

Optimal Non-linear Estate Taxation

C.C. Yang

Academia Sinica

National Chengchi University

Feng Chia University

Xueya Zhao

University of International Business and Economics

Shenghao Zhu*

University of International Business and Economics

December 11, 2023

Abstract

We investigate the optimal inheritance tax rates in the heterogeneous agents model with fat-tailed wealth distributions. We find a formula of the optimal inheritance tax and characterize the stationary wealth distribution. We first show the equity-efficiency trade-off in a simple one-period lifetime dynamic model under non-linear estate taxes. Using a variational approach, we explicitly derive the impact of inheritance taxes on the stationary wealth distribution and the social welfare function. Then we obtain a formula of the optimal inheritance tax, which is expressed by sufficient statistics.

JEL classifications:

Keywords: wealth distribution, non-linear estate taxation, perturbation method, machine learning

*Corresponding author, E-mail: zhushenghao@yahoo.com. We thank Felix Kübler and Serguei Maliar for their comments and suggestions.

1 Introduction

Studying the impact of estate taxation on macroeconomic outcomes is one of the most celebrated policy exercises within the neoclassical growth model: it is important for understanding the implications of fiscal policy, the macroeconomic effects of wars, and the cross-section of countries.

We investigate the optimal inheritance tax rates in the heterogeneous agents model with fat-tailed wealth distributions. We find a formula of the optimal inheritance tax and characterize the stationary wealth distribution. We first show the equity-efficiency trade-off in a simple one-period lifetime dynamic model under non-linear estate taxes. Using a variational approach, we explicitly derive the impact of inheritance taxes on the stationary wealth distribution and the social welfare function. Then we obtain a formula of the optimal inheritance tax, which is expressed by sufficient statistics.

1.1 Literature review

2 The Basic Model

Time is discrete. There is a continuum of infinitely lived households, indexed by i and distributed uniformly over $[0, 1]$. Each household is endowed with one unit of labor, which it supplies inelastically in a competitive labor market. Each household also owns and runs a firm, which employs labor in the competitive labor market. All uncertainty is purely idiosyncratic, and hence all aggregates are deterministic.

2.1 General case on non-linear estate tax

Household In our model, Thus, the household's problem is a deterministic optimization problem. Both the utility functions of the consumption and the capital have the form of constant relative risk aversion (CRRA).

The household's problem is

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \quad (1)$$

s.t.

$$c_t + k_{t+1} = y_t - T(k_{t+1})$$

where β represents the time discount factor, c_t denotes the household's consumption,

$y_t = w_t \exp(z_t) + \pi_t(k_t, z_t, \theta_t) + (1 - \delta)k_t$, $w_t \exp(z_t)$ denotes labor income, w_t represents the wage rate, and $\pi_t(k_t)$ is the profit from the firm that the household operates. k_t denotes household-owned capital, δ denotes the mean depreciation rate of capital k_t , and $T(k_{t+1})$ is the wealth tax scheme imