0$. Both $c^s(y, q)$ and $h^s(y, q)$ are increasing in $y$.

4) To prove that $J(y, q)$ is differentiable at $y_0 \in (0, \infty)$, note that Assumption 2 implies that $c_0 > 0$, which in turn means that $y_0 - h^s(y_0, e)q > 0$. Thus, for any $y$ belonging to a neighborhood $D$ of $y_0$, $h^s(y_0, q)$ is still feasible. Define $H(y, q)$ on $D$ as $H(y, q) = u\left[y - h^s(y_0, q)q, h^s(y_0, e)\right]$. Thus, $H(y, q)$ is concave and differentiable in $y$. Since $h^s(y_0, q)$ is still feasible for all $y \in D$, it follows that

$$H(y, q) \leq \max_{h \in [0,1]} u(y - hq, h) = J(y, q), \forall y \in D,$$

with equality at $y_0$. Now any subgradient $p$ of $J(y, q)$ at $y_0$ must satisfy

$$p(y - y_0) \geq J(y, q) - J(y_0, q) \geq H(y, q) - H(y_0, q), \forall y \in D,$$

where the first inequality uses the definition of a subgradient and the second uses the fact that $H(y, q) \leq J(y, q)$, with equality at $y_0$. Since $H(y, q)$ is differentiable at $y_0$, $p$ is unique. Following Theorem 25.1 posited by Rockafellar

(1970), any concave function with a unique subgradient at an interior point $y_0$ is differentiable at $y_0$. Thus, $J(y, q)$ is differentiable at $y_0$. Furthermore, we know that $J_1(y_0, q) = H_1(y_0, q) = u_1 [c^s(y_0, q), h^s(y_0, q)]$ for $y_0 \in (0, \infty)$. From part 3) of this proposition, $c^s(y_0, q)$ and $h^s(y_0, q)$ are continuous in $y_0 \in (0, \infty)$. Thus, $J_1(y_0, q)$ is continuous in $y_0 \in (0, \infty)$. ■

## 1.2 Proof of Proposition 2

Proof: 1) This is a direct result from Theorems 9.6, 9.7, and 9.8 from the work of Stokey and Lucas (1989).

2) To prove that $V(a, e)$ is differentiable at $a_0 \in (0, \infty)$, note that Assumption 2 implies that $y_0 > 0$, which in turn means that $Ra_0 + ew - a'(a_0, e) > 0$. Thus, for any $a$ belonging to a neighborhood $D$ of $a_0$, $a'(a_0, e)$ is still feasible. Define $W(a, e)$ on $D$ as $W(a, e) = J [Ra + ew - a'(a_0, e), ew] + \beta E[V(a'(a_0, e), e')|e]$. Thus, $W(a, e)$ is concave and differentiable in $a$. Since $a'(a_0, e)$ is still feasible for all $a \in D$, it follows that

$$W(a, e) \leq \max_{a' \in \Gamma(a,e)} \{J(Ra + ew - a', ew) + \beta E[V(a', e')|e]\} = V(a, e), \forall a \in D,$$

with equality at $a_0$. Now any subgradient $p$ of $V(a, e)$ at $a_0$ must satisfy

$$p(a - a_0) \geq V(a, e) - V(a_0, e) \geq W(a, e) - W(a_0, e), \forall a \in D,$$

where the first inequality uses the definition of a subgradient and the second uses the fact that $W(a, e) \leq V(a, e)$, with equality at $a_0$. Since $W(a, e)$ is differentiable at $a_0$, $p$ is unique. By Theorem 25.1 posited by Rockafellar (1970), any concave function with a unique subgradient at an interior point $a_0$ is differentiable at $a_0$. Thus, $V(a, e)$ is differentiable at $a_0$. Furthermore, we know that $V_1(a_0, e) = W_1(a_0, e) = RJ_1 [y(a_0, e), ew]$ for $a_0 \in (0, \infty)$. From part 1) of this proposition we know that $V(a, e)$ is continuous and concave in $a \in [0, \infty)$.

Thus, using Proposition 6.7.4 in Florenzano and Le Van (2001), we know that $\lim_{a\to 0} V_1(a,e) = V_1^+(0,e)$. Therefore, $V(a,e)$ is continuously differentiable in $a \in [0,\infty)$. We already know that $V_1(a,e) = RJ_1\left[y(a,e), ew\right]$ for $a \in (0,\infty)$. By the Theorem of the Maximum, $y(a,e)$ is continuous in $a \in [0,\infty)$. We also know from part 4) of Proposition 1 that $J_1(y, ew)$ is continuous in $y \in (0,\infty)$. Thus we have $V_1(a,e) = RJ_1\left[y(a,e), ew\right]$ for all $a \in [0,\infty)$.

3) By the Theorem of the Maximum, $a'(a,e)$ is continuous in $a$.

The first-order condition (FOC) of the household's problem is

$$J_1\left[y(a,e), ew\right] \geq \beta E[V_1(a'(a,e), e')|e], \text{ with equality if } a'(a,e) > 0. \tag{A.1}$$

Combining FOC (A.1) and $V_1(a,e) = RJ_1\left[y(a,e), ew\right]$ for all $a \in [0,\infty)$, we have the Euler equation of the household's problem,

$$V_1(a,e) \geq \beta RE[V_1(a'(a,e), e')|e], \text{ with equality if } a'(a,e) > 0. \tag{A.2}$$

For fixed $e \in E$ and any $a_2 > a_1 \geq 0$, we know that either $a'(a_1,e) = 0$ or $a'(a_1,e) > 0$. If $a'(a_1,e) = 0$, then $a'(a_2,e) \geq a'(a_1,e)$. If $a'(a_1,e) > 0$, then we have

$$V_1(a_1,e) = \beta RE[V_1(a'(a_1,e), e')|e].$$

Suppose that $a'(a_2,e) < a'(a_1,e)$. Then, from the Euler equation (A.2), we have

$$V_1(a_2,e) \geq \beta RE[V_1(a'(a_2,e), e')|e] > \beta RE[V_1(a'(a_1,e), e')|e] = V_1(a_1,e),$$

which contradicts the fact that $V(a,e)$ is strictly concave in $a$. Thus we have $a'(a_2,e) \geq a'(a_1,e)$.

4) By the Theorem of the Maximum, $y(a,e)$ is continuous in $a$. From part 2) of this proposition we know that $V_1(a,e) = RJ_1\left[y(a,e), ew\right]$ for all $a \in [0,\infty)$. Thus, $y(a,e)$ is strictly increasing in $a$. ■

## 1.3 Proof of Proposition 3

Proof: 1) By part 4) of Proposition 2, $y(a, e)$ is continuous and strictly increasing in $a$. Since $c^s(y, q)$ and $h^s(y, q)$ are continuous and increasing in $y$ by part 3) of Proposition 1, $c(a, e)$ and $h(a, e)$ are continuous and increasing in $a$.

For $e \in E$, $h(a, e)$ is increasing in $a$ and $h(a, e) \in (0, 1]$. Thus, we have $\lim_{a\to\infty} h(a, e) = \bar{h}(e) \in [0, 1]$. We know that $\lim_{a\to\infty} V_1(a, e) = 0$, since $V(a, e)$ is bounded. Thus,

$$\lim_{a\to\infty} u_1 \left[c(a, e), h(a, e)\right] = 0, \tag{A.3}$$

since $V_1(a, e) = Ru_1 \left[c(a, e), h(a, e)\right]$. Suppose that there exists $\{a_m\}_{m=1}^{\infty}$ and $B > 0$ such tha t$\lim_{m\to\infty} a_m = \infty$ and $c(a_m, e) \leq B$ for all $m \geq 1$. Then we have

$$u_1 \left[c(a, e), h(a, e)\right] \geq u_1 \left[B, h(a, e)\right].$$

Thus,

$$\lim_{a\to\infty} u_1 \left[c(a, e), h(a, e)\right] \geq \lim_{a\to\infty} u_1 \left[B, h(a, e)\right] = u_1 \left[B, \bar{h}(e)\right] > 0,$$

which contradicts Equation (A.3). Therefore, we have $\lim_{a\to\infty} c(a, e) = \infty$.

2) Suppose that $h(a, e) < 1$ for all $a > 0$. Then we have

$$u_2 \left[c(a, e), h(a, e)\right] = u_1 \left[c(a, e), h(a, e)\right] ew.$$

From Equation (A.3) we have

$$\lim_{a\to\infty} u_2 \left[c(a, e), h(a, e)\right] = 0. \tag{A.4}$$

If Case A) of Assumption 5 holds, we can pick $\hat{a} > 0$ such that $u_2 \left[c(\hat{a}, e), 1\right] > 0$. We know that $c(a, e) \geq c(\hat{a}, e)$ for $a > \hat{a}$. Thus, $u_{12} \geq 0$ implies that

$$u_2 \left[c(a, e), h(a, e)\right] \geq u_2 \left[c(\hat{a}, e), h(a, e)\right] > u_2 \left[c(\hat{a}, e), 1\right] > 0,$$

which contradicts Equation (A.4). Thus there exists $\tilde{a} > 0$ such that $h(\tilde{a}, e) = 1$. From part 1) of this proposition we know that $h(a, e)$ is increasing in $a$. Thus we have $h(a, e) = 1$ for $a \geq \tilde{a}$.

Since $E$ is a finite set, we have $h(a, e) = 1$ for sufficiently large $a$ and all $e \in E$. ■

## 1.4 Proof of Theorem 1

Proof: Let $d_t = (\beta R)^t V_1(a_t, e_t)$. The Euler equation (4) implies that

$$d_t \geq E_t(d_{t+1}).$$

Thus, $\{d_t\}_{t=0}^{\infty}$ is a nonnegative supermartingale. We know that $V_1(a_t, e_t)$ is finite since $V_1(a_t, e_t) = Ru_1(c_t, h_t)$. Since $d_0 = V_1(a_0, e_0)$, it follows from the Supermartingale Convergence Theorem that there exists a random variable $d_\infty$ with $E(d_\infty) \leq V_1(a_0, e_0)$ such that $\lim_{t\to\infty} d_t = d_\infty$ almost surely. Thus we have $\lim_{t\to\infty}(\beta R)^t V_1(a_t, e_t) = d_\infty$ almost surely. Since $\beta R > 1$, we have

$$\lim_{t\to\infty} V_1(a_t, e_t) = 0 \ a.s. \tag{A.5}$$

Let $D = \{\omega : \liminf_{t\to\infty} a_t(\omega) < \infty\}$. For each $\omega \in D$, there exists a bounded subsequence $\{a_{t_k}(\omega)\}_{k=1}^{\infty}$ and $B(\omega) > 0$ such that $a_{t_k}(\omega) < B(\omega)$ for all $k \geq 0$. Suppose that the probability of $D$ is positive, i.e. $\Pr(D) > 0$. From Equation (A.5), we can pick a path $\omega \in D$ such that $V_1(a_{t_k}(\omega), e_{t_k}(\omega)) \to 0$ as $k \to \infty$. For convenience I omit $\omega$ in the following derivation. Thus we have

$$V_1(a_{t_k}, e_{t_k}) \geq V_1(B, e_{t_k}) \geq \min_{e\in E}\{V_1(B, e)\} > 0, \forall k \geq 0.$$

We have a contradiction. Thus, $\lim_{t\to\infty} a_t = \infty$ almost surely. ■

## 1.5 Proof of Lemma 1

Proof: The Euler equation (4) implies that

$$V_1(a_t, e_t) \geq E_t V_1(a_{t+1}, e_{t+1}).$$

Thus, $\{V_1(a_t, e_t)\}_{t=0}^{\infty}$ is a nonnegative supermartingale. We know that $V_1(a_t, e_t)$ is finite since $V_1(a_t, e_t) = Ru_1(c_t, h_t)$. Since $d_0 = V_1(a_0, e_0)$, it follows from the Supermartingale Convergence Theorem that there exists a random variable $d_\infty$ with $E(d_\infty) \leq V_1(a_0, e_0)$ such that

$$\lim_{t\to\infty} V_1(a_t, e_t) = d_\infty \ a.s.$$

Moreover, $d_\infty$ is finite almost surely, since $E(d_\infty) \leq V_1(a_0, e_0)$. ■

## 1.6 Proof of Proposition 4

Proof: If Case ii) of Assumption 2 holds, $g(\lambda, e) = 0$ for $\lambda \in (0, \infty)$. Thus, it is decreasing in $\lambda$.

If Case i) of Assumption 2 holds, we have

$$g(\lambda, e) = \min\{v(\lambda, e), 1\}, \lambda \in (0, \infty),$$

for $e \in E$. Therefore, we know that $g(\lambda, e)$ is decreasing in $\lambda \in (0, \infty)$ since $\frac{\partial v(\lambda,e)}{\partial \lambda} < 0$ for $\lambda \in (0, \infty)$. ■

## 1.7 Proof of Lemma 2

Proof: If Case ii) of Assumption 2 holds, we have $\bar{\lambda} = 0$. We know that $\xi(\phi, e) = (U')^{-1}(\phi)$ and $g(\phi, e) = 0$ for $\phi > 0$ and all $e \in E$. Therefore, we have

$$\begin{aligned} \chi(\phi, e^1) &= (U')^{-1}(\phi) - e^1 w \\ &> (U')^{-1}(\phi) - e^2 w = \chi(\phi, e^2) \\ &\cdots \\ &> (U')^{-1}(\phi) - e^n w = \chi(\phi, e^n), \end{aligned}$$

for $\phi > 0$. Thus we have $\chi(\phi, e^1) > \chi(\phi, e^2) > \cdots > \chi(\phi, e^n)$ for $\phi > 0$.

If Case i) of Assumption 2 holds, we have $u_{11}u_{22} - u_{21}u_{12} > 0$. Thus we use the Implicit Function Theorem to find continuous functions $\xi(\lambda, e)$ and $v(\lambda, e)$ on $(0, \infty) \times (0, 2e^n)$ such that

$$u_1\left[\kappa(\lambda, e), v(\lambda, e)\right] = \lambda,$$

and

$$u_2\left[\kappa(\lambda, e), v(\lambda, e)\right] = \lambda e w,$$

for $\lambda > 0$ and $e \in (0, 2e^n)$. From the Implicit Function Theorem we also know that

$$\frac{\partial \kappa(\lambda, e)}{\partial e} = -\frac{u_{22}}{u_{11}u_{22} - u_{21}u_{12}}\lambda w > 0,$$

and

$$\frac{\partial v(\lambda, e)}{\partial e} = \frac{u_{11}}{u_{11}u_{22} - u_{21}u_{12}}\lambda w < 0,$$

for $(\lambda, e) \in (0, \infty) \times (0, 2e^n)$.

For $\lambda > 0$, let

$$e_1(\lambda) = \begin{cases} 0, & \text{if } \Phi_1(\lambda) \text{ is empty} \\ \sup \Phi_1(\lambda), & \text{if } \Phi_1(\lambda) \text{ is not empty} \end{cases},$$

where $\Phi_1(\lambda) = \{e \in (0, 2e^n) : v(\lambda, e) \geq 1\}$. Since $\frac{\partial v(\lambda, e)}{\partial e} < 0$ for $e \in (0, 2e^n)$, we define

$$h = g(\lambda, e) = \begin{cases} 1, & e \in (0, e_1(\lambda)] \\ v(\lambda, e), & e \in (e_1(\lambda), 2e^n) \end{cases},$$

and

$$c = \xi(\lambda, e) = \begin{cases} \vartheta^{-1}(\lambda), & e \in (0, e_1(\lambda)] \\ \kappa(\lambda, e), & e \in (e_1(\lambda), 2e^n) \end{cases},$$

where $\vartheta(c) = u_1(c, 1)$. This way we extend the domain of $\xi(\lambda, e)$ and $g(\lambda, e)$ to $(0, \infty) \times (0, 2e^n)$, which contains $(0, \infty) \times E$. We know that $g(\lambda, e) > 0$ for all $(\lambda, e) \in (0, \infty) \times (0, 2e^n)$.

For $\phi \in (0, \bar{\lambda}]$, we have

$$g(\phi, e) = 1, \forall e \in E,$$

and

$$\chi(\phi, e) = \vartheta^{-1}(\phi), \forall e \in E.$$

For $\phi > \bar{\lambda}$, we have $0 < g(\phi, e) = v(\phi, e) < 1$ and $\xi(\phi, e) = \kappa(\phi, e)$ for all $e \in (e_1(\phi), 2e^n)$. Therefore, we have

$$\begin{aligned} \frac{\partial \chi(\phi, e)}{\partial e} &= \frac{\partial \kappa(\phi, e)}{\partial e} + \frac{\partial v(\phi, e)}{\partial e} ew - (1 - h)w \\ &= -\frac{u_{12}u_1 - u_{11}u_2}{u_{11}u_{22} - u_{21}u_{12}} \frac{\phi w}{u_1} - [1 - g(\phi, e)]\, w < 0, \end{aligned}$$

for $e \in (e_1(\phi), 2e^n)$. Suppose that $e_1(\phi) \geq e^n$. Then we have

$$g(\phi, e) = 1, \forall e \in E,$$

since $E \subset (0, e_1(\phi)]$. This is impossible since, by the definition of $\bar{\lambda}$ (9), we know that, for $\phi > \bar{\lambda}$, there exists $e \in E$ such that $g(\phi, e) < 1$. Thererfore, we have $e_1(\phi) < e^n$ for $\phi > \bar{\lambda}$.

For $\phi > \bar{\lambda}$, if there exists $1 \leq i \leq n - 1$ such that $e_1(\phi) \in [e^i, e^{i+1})$, then we have

$$\chi(\phi, e_1(\phi)) > \chi(\phi, e^{i+1}) > \cdots > \chi(\phi, e^n),$$

since $(e_1(\phi), e^n] \subset (e_1(\phi), 2e^n)$ and $\frac{\partial \chi(\phi, e)}{\partial e} < 0$ for $e \in (e_1(\phi), 2e^n)$. Thus we have

$$\chi(\phi, e^1) = \cdots = \chi(\phi, e^i) = \chi(\phi, e_1(\phi)) > \chi(\phi, e^{i+1}) > \cdots > \chi(\phi, e^n),$$

since $\chi(\phi, e^1) = \cdots = \chi(\phi, e^i) = \chi(\phi, e_1(\phi)) = \vartheta^{-1}(\phi)$. If $e_1(\phi) < e^1$, then we have

$$\chi(\phi, e^1) > \chi(\phi, e^2) > \cdots > \chi(\phi, e^n),$$

since $[e^1, e^n] \subset (e_1(\phi), 2e^n)$. ■

## 1.8 Proof of Lemma 3

Proof: Denote

$$\bar{P} = \min_{(e,e')\in E\times E} \{\pi(e'|e)\}.$$

Choose $T$ such that $\beta^T < \frac{1}{4}$. Let

$$\varepsilon_\phi = \min\left\{\left(\bar{P}\right)^T, \frac{\beta^2}{1-\beta}\frac{\chi(\phi,e^1)-\chi(\phi,e^n)}{4}\right\}.$$

Note that $\varepsilon_\phi > 0$. We denote

$$\bar{\alpha} = \beta\chi(\phi,e_t) + \frac{\beta^2}{1-\beta}\frac{\chi(\phi,e^1)+\chi(\phi,e^n)}{2}.$$

Then we show this lemma in two cases.

Case (i) $\alpha \leq \bar{\alpha}$. Pick event $D_1 = \left\{e_t, e_{t+j-1} = e^1 \text{ for } j = 2,3,\cdots,T+1\right\}$. On $D_1$ we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}\chi(\phi,e_{t+j-1})\beta^j \\
= \ & \beta\chi(\phi,e_t) + \sum_{j=2}^{\infty}\chi(\phi,e^1)\beta^j - \sum_{j=T+2}^{\infty}\left[\chi(\phi,e^1)-\chi(\phi,e_{t+j-1})\right]\beta^j \\
\geq \ & \beta\chi(\phi,e_t) + \sum_{j=2}^{\infty}\chi(\phi,e^1)\beta^j - \sum_{j=T+2}^{\infty}\left[\chi(\phi,e^1)-\chi(\phi,e^n)\right]\beta^j \\
= \ & \beta\chi(\phi,e_t) + \frac{\beta^2}{1-\beta}\chi(\phi,e^1) - \frac{\beta^{T+2}}{1-\beta}\left[\chi(\phi,e^1)-\chi(\phi,e^n)\right] \\
= \ & \beta\chi(\phi,e_t) + \frac{\beta^2}{1-\beta}\frac{\chi(\phi,e^1)+\chi(\phi,e^n)}{2} + \frac{\beta^2}{1-\beta}\frac{\chi(\phi,e^1)-\chi(\phi,e^n)}{2} \\
& -\frac{2\beta^{T+2}}{1-\beta}\frac{\chi(\phi,e^1)-\chi(\phi,e^n)}{2} \\
= \ & \beta\chi(\phi,e_t) + \frac{\beta^2}{1-\beta}\frac{\chi(\phi,e^1)+\chi(\phi,e^n)}{2} + (1-2\beta^T)2\frac{\beta^2}{1-\beta}\frac{\chi(\phi,e^1)-\chi(\phi,e^n)}{4} \\
\geq \ & \bar{\alpha} + (1-2\beta^T)2\varepsilon_\phi \\
> \ & \bar{\alpha} + \varepsilon_\phi \\
\geq \ & \alpha + \varepsilon_\phi.
\end{aligned}$$

We know $\Pr(D_1|e_t) = \Pr\left(e_{t+j-1} = e^1 \text{ for } j = 2, 3, \cdots, T+1|e_t\right) \geq \left(\bar{P}\right)^T \geq \varepsilon_\phi$. Thus, $\Pr\left(\sum_{j=1}^{\infty} \chi(\phi, e_{t+j-1})\beta^j > \alpha + \varepsilon_\phi|e_t\right) \geq \Pr(D_1|e_t) \geq \varepsilon_\phi$. We have

$$\begin{aligned}
& \Pr\left(\alpha \leq \sum_{j=1}^{\infty} \chi(\phi, e_{t+j-1})\beta^j \leq \alpha + \varepsilon_\phi \middle| e_t\right) \\
\leq \ & \Pr\left(\sum_{j=1}^{\infty} \chi(\phi, e_{t+j-1})\beta^j \leq \alpha + \varepsilon_\phi \middle| e_t\right) \\
= \ & 1 - \Pr\left(\sum_{j=1}^{\infty} \chi(\phi, e_{t+j-1})\beta^j > \alpha + \varepsilon_\phi \middle| e_t\right) \\
\leq \ & 1 - \varepsilon_\phi.
\end{aligned}$$

Case (ii) $\alpha > \bar{\alpha}$. Pick event $D_2 = \left\{e_t, e_{t+j-1} = e^n \text{ for } j = 2, 3, \cdots, T+1\right\}$. On $D_2$ we have

$$\begin{aligned}
& \sum_{j=1}^{\infty} \chi(\phi, e_{t+j-1})\beta^j \\
= \ & \beta\chi(\phi, e_t) + \sum_{j=2}^{\infty} \chi(\phi, e^n)\beta^j + \sum_{j=T+2}^{\infty} \left[\chi(\phi, e_{t+j-1}) - \chi(\phi, e^n)\right]\beta^j \\
\leq \ & \beta\chi(\phi, e_t) + \sum_{j=2}^{\infty} \chi(\phi, e^n)\beta^j + \sum_{j=T+2}^{\infty} \left[\chi(\phi, e^1) - \chi(\phi, e^n)\right]\beta^j \\
= \ & \beta\chi(\phi, e_t) + \frac{\beta^2}{1-\beta}\chi(\phi, e^n) + \frac{\beta^2\beta^T}{1-\beta}\left[\chi(\phi, e^1) - \chi(\phi, e^n)\right] \\
< \ & \beta\chi(\phi, e_t) + \frac{\beta^2}{1-\beta}\chi(\phi, e^n) + \frac{\beta^2}{1-\beta}\frac{\chi(\phi, e^1) - \chi(\phi, e^n)}{2} \\
= \ & \beta\chi(\phi, e_t) + \frac{\beta^2}{1-\beta}\frac{\chi(\phi, e^1) + \chi(\phi, e^n)}{2} \\
= \ & \bar{\alpha} \\
< \ & \alpha.
\end{aligned}$$

We know $\Pr(D_2|e_t) = \Pr\left(e_{t+j-1} = e^n \text{ for } j = 2, 3, \cdots, T+1|e_t\right) \geq \left(\bar{P}\right)^T \geq \varepsilon_\phi$.

Thus, $\Pr\left(\sum_{j=1}^{\infty}\chi(\phi, e_{t+j-1})\beta^j < \alpha | e_t\right) \geq \Pr(D_2|e_t) \geq \varepsilon_\phi$. We have

$$\begin{aligned} & \Pr\left(\alpha \leq \sum_{j=1}^{\infty}\chi(\phi, e_{t+j-1})\beta^j \leq \alpha + \varepsilon_\phi \middle| e_t\right) \\ \leq \; & \Pr\left(\sum_{j=1}^{\infty}\chi(\phi, e_{t+j-1})\beta^j \geq \alpha \middle| e_t\right) \\ = \; & 1 - \Pr\left(\sum_{j=1}^{\infty}\chi(\phi, e_{t+j-1})\beta^j < \alpha \middle| e_t\right) \\ \leq \; & 1 - \varepsilon_\phi. \end{aligned}$$

■

## 1.9 Proof of Theorem 2

Proof: From Lemma 1 we know that $\lim_{t\to\infty} V_1(a_t, e_t)$ exists and is finite almost surely for $\beta R = 1$. Suppose that $\Pr\left(\lim_{t\to\infty} V_1(a_t, e_t) \leq R\bar{\lambda}\right) < 1$. Thus,

$$\Pr\left(\lim_{t\to\infty} u_1(c_t, h_t) \leq \bar{\lambda}\right) < 1.$$

Then there exists $\psi > \bar{\lambda}$ such that we have $\Pr(\lim_{t\to\infty} u_1(c_t, h_t) \in [\psi, \psi + \delta]) > 0$ for any $\delta > 0$.

For any $\varepsilon > 0$, let $\eta = \frac{1-\beta}{2\beta}\varepsilon$. We may choose $\phi$ and $\delta$, $\bar{\lambda} < \phi < \psi < \phi + \delta$, such that $\Pr(\lim_{t\to\infty} u_1(c_t, h_t) \in [\phi, \phi + \delta]) > 0$ and $\Pr(\lim_{t\to\infty} u_1(c_t, h_t) = \phi) = \Pr(\lim_{t\to\infty} u_1(c_t, h_t) = \phi + \delta) = 0$. At the same time we can have $|\xi(\phi, e) - \xi(\phi + \delta, e)| < \frac{\eta}{2}$ and $|g(\phi, e) - g(\phi + \delta, e)|ew < \frac{\eta}{2}$ for all $e \in E$, since $\xi(\lambda, e)$ and $g(\lambda, e)$ are uniformly continuous on interval $[\psi, \psi + \delta]$.

Define $B = \{\lim_{t\to\infty} u_1(c_t, h_t) \in [\phi, \phi + \delta]\}$. Define $A_\tau = \{u_1(c_\tau, h_\tau) \in [\phi, \phi + \delta]\}$ and $B_\tau = \{u_1(c_t, h_t) \in [\phi, \phi + \delta], t \geq \tau\}$ for $\tau \geq 0$. Thus, $\lim_{\tau\to\infty} \Pr(A_\tau) = \Pr(B) > 0$ and $\lim_{\tau\to\infty} \Pr(B_\tau) = \Pr(B) > 0$. We may choose $\tau < \infty$ such that $\Pr(B_\tau) > (1 - \varepsilon)\Pr(A_\tau) > 0$. If $V_1(a_t, e_t) \in [R\phi, R(\phi + \delta)]$, then $a_t$ is bounded.

We have

$$\Pr\left( \sum_{j=1}^{\infty} \left[c_{\tau+j-1} - (1 - h_{\tau+j-1})e_{\tau+j-1}w\right] R^{-j} - a_\tau = 0 \middle| B_\tau \right) = 1.$$

Thus we have

$$\Pr\left( \begin{array}{c} \sum_{j=1}^{\infty}[c_{\tau+j-1} - \xi(\phi, e_{\tau+j-1}) + \left[h_{\tau+j-1} - g(\phi, e_{\tau+j-1})\right] e_{\tau+j-1}w]R^{-j} \\ + \sum_{j=1}^{\infty}[\xi(\phi, e_{\tau+j-1}) - \left[1 - g(\phi, e_{\tau+j-1})\right] e_{\tau+j-1}w]R^{-j} - a_\tau = 0 \end{array} \middle| B_\tau \right) = 1.$$

Since $\beta R = 1$ and we know that $|\xi(\phi, e) - \xi(\phi + \delta, e)| < \frac{\eta}{2}$ and $|g(\phi, e) - g(\phi + \delta, e)|ew < \frac{\eta}{2}$ for all $e \in E$,

$$\Pr\left( \begin{array}{c} \left|\sum_{j=1}^{\infty}[c_{\tau+j-1} - \xi(\phi, e_{\tau+j-1}) + \left[h_{\tau+j-1} - g(\phi, e_{\tau+j-1})\right] e_{\tau+j-1}w]R^{-j}\right| \\ < \frac{\beta}{1-\beta}\eta = \frac{\varepsilon}{2} \end{array} \middle| B_\tau \right) = 1.$$

Thus,

$$\Pr\left( \left| \sum_{j=1}^{\infty}[\xi(\phi, e_{\tau+j-1}) - (1 - g(\phi, e_{\tau+j-1}))e_{\tau+j-1}w]R^{-j} - a_\tau \right| < \frac{\varepsilon}{2} \middle| B_\tau \right) = 1.$$

Since $\beta R = 1$ and $\chi(\phi, e) = \xi(\phi, e) - [1 - g(\phi, e)]\, ew$, we have

$$\Pr\left( \left| \sum_{j=1}^{\infty} \chi(\phi, e_{\tau+j-1})\beta^j - a_\tau \right| < \frac{\varepsilon}{2} \middle| B_\tau \right) = 1.$$

Let $\alpha = a_\tau - \frac{\varepsilon}{2}$. Since $B_\tau \subset A_\tau$ and $\Pr(B_\tau) > (1 - \varepsilon)\Pr(A_\tau)$, it follows that

$$\Pr\left( \alpha < \sum_{j=1}^{\infty} \chi(\phi, e_{\tau+j-1})\beta^j < \alpha + \varepsilon \middle| A_\tau \right) > 1 - \varepsilon.$$

Let $z^\tau = (e_0, e_1, \cdots, e_\tau)$. Thus, the event

$$\Pr\left( \alpha < \sum_{j=1}^{\infty} \chi(\phi, e_{\tau+j-1})\beta^j < \alpha + \varepsilon \middle| z^\tau \right) > 1 - \varepsilon$$

has a positive probability since $A_\tau$ is measurable with respect to $z^\tau$. Note that $\{e_t\}_{t=0}^{\infty}$ follows a Markov chain. Thus exists $e_\tau \in E$ such that

$$\Pr\left( \alpha < \sum_{j=1}^{\infty} \chi(\phi, e_{\tau+j-1})\beta^j < \alpha + \varepsilon \middle| e_\tau \right) > 1 - \varepsilon,$$

which contradicts Lemma 3. Thus, we have

$$\Pr\left(\lim_{t\to\infty} u_1(c_t, h_t) \leq \bar{\lambda}\right) = 1.$$

If $\bar{\lambda} > 0$, then $g(\lambda, e) = 1$ for $\lambda \in (0, \bar{\lambda}]$ and all $e \in E$. Thus, we have

$$\lim_{t\to\infty} h_t = 1 \ a.s.$$

If $\bar{\lambda} = 0$, then we have

$$\lim_{t\to\infty} V_1(a_t, e_t) = 0 \ a.s. \tag{A.6}$$

Let $D = \{\omega : \liminf_{t\to\infty} a_t(\omega) < \infty\}$. For each $\omega \in D$, there exists a bounded subsequence $\{a_{t_k}(\omega)\}_{k=1}^{\infty}$ and $B(\omega) > 0$ such that $a_{t_k}(\omega) < B(\omega)$ for all $k \geq 0$. Suppose that the probability of $D$ is positive, i.e., $\Pr(D) > 0$. From Equation (A.6), we can pick a path $\omega$ in $D$ such that $V_1(a_{t_k}(\omega), e_{t_k}(\omega)) \to 0$ as $k \to \infty$. For convenience I omit $\omega$ in the following derivation. Thus, we have

$$V_1(a_{t_k}, e_{t_k}) \geq V_1(B, e_{t_k}) \geq \min_{e\in E}\{V_1(B, e)\} > 0, \forall k \geq 0,$$

We have a contradiction. Therefore, we have

$$\lim_{t\to\infty} a_t = \infty \ a.s.$$

■

## 1.10 Proof of Proposition 5

Proof: From the definition of $\bar{k}$ we know that $h(a, e) = 1$ for $a \geq \bar{k} > 0$ and all $e \in E$. For $a \geq \bar{k}$, suppose that

$$a'(a, \hat{e}(a)) = \max_{e\in E}\{a'(a, e)\} > a.$$

Then, we have

$$V_1(a, \hat{e}(a)) = E[V_1(a'(a, \hat{e}(a)), e')|\hat{e}(a)] < E[V_1(a, e')|\hat{e}(a)]. \tag{A.7}$$

Now, the budget constraint (1) becomes

$$c(a,e) + a'(a,e) = Ra. \tag{A.8}$$

Since

$$a'(a,\hat{e}(a)) = \max_{e\in E}\{a'(a,e)\} \geq a'(a,e),$$

we have $c(a,e) \geq c(a,\hat{e}(a))$ for all $e \in E$. Thus,

$$V_1(a,\hat{e}(a)) = Ru_1(c(a,\hat{e}(a)),1) \geq E[Ru_1(c(a,e'),1)|\hat{e}(a)] = E[V_1(a,e')|\hat{e}(a)],$$

which contradicts Equation (A.7). Thus, we have $a(a,e) \leq a$ for $a \geq \bar{k}$ and all $e \in E$.

For $a \geq \bar{k}$, suppose that there exists $e^{(1)} \in E$ such that $a\left(a,e^{(1)}\right) < a$. Then, we have

$$V_1\left(a,e^{(1)}\right) \geq E\left[V_1\left(a\left(a,e^{(1)}\right),e'\right)|e^{(1)}\right] > E\left[V_1\left(a,e'\right)|e^{(1)}\right].$$

Thus, there exists $e^{(2)} \in E$ such that $V_1\left(a,e^{(2)}\right) < V_1\left(a,e^{(1)}\right)$. Since

$$V_1\left(a,e^{(1)}\right) = Ru_1\left(c\left(a,e^{(1)}\right),1\right),$$

and

$$V_1\left(a,e^{(2)}\right) = Ru_1\left(c\left(a,e^{(2)}\right),1\right),$$

we have $c\left(a,e^{(2)}\right) > c\left(a,e^{(1)}\right)$. Therefore, $a\left(a,e^{(2)}\right) < a\left(a,e^{(1)}\right) < a$. Then, we have

$$V_1\left(a,e^{(2)}\right) \geq E\left[V_1\left(a\left(a,e^{(2)}\right),e'\right)|e^{(2)}\right] > E\left[V_1\left(a,e'\right)|e^{(2)}\right].$$

Thus, there exists $e^{(3)} \in E$ such that $V_1\left(a,e^{(3)}\right) < V_1\left(a,e^{(2)}\right) < V_1\left(a,e^{(1)}\right)$. Since

$$V_1\left(a,e^{(2)}\right) = Ru_1\left(c\left(a,e^{(2)}\right),1\right),$$

and

$$V_1\left(a,e^{(3)}\right) = Ru_1\left(c\left(a,e^{(3)}\right),1\right),$$

we have $c\left(a, e^{(3)}\right) > c\left(a, e^{(2)}\right)$. Thus, $a'\left(a, e^{(3)}\right) < a'\left(a, e^{(2)}\right) < a'\left(a, e^{(1)}\right) < a$. By induction, we have $V_1\left(a, e^{(n)}\right) < \cdots < V_1\left(a, e^{(2)}\right) < V_1\left(a, e^{(1)}\right)$ and $a'\left(a, e^{(n)}\right) < \cdots < a'\left(a, e^{(2)}\right) < a'\left(a, e^{(1)}\right) < a$. From $a'\left(a, e^{(n)}\right) < a$ we know that

$$V_1\left(a, e^{(n)}\right) \geq E\left[V_1\left(a\left(a, e^{(n)}\right), e'\right)|e^{(n)}\right] > E\left[V_1\left(a, e'\right)|e^{(n)}\right].$$

This is impossible since $V_1\left(a, e^{(n)}\right) < \cdots < V_1\left(a, e^{(2)}\right) < V_1\left(a, e^{(1)}\right)$. Thus we know that, for $a \geq \bar{k}$, there does not exist $e \in E$ such that $a\left(a, e\right) < a$. Then, we have $a(a, e) = a$ for $a \geq \bar{k}$ and all $e \in E$.

From the budget constraint (A.8) we have $c(a, e) = (R-1)a = ra$ for $a \geq \bar{k}$ and all $e \in E$.

The borrowing constraint implies that $a_{t+1} \geq 0$ for all $t \geq 0$. Since $a'\left(\bar{k}, e\right) = \bar{k}$ for all $e \in E$, we know that

$$a'(a, e) \leq a'\left(\bar{k}, e\right) = \bar{k},$$

for $a \leq \bar{k}$ and all $e \in E$, from part 3) of Proposition 2. If $a_0 \in [0, \bar{k}]$, $a_1 = a'\left(a_0, e_0\right) \leq \bar{k}$. Thus $a_2 = a'\left(a_1, e_1\right) \leq \bar{k}$. By induction, we have $a_t \leq \bar{k}$ for all $t \geq 1$. Thus, $a_t \in [0, \bar{k}]$ for all $t \geq 0$.

If $a_0 \leq \bar{k}$, wealth accumulation is bounded. Thus we have $\lim_{t\to\infty} h_t = 1$ almost surely from Theorem 2. Consequently, we have

$$\Pr\left(\left\{\omega : \liminf_{t\to\infty} h_t(\omega) < 1\right\}\right) = 0.$$

Let $A = \left\{\omega : \liminf_{t\to\infty} a_t(\omega) = a_*(\omega) < \bar{k}\right\}$. Since $a_*(\omega) < \bar{k}$, there exists $e^* \in E$ such that $h(a_*(\omega), e^*) < 1$. We know that $\Pr\left(e_t = e \text{ infinitely often}\right) = 1$ for each $e \in E$. Since $h(a, e)$ is continuous in $a$ by part 1) of Proposition 3, we have $A \subset \{\omega : \liminf_{t\to\infty} h_t(\omega) < 1\}$. Thus,

$$\Pr(A) \leq \Pr\left(\left\{\omega : \liminf_{t\to\infty} h_t(\omega) < 1\right\}\right) = 0.$$

We have $\Pr(A) = 0$. Thus,

$$\Pr\left(\left\{\omega : \liminf_{t\to\infty} a_t(\omega) \geq \bar{k}\right\}\right) = 1.$$

Therefore, we have $\lim_{t\to\infty} a_t = \bar{k}$ almost surely.

From part 2) of Proposition 3 we know that $\bar{k} < \infty$ in Case A) of Assumption 5. ■

## 1.11 Proof of Proposition 6

Proof: If $\bar{k} < \infty$, from Proposition 5, we know that $\Pr\left(\{(a_t, e_t)\}_{t=0}^{\infty} \text{ is bounded}\right) = 1$. Thus, $\lim_{t\to\infty} a_t = \infty$ almost surely implies that $\bar{k} = \infty$.

To prove the other direction, note that $\Pr(\lim_{t\to\infty} a_t = \infty) < 1$ implies that $\Pr(\lim_{t\to\infty} h_t = 1) = 1$ from Theorem 2. Let $D = \{\omega : \liminf_{t\to\infty} a_t(\omega) < \infty\}$. Thus, $\Pr(D) = 1 - \Pr(\lim_{t\to\infty} a_t = \infty) > 0$. We know that $\Pr(e_t = e \text{ infinitely often}) = 1$ for each $e \in E$. Thus we can find $\omega \in D$ such that, for each $e \in E$, there exists a subsequence $\{\left(a_{t_k^e}(\omega), e_{t_k^e}(\omega)\right)\}_{k=1}^{\infty}$, $\lim_{k\to\infty} a_{t_k^e}(\omega) = B(e) < \infty$, $\lim_{k\to\infty} h\left[a_{t_k^e}(\omega), e_{t_k^e}(\omega)\right] = 1$, and $e_{t_k^e}(\omega) = e$ for all $k \geq 1$. From part 1) of Proposition 3 we know that $h(a, e)$ is continuous and increasing in $a$. Thus we have $h(a, e) = 1$ for $a \geq B(e)$ and $e \in E$. Thus, $\bar{k} < \infty$. Therefore, $\bar{k} = \infty$ implies that $\lim_{t\to\infty} a_t = \infty$ almost surely. ■

## 1.12 Proof of Lemma 4

Proof: For $a > 0$, suppose that $a'(a, e) \geq a$ for all $e \in E$. Then we have

$$a'(a, e) \geq a > 0,$$

for all $e \in E$. Thus,

$$V_1(a, e) = \beta RE[V_1(a'(a, e), e')|e] \leq \beta RE[V_1(a, e')|e] < E[V_1(a, e')|e], \quad \text{(A.9)}$$

for all $e \in E$.

Pick $e^{(1)} \in E$. By Equation (A.9) we have

$$V_1\left(a, e^{(1)}\right) < E\left[V_1\left(a, e'\right)|e^{(1)}\right].$$

Thus there exists $e^{(2)} \in E$ such that $V_1\left(a, e^{(1)}\right) < V_1\left(a, e^{(2)}\right)$. It follows from Equation (A.9) that

$$V_1\left(a, e^{(2)}\right) < E\left[V_1\left(a, e'\right)|e^{(2)}\right].$$

Thus there exists $e^{(3)} \in E$ such that $V_1\left(a, e^{(1)}\right) < V_1\left(a, e^{(2)}\right) < V_1\left(a, e^{(3)}\right)$. By induction, we have $V_1\left(a, e^{(1)}\right) < V_1\left(a, e^{(2)}\right) < \cdots < V_1\left(a, e^{(n)}\right)$.

However, Equation (A.9) also implies that

$$V_1\left(a, e^{(n)}\right) < E\left[V_1\left(a, e'\right)|e^{(n)}\right].$$

This is impossible since $V_1\left(a, e^{(1)}\right) < V_1\left(a, e^{(2)}\right) < \cdots < V_1\left(a, e^{(n)}\right)$. Therefore, for $a > 0$, there exists $e \in E$ such that $a'(a, e) < a$. ■

## 1.13 Proof of Proposition 7

Proof: For $a \geq \bar{k}$, suppose that

$$a'(a, \hat{e}(a)) = \max_{e \in E}\{a'(a, e)\} \geq a.$$

Thus we have

$$\begin{aligned} V_1(a, \hat{e}(a)) &= \beta R E[V_1(a'(a, \hat{e}(a)), e')|\hat{e}(a)] \\ &\leq \beta R E[V_1(a, e')|\hat{e}(a)] < E[V_1(a, e')|\hat{e}(a)], \end{aligned} \tag{A.10}$$

since $\beta R < 1$.

We know that $h(a, e) = 1$ for $a \geq \bar{k}$ and all $e \in E$, by part 2) of Proposition 3. Thus, the budget constraint (1) becomes

$$c(a, e) + a'(a, e) = Ra, \ a \geq \bar{k}.$$

We have $c(a,e) \geq c(a,\hat{e}(a))$ for $a \geq \bar{k}$ and all $e \in E$ since

$$a'(a,\hat{e}(a)) = \max_{e \in E}\{a'(a,e)\} \geq a'(a,e).$$

By Lemma 4, there exists $\tilde{e} \in E$ such that $a'(a,\tilde{e}) < a$. We have $c(a,\tilde{e}) > c(a,\hat{e}(a))$, since

$$a'(a,\hat{e}(a)) \geq a > a'(a,\tilde{e}).$$

Thus we have

$$V_1(a,\hat{e}(a)) = Ru_1(c(a,\hat{e}(a)),1) > E[Ru_1(c(a,e'),1)|\hat{e}(a)] = E[V_1(a,e')|\hat{e}(a)],$$

which contradicts Equation (A.10). Thus, we have $a'(a,e) < a$ for $a \geq \bar{k}$ and all $e \in E$.

From part 2) of Proposition 3 we know that $\bar{k} < \infty$ in Case A) of Assumption 5. ■

## 1.14 Proof of Theorem 3

Proof: If Case A) of Assumption 5 holds, we pick $k^b = \bar{k}$. If Case B) of Assumption 5 holds, we know from Proposition 8 that there exists $k^b > 0$ such that $a'(a,e) < a$ for all $a \geq k^b$ and $e \in E$.

Note that $a_0 \leq \max\left\{k^b, a_0\right\}$. From Propositions 7 and 8, we know that $a'\left(k^b, e\right) < k^b$ for all $e \in E$. From part 3) of Proposition 2 we have

$$a'(a,e) \leq a'\left(k^b, e\right) < k^b,$$

for $a \leq k^b$ and all $e \in E$. Thus,

$$a_{t+1} = a'\left(a_t, e_t\right) \leq k^b, \text{ if } a_t \leq k^b. \tag{A.11}$$

If $k^b < a_t \leq a_0$, $a_{t+1} = a'\left(a_t, e_t\right) < a_t \leq a_0$, by Propositions 7 and 8. Thus, $a_t \leq \max\left\{k^b, a_0\right\}$ implies that $a_{t+1} \leq \max\left\{k^b, a_0\right\}$. By mathematical induction, we have $a_t \leq \max\left\{k^b, a_0\right\}$ for all $t \geq 0$.

Case (i) $a_0 \leq k^b$. We have $a_t \leq k^b$ for all $t \geq 0$. Thus,

$$\Pr\left(a_t \leq k^b,\ \forall t \geq 0\right) = 1.$$

Case (ii) $a_0 > k^b$. Define $\theta = \min\left\{a - \hat{a}(a) : a \in [k^b, a_0]\right\} > 0$. The relationship (A.11) implies that the wealth accumulation process $\{a_t\}_{t=0}^{\infty}$ stays in $[0, k^b]$ if it reaches the interval. Additionally, we know that $\hat{a}(a) < a$ if $a \geq k^b$. Given $a_t \geq k^b$, $a_t$ decreases by at least $\theta$ in one step. Thus, starting from $a_0$, the process $\{a_t\}_{t=0}^{\infty}$ reaches $[0, k^b]$ in at most $\left[\frac{a_0 - k^b}{\theta}\right] + 1$ steps. Then it stays in $[0, k^b]$. Thus,

$$\Pr\left(a_t \leq k^b,\ \forall t \geq \left[\frac{a_0 - k^b}{\theta}\right] + 1\right) = 1.$$

Combining Cases (i) and (ii), we have

$$\Pr\left(a_t \leq k^b,\ \forall t \geq I\right) = 1,$$

where

$$I = \begin{cases} 0, & \text{if } a_0 \leq k^b \\ \left[\frac{a_0 - k^b}{\theta}\right] + 1, & \text{if } a_0 > k^b \end{cases}.$$

■

## 1.15 Proof of Proposition 9

Proof: From the definition $\bar{a}$ in Section 2.3 we know that $a'(\bar{a}, e) \leq \hat{a}(\bar{a}) = \bar{a}$ for all $e \in E$. If $a_t \leq \bar{a}$,

$$a_{t+1} = a'(a_t, e_t) \leq a'(\bar{a}, e_t) \leq \bar{a}.$$

Thus we have

$$\Pr\left((a_t, e_t) \in S, \forall t \geq T \mid (a_T, e_T) \in S\right) = 1. \tag{A.12}$$

Equation (A.12) implies that the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ stays in $S$ if it reaches $S$.

Case (i) $(a_0, e_0) \in S$. Thus, $T = 0$ in Equation (A.12). We have

$$\Pr\left((a_t, e_t) \in S, \forall t \geq 0 \mid (a_0, e_0) \in S\right) = 1.$$

Case (ii) $(a_0, e_0) \notin S$. From Proposition 3 we know that there exists $I \geq 1$ such that

$$\Pr\Big((a_t, e_t) \in [0, \bar{k}] \times E,\ \forall t \geq I\Big) = 1.$$

Let

$$\breve{a}(a) = \min_{e \in E}\{a'(a, e)\}.$$

Thus, $\breve{a}(a)$ is continuous in $a$ since $a'(a, e)$ is continuous in $a$ by part 3) of Proposition 2. By Lemma 4, we have $\breve{a}(a) < a$ for all $a > 0$. Let $\gamma = \min\{a - \breve{a}(a) : a \in [\bar{a}, \bar{k}]\}$. Thus, $\gamma > 0$. Given $a_t \in [\bar{a}, \bar{k}]$, $a_t$ could decrease by at least $\gamma$ in one step. Let

$$q = \left[\frac{\bar{k} - \bar{a}}{\gamma}\right] + 1.$$

From Proposition 3 and the Markov property of the process $\{(a_t, e_t)\}_{t=0}^{\infty}$, we know that the process stays in $[0, \bar{k}] \times E$ if it reaches $[0, \bar{k}] \times E$. We have

$$(\bar{a}, \bar{k}] \times E = \Big\{(a, e) : (a, e) \in [0, \bar{k}] \times E \text{ and } (a, e) \notin S\Big\}.$$

For any $(a, e) \in (\bar{a}, \bar{k}] \times E$, we can pick the realization sequence of labor efficiency shocks $e's$ such that $(a, e)$ moves along $(\breve{a}(a), e)$ to reach $S$ in at most $q$ steps. Let

$$\bar{P} = \min_{(e, e') \in E \times E}\{\pi(e'|e)\}.$$

For any $j \geq 1$ we know that

$$\Pr\Big(\exists\, (j+1) \leq t \leq (j+q),\ \text{such that}\ (a_t, e_t) \in S \,|\, \big(a_j, e_j\big) \in (\bar{a}, \bar{k}] \times E\Big) > \big(\bar{P}\big)^q.$$

Thus we have

$$\begin{aligned} & \Pr\Big((a_t, e_t) \in (\bar{a}, \bar{k}] \times E, t = j+1, j+2, \cdots, j+q \Big| \big(a_j, e_j\big) \in (\bar{a}, \bar{k}] \times E\Big) \\ = \;& 1 - \Pr\Big(\exists\, (j+1) \leq t \leq (j+q),\ \text{such that}\ (a_t, e_t) \in S \,|\, \big(a_j, e_j\big) \in (\bar{a}, \bar{k}] \times E\Big) \\ \leq \;& 1 - \big(\bar{P}\big)^q. \end{aligned}$$

Then, we know that

$$
\begin{aligned}
& \Pr\left((a_t, e_t) \notin S, \forall t \geq 1 | (a_0, e_0) \notin S\right) \\
= \;& \Pr\Big((a_t, e_t) \in (\bar{a}, \bar{k}] \times E, \forall t \geq I \Big| (a_0, e_0) \notin S\Big) \\
\leq \;& \Pr\Big((a_t, e_t) \in (\bar{a}, \bar{k}] \times E, t = I, I+1, \cdots, I+nq \Big| (a_0, e_0) \notin S\Big) \\
= \;& \Pr\Big((a_I, e_I) \in (\bar{a}, \bar{k}] \times E \Big| (a_0, e_0) \notin S\Big) \\
& \times \Pr\Big((a_t, e_t) \in (\bar{a}, \bar{k}] \times E, t = I+1, I+2, \cdots, I+q \Big| (a_I, e_I) \in (\bar{a}, \bar{k}] \times E\Big) \\
& \times \Pr\Big((a_t, e_t) \in (\bar{a}, \bar{k}] \times E, t = I+q+1, I+q+2, \cdots, I+2q \Big| \left(a_{I+q}, e_{I+q}\right) \in (\bar{a}, \bar{k}] \times E\Big) \\
& \times \cdots \\
& \times \Pr\left( \begin{array}{c} (a_t, e_t) \in (\bar{a}, \bar{k}] \times E, \\ t = I+(n-1)q+1, \cdots, I+nq \end{array} \middle| \left(a_{I+(n-1)q}, e_{I+(n-1)q}\right) \in (\bar{a}, \bar{k}] \times E \right) \\
\leq \;& \Pr\Big((a_t, e_t) \in (\bar{a}, \bar{k}] \times E, t = I+1, I+2, \cdots, I+q \Big| (a_I, e_I) \in (\bar{a}, \bar{k}] \times E\Big) \\
& \times \Pr\Big((a_t, e_t) \in (\bar{a}, \bar{k}] \times E, t = I+q+1, I+q+2, \cdots, I+2q \Big| \left(a_{I+q}, e_{I+q}\right) \in (\bar{a}, \bar{k}] \times E\Big) \\
& \times \cdots \\
& \times \Pr\left( \begin{array}{c} (a_t, e_t) \in (\bar{a}, \bar{k}] \times E, \\ t = I+(n-1)q+1, \cdots, I+nq \end{array} \middle| \left(a_{I+(n-1)q}, e_{I+(n-1)q}\right) \in (\bar{a}, \bar{k}] \times E \right) \\
\leq \;& \left[1 - \left(\bar{P}\right)^q\right]^n.
\end{aligned}
$$

Letting $n \to \infty$, we have

$$\Pr\left((a_t, e_t) \notin S, \forall t \geq 1 | (a_0, e_0) \notin S\right) = 0.$$

Thus, we know that

$$
\begin{aligned}
& \Pr\left(\exists T \geq 1, \text{ such that } (a_T, e_T) \in S | (a_0, e_0) \notin S\right) \\
= \;& 1 - \Pr\left((a_t, e_t) \notin S, \forall t \geq 1 | (a_0, e_0) \notin S\right) \\
= \;& 1.
\end{aligned}
$$

■

## 1.16 Proof of Lemma 5

Proof: Suppose that $a'(a,e) > 0$ for $a > 0$ and all $e \in E$. Thus, for $a_0 > 0$, we have

$$V_1(a_0, e_0) = (\beta R)^t E_0 V_1(a_t, e_t), \forall t \geq 0.$$

Note that $V_1(a_0, e_0) > 0$. The right-hand side of this equation approaches 0 as $t \to \infty$, since $\beta R < 1$ and $V_1(a,e) < V_1(0,e) < \infty$ for $a > 0$ and all $e \in E$. We have a contradiction. Thus, there exist $\tilde{a} > 0$ and $\tilde{e} \in E$ such that $a'(\tilde{a}, \tilde{e}) = 0$. From part 3) of Proposition 2, we know that $a'(a, \tilde{e})$ is weakly increasing in $a$. Thus, we have $a'(a, \tilde{e}) = 0$ for $a \in [0, \tilde{a}]$. ■

## 1.17 Proof of Theorem 4

Proof: By Theorem 16.0.2 posited by Meyn and Tweedie (2009), $\{(a_t, e_t)\}_{t=0}^{\infty}$ is uniformly ergodic if the state space $S$ is $v_m-$small for some $m$.

**Definition 1** *A set $C \in \mathbf{B}(S)$ is called a small set if there exists $m > 0$ and non-trivial measure $v_m$ on $\mathbf{B}(S)$ such that $P^m(s, B) \geq v_m(B)$ for all $s \in C$ and $B \in \mathbf{B}(S)$.*

Let $\check{a}(a) = \min_{e \in E}\{a'(a,e)\}$. Thus, $\check{a}(a)$ is continuous in $a$ since $a'(a,e)$ is continuous in $a$ by part 3) of Proposition 2. By Lemma 4, we have $\check{a}(a) < a$ for all $a > 0$. By Lemma 5, there exists $\tilde{a} > 0$ such that $\check{a}(a) = 0$ for $a \leq \tilde{a}$. Let $\kappa = \min\{a - \check{a}(a) : a \in [\tilde{a}, \bar{a}]\}$. Thus, $\kappa > 0$. Let

$$m = \left[\frac{\bar{a}}{\kappa}\right] + 1,$$

and

$$\bar{P} = \min_{(e,e') \in E \times E}\{\pi(e'|e)\}.$$

Define a non-trivial measure $v_m$ on $\mathbf{B}(S)$ as, for all $B \in \mathbf{B}(S)$,

$$v_m(B) = \begin{cases} \left(\bar{P}\right)^m, & \text{if } (0, \tilde{e}) \in B \\ 0, & \text{if } (0, \tilde{e}) \notin B \end{cases},$$

where $\tilde{e}$ is defined in Lemma 5.

For all $s \in S$, we can pick the realization sequence of labor efficiency shocks $e's$ such that $(a, e)$ moves along $(\check{a}(a), e)$ to reach state $s^* = (0, \tilde{e})$ in at most $m$ steps. Thus we have $P^m(s, B) \geq v_m(B)$ for all $s \in S$ and $B \in \mathbf{B}(S)$. We conclude that $S$ is $v_m-$small.

Let $\rho = [1 - v_m(S)]^{\frac{1}{m}}$. Thus, we obtain the results of Theorem 4 through using Theorem 16.0.2 presented by Meyn and Tweedie (2009). ■

## 1.18 Proof of Proposition 10

Proof: From Theorem 4 we know that the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ has a unique stationary distribution $\mu$ on $S$. By Theorem 17.0.1 posited by Meyn and Tweedie (2009), the Law of Large Numbers holds for any $\mathbf{B}(S)-$measurable function $f$ satisfying $\int_S |f| d\mu < \infty$, if $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a positive Harris chain.[1] From their Theorem 18.0.2, we know that $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a positive Harris chain if it satisfies the following three conditions:

1) $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a $T-$chain,[2]

2) There exists a reachable state $s^*$, and

3) $\{(a_t, e_t)\}_{t=0}^{\infty}$ is bounded.[3]

By Theorem 6.2.5 posited by Meyn and Tweedie (2009), $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a $T-$chain if every compact set is petite. A slight change in Proof of Theorem 4 can show that every compact set of $S$ is a small set. By Proposition 5.5.3 posited

[1]For the definition of positive Harris chains, see Meyn and Tweedie (2009) (page 231).

[2]For the definition of $T-$chains, see Meyn and Tweedie (2009) (page 124).

[3]Actually, the theorem only requires it to be bounded in probability.

by Meyn and Tweedie (2009), every small set is a petite set.[4] Thus, $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a $T-$chain. Condition 1) is verified.

From Proof of Theorem 4, we know that $s^* = (0, \tilde{e})$, where $\tilde{e}$ is defined in Lemma 5, and is a reachable state. Thus, condition 2) is satisfied.

Condition 3) is obviously satisfied since $S$ is compact. ■

## 1.19 Proof of Proposition 11

Proof: Let $s^* = (0, \tilde{e})$, where $\tilde{e}$ is defined in Lemma 5. Furthermore, let

$$\tau_{s^*} = \min\{t \geq 1 : (a_t, e_t) = s^*\}.$$

By Theorem 10.2.2 (Kac's Theorem) proposed by Meyn and Tweedie (2009), $E_{s^*}[\tau_{s^*}] < \infty$, and $\mu(s^*) = (E_{s^*}[\tau_{s^*}])^{-1}$ if $\{(a_t, e_t)\}_{t=0}^{\infty}$ is $\psi-$irreducible and positive recurrent. From Proof of Proposition 10 we know that $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a $T-$chain and $s^*$ is a reachable state. By Proposition 6.2.1 posited by Meyn and Tweedie (2009), $\{(a_t, e_t)\}_{t=0}^{\infty}$ is $\psi-$irreducible. From Proof of Proposition 10 we know that $\{(a_t, e_t)\}_{t=0}^{\infty}$ is positive Harris recurrent. Thus, it is positive recurrent. Therefore, we have

$$\mu(\{(a, e) : a = 0\}) \geq \mu(s^*) = (E_{s^*}[\tau_{s^*}])^{-1} > 0.$$

■

## 1.20 Proof of Lemma 6

Proof: Since $f(x)$ is a continuous function of $x \in [b, d]$, it is uniformly continuous on $[b, d]$. Thus, for any $\varepsilon > 0$, there exists a subdivision of $[b, d]$, such that $b = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m(\varepsilon)} = d$ and $0 \leq f(\xi_{i+1}) - f(\xi_i) < \frac{\varepsilon}{2}$ for $0 \leq i \leq m(\varepsilon)$. For any $x \in [b, d]$, there exists $i(x)$ such that $0 \leq i(x) <$

[4]For the definition of petitle sets, see Meyn and Tweedie (2009) (page 117).

$m(\varepsilon)$ and $\xi_{i(x)} \leq x \leq \xi_{i(x)+1}$. Since $f_n(s)$ is weakly increasing in $x$, we have $f_n(\xi_{i(x)}) - f(x) \leq f_n(x) - f(x) \leq f_n(\xi_{i(x)+1}) - f(x)$. Thus, $|f_n(x) - f(x)| \leq \max\{|f_n(\xi_{i(x)}) - f(x)|, |f_n(\xi_{i(x)+1}) - f(x)|\}$. For any $0 \leq i \leq m(\varepsilon)$, there exists $N_i$ such that $|f_n(\xi_i) - f(\xi_i)| < \frac{\varepsilon}{2}$ for all $n > N_i$, since $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in [b, d]$. Let $N = \max\{N_0, N_1, \cdots, N_{m(\varepsilon)}\}$. Thus $n > N$ implies that $|f_n(\xi_i) - f(\xi_i)| < \frac{\varepsilon}{2}$ for any $0 \leq i \leq m(\varepsilon)$. We have $|f_n(\xi_{i(x)}) - f(x)| \leq |f_n(\xi_{i(x)}) - f(\xi_{i(x)})| + |f(\xi_{i(x)}) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Similarly, $|f_n(\xi_{i(x)+1}) - f(x)| < \varepsilon$. Therefore, we have $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [b, d]$. Consequently, we know that $\{f_n\}_{n=1}\infty$ converges uniformly to $f$. ■

## 1.21 Proof of Proposition 12

Proof: I study the household's problem in two steps. In step 1, I solve an intratemporal problem. And, in step 2, I solve an intertemporal problem. In step 1, we know that $J(y, q)$, $c^s(y, q)$, and $h^s(y, q)$ are continuous functions of $y$ and $q$, by the Theorem of the Maximum. In step 2, we know that $V(a, e; w, r)$, $y(a, e; w, r)$, and $a'(a, e; w, r)$ are continuous functions of $a$, $e$, $w$, and $r$, by Theorem 1 posited by Dutta et al. (1994). Thus,

$$c(a, e; w, r) = c^s\left[y(a, e; w, r), ew\right] \text{ is continuous in } a, e, w, \text{ and } r,$$

and

$$h(a, e; w, r) = h^s\left[y(a, e; w, r), ew\right] \text{ is continuous in } a, e, w, \text{ and } r.$$

The firm's profit-maximization conditions in Section 3.1 determine a continuous function $w(r)$ between wage rate $w$ and interest rate $r$. Thus, we know that $c(s; r)$, $h(s; r)$, and $a'(s; r)$ are continuous in $s$ and $r$, where $s = (a, e)$. ■

## 1.22 Proof of Lemma 7

Proof: We prove this lemma in two cases.

*Case A) of Assumption 5 holds.*

For $r_0 \in (-1, \bar{r})$, there exists $0 < \bar{k}(r_0) < \infty$ such that $h(a, e; r_0) = 1$ for $a \geq \bar{k}(r_0)$ and all $e \in E$. Thus we have

$$\frac{u_2 \left[c\left(a, e; r_0\right), 1\right]}{u_1 \left[c\left(a, e; r_0\right), 1\right]} \geq ew, \forall e \in E,$$

for $a \geq \bar{k}(r_0)$. We know that $\frac{u_2(c,1)}{u_1(c,1)}$ is strictly increasing in $c$ since $u_{21}u_1 - u_{11}u_2 > 0$ by Assumption 2. From part 1) of Proposition 3, we know that $\lim_{a\to\infty} c\left(a, e; r_0\right) = \infty$. Thus, we can pick a sufficiently large $k^M(r_0) > \bar{k}(r_0)$ such that

$$\frac{u_2 \left[c\left(k^M(r_0), e; r_0\right), 1\right]}{u_1 \left[c\left(k^M(r_0), e; r_0\right), 1\right]} > ew, \forall e \in E.$$

From Proposition 12, we know that $c(a, e; r)$ is continuous in $r$. Therefore, we could find $\varepsilon > 0$ such that, for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$, we have

$$\frac{u_2 \left[c\left(k^M(r_0), e; r\right), 1\right]}{u_1 \left[c\left(k^M(r_0), e; r\right), 1\right]} > ew, \forall e \in E.$$

Thus, we have $h\left[k^M(r_0), e; r\right] = 1$ for all $e \in E$. By the definition of $\bar{k}$ in Equation (6), we know that $\bar{k}(r) \leq k^M(r_0)$, for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$. From the definition of $\bar{a}$ in Equation (11), we know that $\bar{a}(r) < \bar{k}(r)$, for $r \in (-1, \bar{r})$. Thus, $\bar{a}(r) < \bar{k}(r) < k^M(r_0)$, for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$. For all $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$, we find a uniform upper bound $k^M(r_0)$ for asset accumulation such that $[0, \bar{a}(r)] \subset [0, k^M(r_0)]$.

*Case B) of Assumption 5 holds.*

We want to show that there exists $\varepsilon > 0$ and $0 < k^M(r_0) < \infty$ for $r_0 \in (-1, \bar{r})$ such that $\{\mu(r) : r \in (r_0 - \varepsilon, r_0 + \varepsilon)\}$ has common bounded support $[0, k^M(r_0)] \times E$. Suppose that, for some $e \in E$, we can pick sequence $\{(a_m, r_m)\}_{m=1}^{\infty}$ such that

$a'(a_m, e; r_m) \geq a_m$, $\lim_{m\to\infty} a_m = \infty$, and $\lim_{m\to\infty} r_m = r_0$. Thus, we have

$$\begin{aligned} c(a_m, e; r_m) &= (1 + r_m)a_m - a'(a_m, e; r_m) + (1 - h_m)ew(r_m) \\ &\leq (1 + r_m)a_m - a_m + (1 - h_m)ew(r_m) \\ &= r_m a_m + (1 - h_m)ew(r_m) \\ &\leq r_m a_m + ew(r_m). \end{aligned}$$

We have either $\lim_{m\to\infty} r_m a_m = \infty$ or $\liminf_{m\to\infty} r_m a_m = B < \infty$.

If there exists $B < \infty$ such that $\liminf_{m\to\infty} r_m a_m = B$, then we can find a subsequence $\{(a_{m_i}, r_{m_i})\}_{i=1}^{\infty}$ such that $r_{m_i} a_{m_i} < B + 1$ for $i \geq 1$. Thus, we have $c(a_{m_i}, e; r_{m_i}) \leq B + 1 + ew(r_{m_i})$ for $i \geq 1$. For $\epsilon > 0$ we can find integer $I > 0$ such that $r_{m_i} \in (r_0 - \epsilon, r_0 + \epsilon)$ for all $i \geq I$. Denote $\bar{w} = \max\{w(r) : r \in [r_0 - \epsilon, r_0 + \epsilon]\}$. We know that $\bar{w} < \infty$ since $w(r)$ is continuous in $r$. From part 1) of Proposition 3, we know that $\lim_{a\to\infty} c(a, e; r_0) = \infty$. Thus we can find $A$ such that $c(A, e; r_0) > B + 1 + e\bar{w}$. Since $\lim_{i\to\infty} a_{m_i} = \infty$, there exits integer $\tilde{I} > 0$ such that $a_{m_i} > A$ for all $i \geq \tilde{I}$. Thus we have $c(a_{m_i}, e; r_{m_i}) \geq c(A, e; r_{m_i})$ for all $i \geq \tilde{I}$. Since $c(A, e; r)$ is continuous in $r$ from Proposition 12, we can find $\hat{i} \geq \max\left\{I, \tilde{I}\right\}$ such that $c(A, e; r_{m_{\hat{i}}}) > B + 1 + e\bar{w}$. Therefore,

$$c(a_{m_{\hat{i}}}, e; r_{m_{\hat{i}}}) \geq c(A, e; r_{m_{\hat{i}}}) > B + 1 + e\bar{w} \geq B + 1 + ew(r_{m_{\hat{i}}}) \geq c(a_{m_{\hat{i}}}, e; r_{m_{\hat{i}}}).$$

We have a contradiction.

If $\lim_{m\to\infty} r_m a_m = \infty$, then we have $r_0 > 0$. Thus we could find $\epsilon > 0$ such that $r_0 - \epsilon > 0$ and $\beta(1 + r_0 + \epsilon) < 1$. Denote $\bar{w} = \max\{w(r) : r \in [r_0 - \epsilon, r_0 + \epsilon]\}$. Thus we have $r_m a_m + ew(r_m) \leq r_m a_m + e\bar{w}$. Letting $\Delta = 0$ in Case B) of Assumption 5, we have

$$\Psi(c, 0) = \max_{h,h'\in[0,1]} \left\{ \frac{u_1(c, h')}{u_1(c, h)} \right\}.$$

Thus, for $\bar{\varepsilon} = \frac{1}{2}\left(\frac{1}{\beta(1+r_0+\epsilon)} - 1\right)$, there exists $\overline{\overline{C}} > 0$ such that

$$\frac{u_1(c, h')}{u_1(c, h)} < 1 + \bar{\varepsilon}, \forall h, h' \in [0, 1],$$

for all $c \geq \overline{\overline{C}}$. From Proof of Proposition 8, we know that there exists

$$\overline{\overline{A}} = \frac{\overline{\overline{C}}}{r_0 - \epsilon} > 0,$$

such that

$$c(a, e; r) \geq ra, \forall e \in E, \forall a \geq \overline{\overline{A}}, \forall r \in (r_0 - \epsilon, r_0 + \epsilon).$$

Thus we have $a'(a_m, e; r_m) \geq a_m \geq \overline{\overline{A}} > 0$ for $a_m \geq \overline{\overline{A}}$. Therefore, we know that

$$c(a'(a_m, e; r_m), e'; r_m) \geq r_m a'(a_m, e; r_m) \geq r_m a_m, \forall e \in E,$$

for $a_m \geq \overline{\overline{A}}$ and $r_m \in (r_0 - \epsilon, r_0 + \epsilon)$. Consequently, we have

$$\begin{aligned}
\Phi\left[c(a_m, e; r_m), ew(r_m)\right] &= \beta(1 + r_m)E\left[\Phi(c(a'(a_m, e; r_m), e'; r_m), e'w(r_m))|e\right] \\
&\leq \beta(1 + r_m)E\left[\Phi(r_m a_m, e'w(r_m))|e\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\Phi\left[r_m a_m + e\bar{w}, ew(r_m)\right] &\leq \Phi\left[r_m a_m + ew(r_m), ew(r_m)\right] \\
&\leq \Phi\left[c(a_m, e; r_m), ew(r_m)\right] \\
&\leq \beta(1 + r_m)E\left[\Phi(r_m a_m, e'w(r_m))|e\right].
\end{aligned}$$

Therefore, we have

$$E\left[\left.\frac{\Phi\left[r_m a_m, e'w(r_m)\right]}{\Phi\left[r_m a_m + e\bar{w}, ew(r_m)\right]}\right| e\right] \geq \frac{1}{\beta(1 + r_m)} \geq \frac{1}{\beta(1 + r_0 + \epsilon)},$$

which implies that there exists $e' \in E$ and a subsequence $\{(a_{m_i}, r_{m_i})\}_{i=1}^{\infty}$ such that

$$\begin{aligned}
\max_{h,h' \in [0,1]} \left\{\frac{u_1(r_{m_i} a_{m_i}, h')}{u_1(r_{m_i} a_{m_i} + e\bar{w}, h)}\right\} &\geq \frac{u_1\left[r_{m_i} a_{m_i}, j(r_{m_i} a_{m_i}, e'w(r_{m_i}))\right]}{u_1\left[r_{m_i} a_{m_i} + e\bar{w}, j(r_{m_i} a_{m_i} + e\bar{w}, ew(r_{m_i}))\right]} \\
&\geq \frac{1}{\beta(1 + r_0 + \epsilon)} > 1,
\end{aligned}$$

since $E$ is a finite set. Therefore, we have

$$\limsup_{c \to \infty} \Psi(c, e\bar{w}) \geq \frac{1}{\beta(1 + r_0 + \epsilon)} > 1,$$

which contradicts Case B) of Assumption 5.

Consequently, we know that there exists $\varepsilon > 0$ and $0 < k^M(r_0) < \infty$ for $r_0 \in (-1, \bar{r})$ such that

$$a'(a, e; r) < a, \forall e \in E, \forall a \geq k^M(r_0),$$

for all $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$. From Propoposition 8 and the definiton of $\bar{a}$ in Equation (11), we know that $\bar{a}(r) < k^M(r_0)$, for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$. For all $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$, we find a uniform upper bound $k^M(r_0)$ for asset accumulation such that $[0, \bar{a}(r)] \subset [0, k^M(r_0)]$.

Now we extend measure $\mu(r)$ from $[0, \bar{a}(r)] \times E$ to $[0, k^M(r_0)] \times E$. The unique stationary distribution on $[0, k^M(r_0)] \times E$ is constructed by combining the stationary distribution $\mu(r)$ on $[0, \bar{a}(r)] \times E$ and zero measure on $(\bar{a}(r), k^M(r_0)] \times E$. Without causing confusion, I still use $\mu(r)$ to represent the unique stationary distribution with extended support. Now the collection of the extended measure, $\{\mu(r) : r \in (r_0 - \varepsilon, r_0 + \varepsilon)\}$, has common bounded support $[0, k^M(r_0)] \times E$. ■

## 1.23 Proof of Theorem 6

Proof: From Lemma 7, we know that there exists $k^M(r_0)$ for each $r_0 \in (-1, \bar{r})$, such that $[0, k^M(r_0)] \times E$ containing $S = [0, \bar{a}(r)] \times E$ for all $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$. Thus, $[0, k^M(r_0)] \times E$ is a common bounded support for $\{\mu(r) : r \in (r_0 - \varepsilon, r_0 + \varepsilon)\}$, and $\mu(r)$ is the unique stationary distribution on $[0, k^M(r_0)] \times E$ for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$. We use Theorem 12.13 presented by Stokey and Lucas (1989) to show that $\{\mu(r_m)\}_{m=1}^{\infty}$ converges weakly to $\mu(r_0)$ as $r_m \to r_0$.

*Verification of Conditions (a), (b), and (c) of Theorem 12.13 posited by Stokey and Lucas (1989)*

Condition (a) is satisfied since $[0, k^M(r_0)] \times E$ is compact.

For sequence $\{(s_m, r_m)\}_{m=1}^{\infty}$ where $s_m = (a_m, e_m)$, suppose that $(s_m, r_m) \to$

$(s_0, r_0)$, where $s_0 = (a_0, e_0)$, as $m \to \infty$. For any bounded continuous function $f$ on $[0, k^M(r_0)] \times E$, we have

$$\begin{aligned}
&\int_{[0,k^M(r_0)]\times E} f(s')P_{r_m}(s_m, s') \\
= &\int_{[0,k^M(r_0)]\times E} f(a', e')P_{r_m}\left[(a_m, e_m), (a', e')\right] \\
= &\int_E f\left[a'(a_m, e_m; r_m), e'\right] P(e_m, e') \\
= &\sum_{i=1}^{n} f\left[a'(a_m, e_m; r_m), e^i\right] \pi(e^i|e_m),
\end{aligned}$$

since $P(e_m, e') = \pi(e'|e_m)$ for all $e' \in E$ by Assumption 4. We have $e_m = e_0$ for all large enough $m's$ since $E$ is a finite set. Thus, we have

$$\begin{aligned}
&\lim_{m\to\infty} \int_{[0,k^M(r_0)]\times E} f(s')P_{r_m}(s_m, s') \\
= &\lim_{m\to\infty} \sum_{i=1}^{n} f\left[a'(a_m, e_m; r_m), e^i\right] \pi(e^i|e_m) \\
= &\lim_{m\to\infty} \sum_{i=1}^{n} f\left[a'(a_m, e_0; r_m), e^i\right] \pi(e^i|e_0) \\
= &\sum_{i=1}^{n} f\left[a'(a_0, e_0; r_0), e^i\right] \pi(e^i|e_0),
\end{aligned}$$

where the last line uses that fact that $f\left[a'(a, e_0; r), e^i\right]$ is a continuous function of $(a, r)$ for all $1 \leq i \leq n$. This is true since $f(a', e')$ is continuous in $(a', e')$ and, due to Proposition 12, $a'(a, e; r)$ is a continuous function of $(a, e, r)$. Therefore, we have

$$\begin{aligned}
&\lim_{m\to\infty} \int_{[0,k^M(r_0)]\times E} f(s')P_{r_m}(s_m, s') \\
= &\sum_{i=1}^{n} f\left[a'(a_0, e_0; r_0), e^i\right] \pi(e^i|e_0) \\
= &\int_E f\left[a'(a_0, e_0; r_0), e'\right] P(e_0, e') \\
= &\int_{[0,k^M(r_0)]\times E} f(a', e')P_{r_0}\left[(a_0, e_0), (a', e')\right] \\
= &\int_{[0,k^M(r_0)]\times E} f(s')P_{r_0}(s_0, s').
\end{aligned}$$

Thus, $\{P_{r_m}(s_m, \cdot)\}_{m=1}^{\infty}$ converges weakly to $P_{r_0}(s_0, \cdot)$. Condition (b) is satisfied.

Condition (c) is satisfied since $\mu(r_m)$ is the unique stationary distribution on $[0, k^M(r_0)] \times E$ for each $m \geq 1$.

Thus Theorem 12.13 posited by Stokey and Lucas (1989) implies that $\{\mu(r_m)\}_{m=1}^{\infty}$ converges weakly to $\mu(r_0)$ as $r_m \to r_0$. Thus, we have

$$\lim_{m\to\infty} \int_{[0,k^M(r_0)]\times E} a d\mu(r_m) = \int_{[0,k^M(r_0)]\times E} a d\mu(r_0).$$

We know that $\int_{[0,k^M(r_0)]\times E} a d\mu(r) = \int_S a d\mu(r) = A(r)$ for all $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ since $\mu((\bar{a}(r), k^M(r_0)] \times E) = 0$. Therefore, we have $\lim_{m\to\infty} A(r_m) = A(r_0)$.

Since $\mu((\bar{a}(r), k^M(r_0)] \times E) = 0$, we have

$$\begin{aligned} L(r_m) &= \int_S e\left[1 - h(s; r_m)\right] d\mu(r_m) \\ &= \int_{[0,k^M(r_0)]\times E} e\left[1 - h(s; r_m)\right] d\mu(r_m) \\ &= \int_{[0,k^M(r_0)]\times E} e d\mu(r_m) - \int_{[0,k^M(r_0)]\times E} e h(s; r_m) d\mu(r_m). \end{aligned}$$

The first term $\int_{[0,k^M(r_0)]\times E} e d\mu(r_m)$ converges to $\int_{[0,k^M(r_0)]\times E} e d\mu(r_0)$ as $r_m \to r_0$, since $\mu(r_m)$ converges weakly to $\mu(r_0)$ as $r_m \to r_0$. We only need to show that $\int_{[0,k^M(r_0)]\times E} e h(s; r_m) d\mu(r_m) \to \int_{[0,k^M(r_0)]\times E} e h(s; r_0) d\mu(r_0)$ as $r_m \to r_0$. For fixed $e \in E$, $h(a, e; r_m)$ is a function on $[0, k^M(r_0)]$. By part 1) of Proposition 3, Lemma 6, and Proposition 12, $h(a, e; r_m)$ uniformly converges to $h(a, e; r_0)$ as $r_m \to r_0$. Thus, for $\delta > 0$, we have

$$\max_{a\in[0,k^M]} |h(a, e; r_m) - h(a, e; r_0)| < \frac{\delta}{2e^n}, \forall e \in E,$$

for sufficiently large $m$. Therefore, we have

$$\max_{(a,e)\in[0,k^M]\times E} \{e|h(a, e; r_m) - h(a, e; r_0)|\} < \frac{\delta}{2},$$

for sufficiently large $m$. We also have

$$\left| \int_{[0,k^M(r_0)]\times E} e h(a, e; r_0) d\mu(r_m) - \int_{[0,k^M(r_0)]\times E} e h(a, e; r_0) d\mu(r_0) \right| < \frac{\delta}{2},$$

for sufficiently large $m$, since $eh(a,e;r_0)$ is a bounded continuous function on $[0,k^M]\times E$ and $\mu(r_m)$ converges weakly to $\mu(r_0)$ as $r_m \to r_0$. Thus, we have

$$\begin{aligned}
& \left|\int_{[0,k^M(r_0)]\times E} eh(a,e;r_m)d\mu(r_m) - \int_{[0,k^M(r_0)]\times E} eh(a,e;r_0)d\mu(r_0)\right| \\
\leq & \left|\int_{[0,k^M(r_0)]\times E} eh(a,e;r_m)d\mu(r_m) - \int_{[0,k^M(r_0)]\times E} eh(a,e;r_0)d\mu(r_m)\right| \\
& + \left|\int_{[0,k^M(r_0)]\times E} eh(a,e;r_0)d\mu(r_m) - \int_{[0,k^M(r_0)]\times E} eh(a,e;r_0)d\mu(r_0)\right| \\
\leq & \int_{[0,k^M(r_0)]\times E} e|h(a,e;r_m) - h(a,e;r_0)|d\mu(r_m) \\
& + \left|\int_{[0,k^M(r_0)]\times E} eh(a,e;r_0)d\mu(r_m) - \int_{[0,k^M(r_0)]\times E} eh(a,e;r_0)d\mu(r_0)\right| \\
< & \int_{[0,k^M(r_0)]\times E} \frac{\delta}{2}d\mu(r_m) + \frac{\delta}{2} \\
= & \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\end{aligned}$$

for sufficiently large $m$. Thus we know that $\int_{[0,k^M(r_0)]\times E} eh(s;r_m)d\mu(r_m) \to \int_{[0,k^M(r_0)]\times E} eh(s;r_0)d\mu(r_0)$ as $r_m \to r_0$. Therefore, $\lim_{m\to\infty} L(r_m) = L(r_0)$. ■

## 1.24 Proof of Proposition 13

Proof: From Proposition11, we have $\mu_r(\{(a,e) : a = 0\}) > 0$ for $r \in (-1,\bar{r})$. By Assumption 2 we know that $h(0,e;r) < 1$ for all $e \in E$. Thus, $L(r) > 0$ for $r \in (-1,\bar{r})$. Since

$$\zeta(r) = \frac{A(r)}{L(r)},$$

we know that $\zeta(r)$ is a continuous function of $r \in (-1,\bar{r})$.

From Proposition 6 we know that either $\Pr(\lim_{t\to\infty} a_t = \infty) = 1$ or $\Pr\left(\{(a_t,e_t)\}_{t=0}^{\infty} \text{ is bounded}\right) = 1$ for $\beta R = 1$. We discuss the limit of $\zeta(r)$ as $r \uparrow \bar{r}$ in these two situations.

$\Pr(\lim_{t\to\infty} a_t = \infty) = 1$ *for* $\beta R = 1$.

In this case we want to show that $\lim_{r\uparrow\bar{r}} A(r) = \infty$. Suppose that this is not true. Then there exists $B > 0$ and sequence $\{r_m\}_{m=1}^{\infty}$ such that $r_m \uparrow \bar{r}$ and

$A(r_m) < B$ for all $m \geq 1$. Thus, for any $\hat{k} > 0$, we have

$$\begin{aligned}
& \hat{k}\mu_{r_m}\left\{(a,e) : a > \hat{k}\right\} \\
\leq & \int_{(\hat{k},\infty)\times E} a d\mu(r_m) \\
\leq & \int_{[0,\infty)\times E} a d\mu(r_m) \\
= & \int_S a d\mu(r_m) \\
= & A(r_m) \\
< & B,
\end{aligned}$$

for all $m \geq 1$. Thus, we have

$$\mu_{r_m}\left\{(a,e) : a > \hat{k}\right\} < \frac{B}{\hat{k}}, \forall m \geq 1.$$

We thus know that $\{\mu(r_m)\}_{m=1}^{\infty}$ is tight. Condition (d) of Theorem 7 holds.

Conditions (a) and (c) of Theorem 7 obviously hold. We can also verify condition (b) of Theorem 7, using the same procedure as that in Proof of Theorem 6. For sequence $\{(x_m, r_m)\}_{m=1}^{\infty}$ where $x_m = (a_m, e_m)$, suppose that $\lim_{m\to\infty} x_m = x_0 = (a_0, e_0)$ and $r_m \uparrow \bar{r}$ . For any bounded continous function $f$ on $[0,\infty) \times E$, we have

$$\begin{aligned}
& \lim_{m\to\infty} \int_{[0,\infty)\times E} f(x')P_{r_m}(x_m, x') \\
= & \sum_{i=1}^{n} f\left[a'(a_0, e_0; \bar{r}), e^i\right]\pi(e^i|e_0) \\
= & \int_E f\left[a'(a_0, e_0; \bar{r}), e'\right] P(e_0, e') \\
= & \int_{[0,\infty)\times E} f(a', e')P_{\bar{r}}\left[(a_0, e_0), (a', e')\right] \\
= & \int_{[0,\infty)\times E} f(x')P_{\bar{r}}(x_0, x').
\end{aligned}$$

Thus, $\{P_{r_m}(x_m, \cdot)\}_{m=1}^{\infty}$ converges weakly to $P_{\bar{r}}(x_0, \cdot)$. Condition (b) is satisfied.

Thus, from Theorem 7, we know that there exists a subsequence $\{r_{m_i}\}_{i=1}^{\infty}$ and a proability measure $\hat{\mu}$ such that $\{\mu(r_{m_i})\}_{i=1}^{\infty}$ converges weakly to $\hat{\mu}$ and $\hat{\mu}$ is a stationary distribution for $P_{\bar{r}}(\cdot, \cdot)$ on $[0, \infty) \times E$. This contradicts $\Pr(\lim_{t\to\infty} a_t = \infty) = 1$ for $\beta R = 1$.

$\Pr\left(\{(a_t, e_t)\}_{t=0}^{\infty} \text{ is bounded}\right) = 1 \, for \, \beta R = 1.$

In this case, Proposition 6 implies that there exists $\bar{k}(\bar{r}) < \infty$ such that $h(a, e) = 1$ for $a \geq \bar{k}(\bar{r})$ and $e \in E$. Following the same procedure as that in the first part of Proof of Lemma 7, we pick a sufficiently large $k^M(\bar{r}) > \bar{k}(\bar{r})$ such that

$$\frac{u_2\left[c\left(k^M(\bar{r}), e; \bar{r}\right), 1\right]}{u_1\left[c\left(k^M(\bar{r}), e; \bar{r}\right), 1\right]} > ew, \forall e \in E.$$

From Proposition 12, we know that $c(a, e; r)$ is continuous in $r$ at $\bar{r}$. Therefore, we could find $\varepsilon > 0$ such that, for $r \in (\bar{r} - \varepsilon, \bar{r})$, we have

$$\frac{u_2\left[c\left(k^M(\bar{r}), e; r\right), 1\right]}{u_1\left[c\left(k^M(\bar{r}), e; r\right), 1\right]} > ew, \forall e \in E.$$

Thus we have $h\left[k^M(r_0), e; r\right] = 1$ for all $e \in E$. By the definition of $\bar{k}$ in Equation (6), we know that $\bar{k}(r) \leq k^M(\bar{r})$, for $r \in (\bar{r} - \varepsilon, \bar{r})$. From Proposition 7 and the definition of $\bar{a}$ in Equation (11), we know that $\bar{a}(r) < \bar{k}(r)$, for $r \in (-1, \bar{r})$. Thus, $\bar{a}(r) < \bar{k}(r) < k^M(r_0)$, for $r \in (\bar{r} - \varepsilon, \bar{r})$. For all $r \in (\bar{r} - \varepsilon, \bar{r})$, we find a uniform upper bound $k^M(\bar{r})$ for asset accumulation such that $[0, \bar{a}(r)] \subset [0, k^M(\bar{r})]$. We then use the same procedure as that in Proof of Lemma 7 to extend measure $\mu(r)$ on $[0, \bar{a}(r)] \times E$ to $[0, k^M(\bar{r})] \times E$. The unique stationary distribution on $[0, k^M(\bar{r})] \times E$ is constructed by combining the stationary distribution $\mu(r)$ on $[0, \bar{a}(r)] \times E$ and zero measure on $(\bar{a}(r), k^M(\bar{r})] \times E$. The collection of the extended measure, $\{\mu(r) : r \in (\bar{r} - \varepsilon, \bar{r})\}$, has common bounded support $[0, k^M(\bar{r})] \times E$. For squence $\{r_m\}_{m=1}^{\infty}$ such that $r_m \uparrow \bar{r}$, without loss of generality, we assume that $r_m \in (\bar{r} - \varepsilon, \bar{r})$ for all $m \geq 1$. Since $[0, k^M(\bar{r})] \times E$ is bounded, we know that $\{\mu(r_m)\}_{m=1}^{\infty}$ is tight. Condition (d) of Thoerem 7 holds.

Conditions (a) and (c) of Theorem 7 obviously hold. We can also verify condition (b) of Theorem 7 as above. Thus Thereom 7 implies that there exists a subsequence $\{r_{m_i}\}_{i=1}^{\infty}$ such that $\{\mu(r_{m_i})\}_{i=1}^{\infty}$ on $[0, k^M(\bar{r})] \times E$ converges weakly to a stationary distribution $\mu(\bar{r})$ on $[0, k^M(\bar{r})] \times E$. Moreover, we know that $\lim_{t\to\infty} h_t = 1$ almost surely in this case. Even though there could be infinitely many stationary distributions on $[0, k^M(\bar{r})] \times E$ for $\bar{r}$, we have $\mu_{\bar{r}}(\{(a, e) : h(a, e) = 1\}) = 1$ for any stationary distribution $\mu(\bar{r})$ on $[0, k^M(\bar{r})] \times E$. Following the same procedure as that in Proof of Theorem 6, we have

$$\begin{aligned}
\lim_{i\to\infty} L(r_{m_i}) &= \lim_{i\to\infty} \int_S e\,[1 - h(s; r_{m_i})]\, d\mu(r_{m_i}) \\
&= \lim_{i\to\infty} \int_{[0,k^M(\bar{r})]\times E} e\,[1 - h(s; r_{m_i})]\, d\mu(r_{m_i}) \\
&= \int_{[0,k^M(\bar{r})]\times E} e\,[1 - h(s; \bar{r})]\, d\mu(\bar{r}) = 0,
\end{aligned}$$

We know from Proposition 5 that $\lim_{t\to\infty} a_t = \bar{k}(\bar{r}) > 0$ if $a_0 \in [0, \bar{k}(\bar{r})]$, and $a_t = a_0$ for all $t \geq 0$ if $a_0 > \bar{k}(\bar{r})$. Consequently, we have $\mu_{\bar{r}}(\{(a, e) : a > 0\}) > 0$. Thus,

$$\begin{aligned}
\lim_{i\to\infty} A(r_{m_i}) = \lim_{i\to\infty} \int_S a d\mu(r_{m_i}) &= \lim_{i\to\infty} \int_{[0,k^M(\bar{r})]\times E} a d\mu(r_{m_i}) \\
&= \int_{[0,k^M(\bar{r})]\times E} a d\mu(\bar{r}) > 0.
\end{aligned}$$

Therefore, we know that $\lim_{r\uparrow\bar{r}} L(r) = 0$ and $\liminf_{r\uparrow\bar{r}} A(r) > 0$.

Finally, we have

$$\lim_{r\uparrow\bar{r}} \zeta(r) = \lim_{r\uparrow\bar{r}} \frac{A(r)}{L(r)} = \infty.$$

■

### 1.25 Proof of Theorem 8

Proof: From Proposition 13 we know that $\zeta(r)$ is a continuous function of $r \in (-1, \bar{r})$. We also know that

$$\lim_{r \uparrow \bar{r}} \zeta(r) = \infty.$$

The firm's profit-maximization problem gives us a downward continuous curve of $D(r) = \frac{K}{L}(r)$. Thus, we have

$$\lim_{r \downarrow -\delta} D(r) = \infty,$$

and

$$\lim_{\frac{K}{L} \downarrow 0} r = \infty.$$

There thus exists at least an intersection of these two curves. Additionally, we know that $-\delta < r < \bar{r}$ and $\frac{K}{L} > 0$ in the stationary equilibrium. ■

# 2 Appendix B

### 2.1 Proof of Proposition 8

Proof: If Case ii) of Assumption 2 holds, $r \leq 0$ implies that $c(a, e) > 0 \geq ra$ for $a \geq 0$ and all $e \in E$. We know that $c(a, e) \geq ra$ for $r > 0$, from Proposition 2 posited by Acikgöz (2018). From Proposition 4 posited by Acikgöz (2018), we also know that there exists $k^b > 0$ such that $a'(a, e) < a$ for $a \geq k^b$ and all $e \in E$.

Next I will concentrate on Case i) of Assumption 2.[5] If the borrowing constraint is binding, the indirect utility function $J(Ra+ew, ew)$ of the intratemporal problem is

$$J(Ra + ew, ew) = \max_{c,h} u(c, h)$$

[5]If Case ii) of Assumption 2 holds, we define $\Phi(c, q) = U'(c)$ for all $q > 0$. All results in the following steps also hold.

$$s.t.\ c + hew = Ra + ew,\ \ h \in [0,1].$$

The optimal solutions of this problem are $c^s(Ra + ew, ew)$ and $h^s(Ra + ew, ew)$. We define

$$\psi(a, e) = u_1 \left[c^s(Ra + ew, ew), h^s(Ra + ew, ew)\right],$$

for $(a, e) \in \mathbb{R}_+ \times E$.

If Case i) of Assumption 2 holds, for $q > 0$, there exists function $\varphi(c, q)$ such that

$$u_2 \left[c, \varphi(c, q)\right] = u_1 \left[c, \varphi(c, q)\right] q,$$

by the Implicit Function Theorem. We also know that $\frac{\partial \varphi(c,q)}{\partial c} = \frac{u_{21}u_1 - u_{11}u_2}{u_{12}u_2 - u_{22}u_1} > 0$ for $c > 0$. For $q > 0$, let

$$\sigma_1(q) = \begin{cases} \infty, & \text{if } \Upsilon(q) \text{ is empty} \\ \inf \Upsilon(q), & \text{if } \Upsilon(q) \text{ is not empty} \end{cases},$$

where $\Upsilon(q) = \{c > 0 : \varphi(c, q) \geq 1\}$. Therefore, we have

$$j(c, q) = \begin{cases} \varphi(c, q), & c \in (0, \sigma_1(q)] \\ 1, & c \in (\sigma_1(q), \infty) \end{cases},$$

and

$$\Phi(c, q) = u_1 \left[c, j(c, q)\right] = \begin{cases} u_1 \left[c, \varphi(c, q)\right], & c \in (0, \sigma_1(q)] \\ u_1(c, 1), & c \in (\sigma_1(q), \infty) \end{cases}.$$

For $(a, e) \in \mathbb{R}_+ \times E$, we also observe that

$$h^s(Ra + ew, ew) = j\left[c^s(Ra + ew, ew), ew\right],$$

$$Ra - c^s(Ra + ew, ew) + (1 - j\left[c^s(Ra + ew, ew), ew\right])ew = 0,$$

and

$$\begin{aligned} & \Phi\left[c^s(Ra + ew, ew), ew\right] \\ = \;& u_1 \left[c^s(Ra + ew, ew), j\left[c^s(Ra + ew, ew), ew\right]\right] \\ = \;& u_1 \left[c^s(Ra + ew, ew), h^s(Ra + ew, ew)\right] \\ = \;& \psi(a, e). \end{aligned}$$

Thus, we have $\psi(a,e) = \Phi\left[c^s(Ra+ew, ew), ew\right] \leq \Phi\left[c^s(ew, ew), ew\right] = \psi(0,e) < \infty$ for $(a,e) \in \mathbb{R}_+ \times E$.

Let $\mathcal{L}$ be the set of functions $c : \mathbb{R}_+ \times E \to \mathbb{R}_+$ such that $c(a,e)$ is increasing in $a$, $0 < c(a,e) \leq c^s(Ra+ew, ew)$, and

$$\sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[c(a,e), ew\right] - \psi(a,e)\right| < \infty.$$

For any $c \in \mathcal{L}$, we have

$$\begin{aligned}
& \sup_{(a,e)\in\mathbb{R}_+\times E} \Phi\left[c(a,e), ew\right] \\
\leq & \sup_{(a,e)\in\mathbb{R}_+\times E} \psi(a,e) + \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[c(a,e), ew\right] - \psi(a,e)\right| \\
\leq & \max_{e\in E}\{\psi(0,e)\} + \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[c(a,e), ew\right] - \psi(a,e)\right| \\
< & \infty.
\end{aligned}$$

Thus, $\Phi\left[c(a,e), ew\right]$ is a bounded function of $(a,e) \in \mathbb{R}_+ \times E$.

Define operator $K$ on $\mathcal{L}$ by

$$\begin{aligned}
& \Phi\left[Kc(a,e), ew\right] \\
= & \max\left\{\beta RE\left[\Phi(c(Ra - Kc(a,e) + (1 - j(Kc(a,e), ew))ew, e'), e'w)|e\right], \psi(a,e)\right\}.
\end{aligned}$$

*Claim C1: For $q > 0$, $\Phi(c,q)$ is strictly decreasing in $c \in (0,\infty)$.*

Proof of Claim C1: For $0 < c < \sigma_1(q)$, we have $j(c,q) = \varphi(c,q)$. Thus,

$$\begin{aligned}
\frac{\partial\Phi(c,q)}{\partial c} &= u_{11} + u_{12}\frac{\partial\varphi(c,q)}{\partial c} \\
&= u_{11} + u_{12}\frac{u_{21}u_1 - u_{11}u_2}{u_{12}u_2 - u_{22}u_1} \\
&= -u_1\frac{u_{11}u_{22} - u_{21}u_{12}}{u_{12}u_2 - u_{22}u_1} < 0.
\end{aligned}$$

For $0 < c_1 < \sigma_1(e) < c_2$, we have $j(\sigma_1(q), q) = j(c_2, q) = 1$. Thus,

$$\begin{aligned}\Phi(c_1, q) &> \Phi(\sigma_1(q), q) \\ &= u_1\left[\sigma_1(q), j(\sigma_1(q), q)\right] \\ &= u_1\left[\sigma_1(q), 1\right] \\ &> u_1\left[c_2, 1\right] \\ &= u_1\left[c_2, j(c_2, q)\right] = \Phi(c_2, q).\end{aligned}$$

For $c > \sigma_1(q)$, we have $j(c, q) = 1$ and $\Phi(c, q) = u_1(c, 1)$. Thus,

$$\frac{\partial \Phi(c, q)}{\partial c} = u_{11}(c, 1) < 0.$$

Therefore, for $0 < c_1 < c_2$, we have $\Phi(c_1, q) > \Phi(c_2, q)$. ■

From Claim C1 we know that $\psi(a, e) = \Phi\left[c^s(Ra + ew, ew), ew\right]$ is decreasing in $a$.

*Claim C2: For $c \in \mathcal{L}$, $Kc$ is a well-defined function and $Kc \in \mathcal{L}$.*

Proof of Claim C2: Fix $(a, e) \in \mathbb{R}_+ \times E$ and $c \in \mathcal{L}$. Let

$$\Pi(x) = \max\left\{\beta R E\left[\Phi(c(Ra - x + (1 - j(x, ew))ew, e'), e'w)|e\right], \psi(a, e)\right\},$$

for $0 < x \leq c^s(Ra + ew, ew)$. Thus, $\Pi(x) \geq \psi(a, e)$ for $0 < x \leq c^s(Ra + ew, ew)$. Furthermore, $\Pi(x)$ is increasing in $x$ since we know that $\Phi(x, ew)$ is decreasing in $x$ from Claim C1. We also know that $\Phi(x, ew)$ is strictly decreasing in $x$, $\lim_{x\to 0} \Phi(x, ew) = \lim_{x\to 0} u_1\left[x, \varphi(x, ew)\right] = \infty$, and $\Phi(c^s(Ra + ew, ew), ew) = \psi(a, e)$. Thus, we have a unique solution $0 < x^* \leq c^s(Ra + ew, ew)$ for the quation

$$\Phi(x, ew) = \Pi(x).$$

Let $Kc(a, e) = x^*$. Thus, $Kc$ is a well-defined function.

We know that $0 < Kc(a, e) \leq c^s(Ra+ew, ew)$ since $0 < x^* \leq c^s(Ra+ew, ew)$. To show that $Kc(a, e)$ is increasing in $a$, we suppose that $Kc(a_1, e) > Kc(a_2, e)$

for $0 \leq a_1 < a_2$. Thus,

$$\begin{aligned}
& \Phi\left[Kc(a_1, e), ew\right] \\
< \; & \Phi\left[Kc(a_2, e), ew\right] \\
= \; & \max\left\{\beta R E\left[\Phi(c(Ra_2 - Kc(a_2, e) + (1 - j(Kc(a_2, e), ew))ew, e'), e'w)|e\right], \psi(a_2, e)\right\} \\
\leq \; & \max\left\{\beta R E\left[\Phi(c(Ra_1 - Kc(a_1, e) + (1 - j(Kc(a_1, e), ew))ew, e'), e'w)|e\right], \psi(a_1, e)\right\} \\
= \; & \Phi\left[Kc(a_1, e), ew\right].
\end{aligned}$$

We have a contradiction. Therefore, $Kc(a_1, e) \leq Kc(a_2, e)$.

From $\psi(a, e) = \Phi\left[c^s(Ra + ew, ew), ew\right]$ we have $\psi(0, e) = \Phi\left[c^s(ew, ew), ew\right] = u_1\left[c^s(ew, ew), h^s(ew, ew)\right] < \infty$. We also know that $\Phi\left[Kc(a, e), ew\right] \geq \psi(a, e)$. Thus

$$\begin{aligned}
& \left|\Phi\left[Kc(a, e), ew\right] - \psi(a, e)\right| \\
= \; & \Phi\left[Kc(a, e), ew\right] - \psi(a, e) \\
\leq \; & \max\left\{E\left[\Phi(c(Ra - Kc(a, e) + (1 - j(Kc(a, e), ew))ew, e'), e'w)|e\right], \psi(a, e)\right\} - \psi(a, e) \\
\leq \; & \max\left\{E\left[\Phi(c(0, e'), e'w)|e\right], \psi(a, e)\right\} - \psi(a, e) \\
\leq \; & \max\left\{E\left[\Phi(c(0, e'), e'w)|e\right] - \psi(a, e), 0\right\} \\
\leq \; & E\left[\Phi(c(0, e'), e'w)|e\right] \\
\leq \; & \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[c(a, e), ew\right] - \psi(a, e)\right| + \max_{e\in E}\left\{\psi(0, e)\right\} \\
< \; & \infty.
\end{aligned}$$

Therefore, we have

$$\sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[Kc(a, e), ew\right] - \psi(a, e)\right| < \infty.$$

■

For $c, d \in \mathcal{L}$, define

$$\rho(c, d) = \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[c(a, e), ew\right] - \Phi\left[d(a, e), ew\right]\right|.$$

Thus, we have $\rho(c,d) \geq 0$. We also know that

$$\begin{aligned}\rho(c,d) &= \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[c(a,e),ew\right] - \Phi\left[d(a,e),ew\right]\right| \\ &\leq \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[c(a,e),ew\right] - \psi(a,e)\right| + \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[d(a,e),ew\right] - \psi(a,e)\right| \\ &< \infty.\end{aligned}$$

Apparently, $\rho(c,d) = \rho(c,d)$. If $\rho(c,d) = 0$, we have

$$\Phi\left[c(a,e),ew\right] = \Phi\left[d(a,e),ew\right], \forall(a,e) \in \mathbb{R}_+ \times E.$$

Thus,

$$c(a,e) = d(a,e), \forall(a,e) \in \mathbb{R}_+ \times E,$$

since we know that $\Phi(c,q)$ is strictly decreasing in $c \in (0,\infty)$ from Claim C1. For $b,c,d \in \mathcal{L}$, we have

$$\begin{aligned}\rho(b,d) &= \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[b(a,e),ew\right] - \Phi\left[d(a,e),ew\right]\right| \\ &\leq \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[b(a,e),ew\right] - \Phi\left[c(a,e),ew\right]\right| \\ &\quad + \sup_{(a,e)\in\mathbb{R}_+\times E} \left|\Phi\left[c(a,e),ew\right] - \Phi\left[d(a,e),ew\right]\right| \\ &= \rho(b,c) + \rho(c,d).\end{aligned}$$

Therefore, $(\mathcal{L},\rho)$ is a metric space.

*Claim C3: Metric space $(\mathcal{L},\rho)$ is complete.*

Proof of Claim C3: Suppose that $\{c_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{L},\rho)$. Thus, for each $(a,e) \in \mathbb{R}_+ \times E$, $\{\Phi\left[c_m(a,e),ew\right]\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it has a finite limit $t(a,e)$. For $\varepsilon > 0$, we choose $M_\varepsilon$ such that $m,n \geq M_\varepsilon$ implies that $\rho(c_m,c_n) < \frac{\varepsilon}{2}$, since $\{c_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{L},\rho)$. For each $(a,e) \in \mathbb{R}_+ \times E$ and $m, n \geq M_\varepsilon$, we have

$$
\begin{aligned}
|\Phi[c_m(a,e), ew] - t(a,e)| &\leq |\Phi[c_m(a,e), ew] - \Phi[c_n(a,e), ew]| \\
&\quad + |\Phi[c_n(a,e), ew] - t(a,e)| \\
&\leq \sup_{(a,e)\in\mathbb{R}_+\times E} |\Phi[c_m(a,e), ew] - \Phi[c_n(a,e), ew]| \\
&\quad + |\Phi[c_n(a,e), ew] - t(a,e)| \\
&\leq \rho(c_m, c_n) + |\Phi[c_n(a,e), ew] - t(a,e)| \\
&< \frac{\varepsilon}{2} + |\Phi[c_n(a,e), ew] - t(a,e)|.
\end{aligned}
$$

Since $\lim_{m\to\infty} \Phi[c_m(a,e), ew] = t(a,e)$ for each $(a,e) \in \mathbb{R}_+ \times E$, we can choose $n$ separately for each fixed $(a,e) \in \mathbb{R}_+ \times E$ such that $|\Phi[c_n(a,e), ew] - t(a,e)| < \frac{\varepsilon}{2}$. Therefore, we have

$$
\sup_{(a,e)\in\mathbb{R}_+\times E} |\Phi[c_m(a,e), ew] - t(a,e)| \leq \varepsilon, \tag{A.13}
$$

for $m \geq M_\varepsilon$.

For each $(a,e) \in \mathbb{R}_+ \times E$, we pick $c_0(a,e) > 0$ such that

$$
\Phi[c_0(a,e), ew] = t(a,e). \tag{A.14}
$$

Since $\Phi[c_m(a,e), ew] \geq \psi(a,e) = \Phi[c^s(Ra + ew, ew), ew]$ for all $m \geq 1$, we have $t(a,e) \geq \psi(a,e) = \Phi[c^s(Ra + ew, ew), ew]$. Thus we have $0 < c_0(a,e) \leq c^s(Ra+ew, ew)$. $t(a,e)$ is decreasing in $a$ since $\Phi[c_m(a,e), ew]$ is decreasing in $a$. Therefore, $c_0(a,e)$ is increasing in $a$. Combining Equations (A.13) and (A.14) we have

$$
\rho(c_m, c_0) = \sup_{(a,e)\in\mathbb{R}_+\times E} |\Phi[c_m(a,e), ew] - \Phi[c_0(a,e), ew]| \leq \varepsilon,
$$

for $m \geq M_\varepsilon$. Thus we have

$$\begin{aligned}
& \sup_{(a,e)\in\mathbb{R}_+\times E} |\Phi[c_0(a,e),ew] - \psi(a,e)| \\
\leq & \sup_{(a,e)\in\mathbb{R}_+\times E} |\Phi[c_0(a,e),ew] - \Phi[c_m(a,e),ew]| \\
& + \sup_{(a,e)\in\mathbb{R}_+\times E} |\Phi[c_m(a,e),ew] - \psi(a,e)| \\
\leq & \ \varepsilon + \sup_{(a,e)\in\mathbb{R}_+\times E} |\Phi[c_m(a,e),ew] - \psi(a,e)| \\
< & \ \infty,
\end{aligned}$$

since $c_m \in$ implies that $\sup_{(a,e)\in\mathbb{R}_+\times E} |\Phi[c_m(a,e),ew] - \psi(a,e)| < \infty$. Thus, the Cauchy sequence $\{c_m\}_{m=1}^{\infty}$ converges to $c_0 \in \mathcal{L}$. Therefore, $(\mathcal{L},\rho)$ is a complete metric space. ■

*Claim C4: $\rho(Kc,Kd) \leq \beta R\rho(c,d)$ for all $c,d \in \mathcal{L}$.*

Proof of Claim C4: Pick any $c,d \in$. For each $(a,e) \in \mathbb{R}_+ \times E$, we have

$$\begin{aligned}
& \Phi[Kc(a,e),ew] \\
= & \max\{\beta RE[\Phi(c(Ra - Kc(a,e) + (1 - j(Kc(a,e),ew))ew,e'),e'w)|e],\psi(a,e)\},
\end{aligned}$$

and

$$\begin{aligned}
& \Phi[Kd(a,e),ew] \\
= & \max\{\beta RE[\Phi(d(Ra - Kd(a,e) + (1 - j(Kd(a,e),ew))ew,e'),e'w)|e],\psi(a,e)\}.
\end{aligned}$$

Without loss of generality, we assume that $Kc(a,e) \geq Kd(a,e)$. Thus,

$$\begin{aligned}
& \Phi[Kd(a,e),ew] \\
= & \max\{\beta RE[\Phi(d(Ra - Kd(a,e) + (1 - j(Kd(a,e),ew))ew,e'),e'w)|e],\psi(a,e)\} \\
\leq & \max\{\beta RE[\Phi(d(Ra - Kc(a,e) + (1 - j(Kc(a,e),ew))ew,e'),e'w)|e],\psi(a,e)\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \Phi[Kd(a,e),ew] - \Phi[Kc(a,e),ew] \\
\leq & \max\{\beta RE[\Phi(d(Ra - Kc(a,e) + (1 - j(Kc(a,e),ew))ew,e'),e'w)|e],\psi(a,e)\} \\
& - \max\{\beta RE[\Phi(c(Ra - Kc(a,e) + (1 - j(Kc(a,e),ew))ew,e'),e'w)|e],\psi(a,e)\}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
& |\Phi\left[Kc(a,e),ew\right] - \Phi\left[Kd(a,e),ew\right]| \\
\leq & \left| \begin{array}{c} \beta RE\left[\Phi(c(Ra - Kc(a,e) + (1 - j(Kc(a,e),ew))ew, e'), e'w)|e\right] \\ -\beta RE\left[\Phi(d(Ra - Kc(a,e) + (1 - j(Kc(a,e),ew))ew, e'), e'w)|e\right] \end{array} \right| \\
\leq & \ \beta RE\left[ \left| \begin{array}{c} \Phi(c(Ra - Kc(a,e) + (1 - j(Kc(a,e),ew))ew, e'), e'w) \\ -\Phi(d(Ra - Kc(a,e) + (1 - j(Kc(a,e),ew))ew, e'), e'w) \end{array} \right| e \right] \\
\leq & \ \beta R\left( \sup_{(a',e')\in\mathbb{R}_+\times E} |\Phi\left[c(a',e'),e'w\right] - \Phi\left[d(a',e'),e'w\right]| \right) \\
= & \ \beta R\rho(c,d).
\end{aligned}
$$

Therefore, we have $\rho(Kc, Kd) \leq \beta R\rho(c, d)$. ■

By Theorem 3.2 (Contraction Mapping Theorem) in Stokey and Lucas (1989), we know that the operator $K$ has a unique fixed point $c \in \mathcal{L}$.[6] Starting from any $c^1 \in \mathcal{L}$, we generate a sequence $\left\{c^i\right\}_{i=1}^{\infty}$ by letting $c^{i+1} = Kc^i$ for all $i \geq 1$. We also know that $\lim_{i\to\infty} \rho(c^i, c) = 0$. This fixed point $c$ is the candidate optimal policy function of the original dynamic uitlity maximization problem.

If Case B) of Assumption 5 holds, we have

$$
\limsup_{c\to\infty} \Psi(c, \Delta) \leq 1, \forall\Delta \geq 0,
$$

where

$$
\Psi(c, \Delta) = \max_{h,h'\in[0,1]} \left\{ \frac{u_1(c, h')}{u_1(c + \Delta, h)} \right\}.
$$

[6]An important implication of this contraction-mapping argument is that $u_1(c, h)$ is bounded. Furthermore, we know that $u_1\left[c(0, e), h(0, e)\right]$ is bounded for all $e \in E$. Thus, $\min_{e\in E} \{c(0, e)\} > 0$ is the lower bound of consumption. To use this contraction-mapping argument, we do not need Assumption 5. Moreover, this argument does not need the assumption that the utility function $u(c, h)$ has a lower bound. Li and Stachurski (2014), Acikgöz (2018), and Stachurski and Toda (2019) apply this contraction-mapping argument to income fluctuation problems with exogenous labor supply.

Letting $\Delta = 0$, we have

$$\Psi(c, 0) = \max_{h,h' \in [0,1]} \left\{ \frac{u_1(c, h')}{u_1(c, h)} \right\}.$$

Thus, for $\varepsilon = \frac{1}{2}\left(\frac{1}{\beta R} - 1\right)$, there exists $\bar{C} > 0$ such that

$$\frac{u_1(c, h')}{u_1(c, h)} < 1 + \varepsilon, \forall h, h' \in [0, 1],$$

for all $c \geq \bar{C}$. Thus, there exists

$$\bar{A} = \frac{\bar{C}}{r} > 0,$$

such that

$$\frac{u_1(ra, h')}{u_1(ra, h)} < 1 + \varepsilon, \forall h, h' \in [0, 1], \forall a \geq \bar{A}.$$

*Claim C5: The fixed point of K satisfies*

$$c(a, e) \geq ra, \forall e \in E,$$

*for* $a \geq \bar{A}$.

Proof of Claim C5: If $r \leq 0$, then we have $c(a, e) > 0 \geq ra$ for $a \geq 0$ and all $e \in E$.

If $r > 0$, we pick $c^1 \in \mathcal{L}$, such that $c^1(a, e) = c^s(Ra + ew, ew)$ for $a \geq 0$ and all $e \in E$. We have

$$\begin{aligned} c^1(a, e) &= c^s(Ra + ew, ew) \\ &= Ra + (1 - j\,[c^s(Ra + ew, ew), ew])ew \\ &\geq ra, \end{aligned}$$

for $a \geq 0$ and all $e \in E$.

For $i \geq 1$, suppose that

$$c^i(a, e) \geq ra, \forall (a, e) \in [\bar{A}, \infty) \times E.$$

We want to show that

$$c^{i+1}(a,e) = Kc^i(a,e) \geq ra, \forall (a,e) \in [\bar{A}, \infty) \times E.$$

Suppose that this is not true. Then we know that there exists $(a,e) \in [\bar{A}, \infty) \times E$ such that

$$c^{i+1}(a,e) = Kc^i(a,e) < ra.$$

Thus,

$$\begin{aligned}
& c^i(Ra - Kc^i(a,e) + (1 - j(Kc^i(a,e), ew))ew, e') \\
\geq\ & r\left[Ra - Kc^i(a,e) + (1 - j(Kc^i(a,e), ew))ew\right] \\
\geq\ & r\left[Ra - Kc^i(a,e)\right] \\
>\ & r(Ra - ra) \\
=\ & ra.
\end{aligned}$$

Therefore, we have

$$\Phi(Kc^i(a,e), ew) = \beta RE\left[\Phi(c^i(Ra - Kc^i(a,e) + (1 - j(Kc^i(a,e), ew))ew, e'), e'w)|e\right],$$

since $Ra - Kc^i(a,e) + (1 - j(Kc^i(a,e), ew))ew > a \geq \bar{A} > 0$. Thus,

$$\begin{aligned}
\Phi(ra, ew) &< \Phi(Kc^i(a,e), ew) \\
&= \beta RE\left[\Phi(c^i(Ra - Kc^i(a,e) + (1 - j(Kc^i(a,e), ew))ew, e'), e'w)|e\right] \\
&< \beta RE\left[\Phi(ra, e'w)|e\right].
\end{aligned}$$

Thus we have

$$\begin{aligned}
1 &< \beta RE\left[\left.\frac{\Phi(ra, e'w)}{\Phi(ra, ew)}\right| e\right] \\
&= \beta RE\left[\left.\frac{u_1\left[ra, j(ra, e'w)\right]}{u_1\left[ra, j(ra, ew)\right]}\right| e\right] \\
&< \beta R(1 + \varepsilon) = \frac{1}{2}(\beta R + 1) < 1.
\end{aligned}$$

We have a contradiction.

By mathetical induction, we have, for all $(a, e) \in [\bar{A}, \infty) \times E$,

$$c^i(a, e) \geq ra, \forall i \geq 1.$$

Thus we have

$$\Phi(c^i(a, e), ew) \leq \Phi(ra, ew), \forall i \geq 1.$$

since we know from Claim C1 that $\Phi(\cdot, ew)$ is a strictly decreasing function. Since $\lim_{i\to\infty} \rho(c^i, c) = 0$ implies that $\lim_{i\to\infty} \Phi(c^i(a, e), ew) = \Phi(c(a, e), ew)$, we have $\Phi(c(a, e), ew) \leq \Phi(ra, ew)$, i.e.

$$c(a, e) \geq ra.$$

■

*Claim C6: The first-order conditions*

$$u_1(c_t, h_t) \geq \beta R E_t u_1(c_{t+1}, h_{t+1}), \text{ with equality if } a_{t+1} > 0, \tag{A.15}$$

$$u_2(c_t, h_t) \geq u_1(c_t, h_t) ew, \text{ with equality if } h_t < 1, \tag{A.16}$$

*and the transversality condition*

$$\lim_{t\to\infty} E_0 \beta^t u_1(c_t, h_t) a_{t+1} = 0, \tag{A.17}$$

*are sufficient for the optimal solution of the original dynamic utility maximization problem.*

Proof of Claim C6: For $a_0 \geq 0$, $\{(c_t, h_t, a_{t+1})\}_{t=0}^{\infty}$ is a feasible sequence satisfying

$$c_t + a_{t+1} = Ra_t + (1 - h_t)e_t w, \forall t \geq 0,$$

and

$$a_{t+1} \geq 0, \forall t \geq 0.$$

The path $\{(c_t, h_t, a_{t+1})\}_{t=0}^{\infty}$ satisfies the first-order conditions and the transversality condition. $\left\{(\hat{c}_t, \hat{h}_t, \hat{a}_{t+1})\right\}_{t=0}^{\infty}$ is an alternative feasible sequence starting from $\hat{a}_0 = a_0$ and satisfying

$$\hat{c}_t + \hat{a}_{t+1} = R\hat{a}_t + (1 - \hat{h}_t)e_t w, \forall t \geq 0,$$

and

$$\hat{a}_{t+1} \geq 0, \forall t \geq 0.$$

From the budget constraints, we have

$$c_t - \hat{c}_t = R\left(a_t - \hat{a}_t\right) - \left(a_{t+1} - \hat{a}_{t+1}\right) - \left(h_t - \hat{h}_t\right) e_t w.$$

Since $u(c, h)$ is strictly concave in $c$ and $h$, we have

$$\begin{aligned}
& u(c_t, h_t) - u(\hat{c}_t, \hat{h}_t) \\
\geq \; & u_1(c_t, h_t)(c_t - \hat{c}_t) + u_2(c_t, h_t)(h_t - \hat{h}_t) \\
\geq \; & u_1(c_t, h_t)\left[R\left(a_t - \hat{a}_t\right) - \left(a_{t+1} - \hat{a}_{t+1}\right) - \left(h_t - \hat{h}_t\right) e_t w\right] + u_2(c_t, h_t)(h_t - \hat{h}_t) \\
\geq \; & u_1(c_t, h_t)\left[R\left(a_t - \hat{a}_t\right) - \left(a_{t+1} - \hat{a}_{t+1}\right)\right] + \left[u_2(c_t, h_t) - u_1(c_t, h_t)e_t w\right]\left(h_t - \hat{h}_t\right).
\end{aligned}$$

From the labor-leisure decision equation (A.16), we know that $h_t < 1$ implies that $u_2(c_t, h_t) - u_1(c_t, h_t)e_t w = 0$. Furthermore, $h_t = 1$ implies that $h_t - \hat{h}_t \geq 0$. In these two cases we have

$$\left[u_2(c_t, h_t) - u_1(c_t, h_t)e_t w\right]\left(h_t - \hat{h}_t\right) \geq 0.$$

Therefore, we have

$$\begin{aligned}
& u(c_t, h_t) - u(\hat{c}_t, \hat{h}_t) \\
\geq \; & u_1(c_t, h_t)\left[R\left(a_t - \hat{a}_t\right) - \left(a_{t+1} - \hat{a}_{t+1}\right)\right].
\end{aligned}$$

For $T \geq 1$ we have

$$
\begin{aligned}
& E_0{}_{t=0}^{T}\beta^t \left[u(c_t, h_t) - u(\hat{c}_t, \hat{h}_t)\right] \\
\geq \; & E_0{}_{t=0}^{T}\beta^t u_1(c_t, h_t)\left[R\left(a_t - \hat{a}_t\right) - \left(a_{t+1} - \hat{a}_{t+1}\right)\right] \\
= \; & E_0{}_{t=0}^{T-1}\beta^t \left[u_1(c_t, h_t) - \beta R E_t u_1(c_{t+1}, h_{t+1})\right]\left(\hat{a}_{t+1} - a_{t+1}\right) \\
& - E_0 \beta^T u_1(c_T, h_T)\left(a_{T+1} - \hat{a}_{T+1}\right).
\end{aligned}
$$

From the Euler equation (A.15) we know that $a_{t+1} > 0$ implies that $u_1(c_t, h_t) - \beta R E_t u_1(c_{t+1}, h_{t+1}) = 0$. Moreover, $a_{t+1} = 0$ implies that $\hat{a}_{t+1} - a_{t+1} \geq 0$. In these two cases we have

$$
\left[u_1(c_t, h_t) - \beta R E_t u_1(c_{t+1}, h_{t+1})\right]\left(\hat{a}_{t+1} - a_{t+1}\right) \geq 0.
$$

Therefore, we have

$$
E_0{}_{t=0}^{T-1}\beta^t \left[u_1(c_t, h_t) - \beta R E_t u_1(c_{t+1}, h_{t+1})\right]\left(\hat{a}_{t+1} - a_{t+1}\right) \geq 0.
$$

Thus, we have

$$
\begin{aligned}
E_0{}_{t=0}^{T}\beta^t \left[u(c_t, h_t) - u(\hat{c}_t, \hat{h}_t)\right] & \geq -E_0 \beta^T u_1(c_T, h_T)\left(a_{T+1} - \hat{a}_{T+1}\right) \\
& \geq -E_0 \beta^T u_1(c_T, h_T) a_{T+1},
\end{aligned}
$$

since $\hat{a}_{T+1} \geq 0$. By the transversality condition (A.17), we have

$$
E_0{}_{t=0}^{\infty}\beta^t \left[u(c_t, h_t) - u(\hat{c}_t, \hat{h}_t)\right] \geq -\lim_{T \to \infty} E_0 \beta^T u_1(c_T, h_T) a_{T+1} = 0.
$$

Thus, the path $\{(c_t, h_t, a_{t+1})\}_{t=0}^{\infty}$ is optimal. ■

Now I verify that the fixed point of operator $K$ satisfies all the conditions in Claim C6. By the construction of the opertaor $K$, its fixed point $c \in \mathcal{L}$ satisfies the first-order condtitions (A.15) and (A.16). We only need to verify the transversality condition (A.17). For any $c \in \mathcal{L}$, $\Phi\left[c(a, e), ew\right]$ is a bounded function of $(a, e) \in \mathbb{R}_+ \times E$. Thus, $\{u_1(c_t, h_t)\}_{t=0}^{\infty}$ is bounded. Then, we only need to show

$$\lim_{t\to\infty} E_0\beta^t a_{t+1} = 0.$$

From Claim C5 we have

$$\begin{aligned} a_{t+1} &= Ra_t - c_t + (1 - h_t)e_t w \\ &\leq Ra_t - ra_t + (1 - h_t)e_t w \\ &\leq a_t + e_t w \\ &\leq a_t + e^n w, \end{aligned}$$

for all $t \geq 0$. Thus, we have

$$a_{t+1} \leq a_0 + (t+1)e^n w.$$

Apparently, we have $\lim_{t\to\infty} E_0\beta^t a_{t+1} = 0$.

Suppose that, for some $e \in E$, we can pick sequence $\{a_m\}_{m=1}^{\infty}$ such that $a'(a_m, e) \geq a_m$ for $m \geq 1$, and $\lim_{m\to\infty} a_m = \infty$. Thus, we have

$$\begin{aligned} c(a_m, e) &= Ra_m - a'(a_m, e) + (1 - h_m)ew \\ &\leq Ra_m - a_m + (1 - h_m)ew \\ &= ra_m + (1 - h_m)ew \\ &\leq ra_m + ew. \end{aligned}$$

If $r \leq 0$, then $c(a_m, e) \leq ew$ for $m \geq 1$. We have a contradiction since $\lim_{m\to\infty} a_m = \infty$ implies that $\lim_{m\to\infty} c(a_m, e) = \infty$ from part 1) of Proposition 3.

If $r > 0$, we have $a'(a_m, e) \geq a_m \geq \bar{A} > 0$ for $a_m \geq \bar{A}$. Thus, we know that

$$c(a'(a_m, e), e') \geq ra'(a_m, e) \geq ra_m, \forall e \in E,$$

from Claim C5. Therefore, we have

$$\Phi\left[c(a_m, e), ew\right] = \beta RE\left[\Phi(c(a'(a_m, e), e'), e'w)|e\right] \leq \beta RE\left[\Phi(ra_m, e'w)|e\right].$$

Thus,

$$\Phi(ra_m + ew, ew) \leq \Phi\left[c(a_m, e), ew\right] \leq \beta RE\left[\Phi(ra_m, e'w)|e\right].$$

Therefore, we have

$$E\left[\left.\frac{\Phi(ra_m, e'w)}{\Phi(ra_m + ew, ew)}\right| e\right] \geq \frac{1}{\beta R},$$

which implies that there exists $e' \in E$ and a subsequence $\{a_{m_i}\}_{i=1}^{\infty}$ such that

$$\max_{h,h' \in [0,1]} \left\{\frac{u_1(ra_{m_i}, h')}{u_1(ra_{m_i} + ew, h)}\right\} \geq \frac{u_1\left[ra_{m_i}, j(ra_{m_i}, e'w)\right]}{u_1\left[ra_{m_i} + ew, j(ra_{m_i} + ew, ew)\right]} \geq \frac{1}{\beta R} > 1,$$

since $E$ is a finite set. Therefore, we have

$$\limsup_{c \to \infty} \Psi(c, ew) \geq \frac{1}{\beta R} > 1,$$

which contradicts Case B) of Assumption 5.

Consequently, we know that there exists $k^b > 0$ such that

$$a'(a, e) < a, \forall e \in E,$$

for $a \geq k^b$. ■

# 3 Appendix C

## 3.1 Proof of Theorem 7

Proof: For any bounded continuous function $f$ on $X$, define

$$(T_\theta f)(x) = \int_X f(x')P_\theta(x, dx'), \forall x \in X, \forall \theta \in \Theta,$$

and

$$\langle f, \lambda \rangle = \int_X f(x)\lambda(dx), \forall \lambda \in \Lambda(X, \mathbf{B}(X)).$$

Define operator $T_\theta^*$ on $\Lambda(X, \mathbf{B}(X))$ by

$$(T_\theta^* \lambda)(B) = \int_X P_\theta(x, B)\lambda(dx), \forall B \in \mathbf{B}(X).$$

From Theorem 8.3 and its corollary, posited by Stoky and Lucas (1989), we have

$$\langle T_\theta f, \lambda \rangle = \langle f, T_\theta^* \lambda \rangle, \forall \lambda \in \Lambda(X, \mathbf{B}(X)).$$

Condition (b) implies that $(T_\theta f)(x)$ is continuous in $(x, \theta)$. Let $\hat{\Theta} \subset \Theta$ be a compact set containing $\{\theta_n\}_{n=1}^\infty$ and $\theta_0$. Thus, it is uniformly continuous on the compact set $C \times \hat{\Theta}$, where $C$ is a compact subset of $X$. Condition (c) implies that

$$(T_{\theta_n}^* \mu_n)(B) = \mu_n(B), \forall B \in \mathbf{B}(X).$$

For $\varepsilon > 0$, condition (d) implies that we can pick compact set $C \subset X$ such that

$$\mu_n(X \backslash C) \leq \frac{\varepsilon}{4\|f\|}, \forall n \geq 1,$$

where $\|f\| = \sup_{x \in X} |f(x)| < \infty$ is the sup norm of $f$. Since $\{\theta_n\}_{n=1}^\infty$ and $\theta_0$ lie in $\hat{\Theta}$ and $\lim_{n \to \infty} \theta_n = \theta_0$, it follows from the uniform continuity of $(T_\theta f)(x)$ on $C \times \hat{\Theta}$ that there exists $N \geq 1$ such that

$$|(T_{\theta_n} f)(x) - (T_{\theta_0} f)(x)| < \frac{\varepsilon}{2}, \forall x \in C, \forall n \geq N.$$

Thus we have

$$
\begin{aligned}
& |\langle T_{\theta_n} f, \mu_n \rangle - \langle T_{\theta_0} f, \mu_n \rangle| \\
= \; & |\langle T_{\theta_n} f - T_{\theta_0} f, \mu_n \rangle| \\
\leq \; & \langle |T_{\theta_n} f - T_{\theta_0} f|, \mu_n \rangle \\
= \; & \int_X \left|(T_{\theta_n} f)(x) - (T_{\theta_0} f)(x)\right| \mu_n(dx) \\
= \; & \int_C \left|(T_{\theta_n} f)(x) - (T_{\theta_0} f)(x)\right| \mu_n(dx) + \int_{X \setminus C} \left|(T_{\theta_n} f)(x) - (T_{\theta_0} f)(x)\right| \mu_n(dx) \\
\leq \; & \int_X \frac{\varepsilon}{2} \mu_n(dx) + \int_{X \setminus C} \left|(T_{\theta_n} f)(x) - (T_{\theta_0} f)(x)\right| \mu_n(dx) \\
\leq \; & \frac{\varepsilon}{2} + \int_{X \setminus C} \left[\left|(T_{\theta_n} f)(x)\right| + \left|(T_{\theta_0} f)(x)\right|\right] \mu_n(dx) \\
\leq \; & \frac{\varepsilon}{2} + \int_{X \setminus C} 2\|f\| \mu_n(dx) \\
\leq \; & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

since $|(T_\theta f)(x)| = \left|\int_X f(x') P_\theta(x, dx')\right| \leq \|f\|$. Therefore, we have

$$|\langle T_{\theta_n} f, \mu_n \rangle - \langle T_{\theta_0} f, \mu_n \rangle| < \varepsilon, \forall n \geq N.$$

That is,

$$\lim_{n \to \infty} |\langle T_{\theta_n} f, \mu_n \rangle - \langle T_{\theta_0} f, \mu_n \rangle| = 0. \tag{A.18}$$

We know that $\{\mu_n\}_{n=1}^\infty$ is tight from condition (d). From Theorem 5.1 posited by Billingsley (1999), we know that it has a weakly convergent subsequence. Let $\{\mu_{n_i}\}_{i=1}^\infty$ be such a subsequence, and let $\hat{\mu}$ be its limit. Thus, for any bounded continuous function $f$ on $X$, we have

$$
\begin{aligned}
& |\langle f, \hat{\mu} \rangle - \langle T_{\theta_0} f, \hat{\mu} \rangle| \\
\leq \; & |\langle f, \hat{\mu} \rangle - \langle f, \mu_{n_i} \rangle| + |\langle f, \mu_{n_i} \rangle - \langle T_{\theta_0} f, \mu_{n_i} \rangle| + |\langle T_{\theta_0} f, \mu_{n_i} \rangle - \langle T_{\theta_0} f, \hat{\mu} \rangle|.
\end{aligned}
$$

Since $f$ and $T_{\theta_0} f$ are bounded continuous functions on $X$, and $\{\mu_{n_i}\}_{i=1}^\infty$ converges weakly to $\hat{\mu}$, we have $\lim_{i \to \infty} |\langle f, \hat{\mu} \rangle - \langle f, \mu_{n_i} \rangle| = 0$ and $\lim_{i \to \infty} |\langle T_{\theta_0} f, \mu_{n_i} \rangle -$

$\langle T_{\theta_0} f, \hat{\mu} \rangle | = 0$. By Equation (A.18) we also have

$$\begin{aligned}
\lim_{i\to\infty} | \langle f, \mu_{n_i} \rangle - \langle T_{\theta_0} f, \mu_{n_i} \rangle | &= \lim_{i\to\infty} | \left\langle f, T^*_{\theta_{n_i}} \mu_{n_i} \right\rangle - \langle T_{\theta_0} f, \mu_{n_i} \rangle | \\
&= \lim_{i\to\infty} | \left\langle T_{\theta_{n_i}} f, \mu_{n_i} \right\rangle - \langle T_{\theta_0} f, \mu_{n_i} \rangle | \\
&= 0.
\end{aligned}$$

Thus, for any bounded continuous function $f$ on $X$, we have

$$\langle f, \hat{\mu} \rangle = \langle T_{\theta_0} f, \hat{\mu} \rangle = \left\langle f, T^*_{\theta_0} \hat{\mu} \right\rangle.$$

Hence, by Corollary 2 to Theorem 12.6 proposed by Stokey and Lucas (1989), we have

$$\hat{\mu}(B) = (T^*_0 \hat{\mu})(B), \forall B \in \mathbf{B}(X).$$

Thus, $\hat{\mu}$ is a fixed point of $P_{\theta_0}(\cdot, \cdot)$. ■

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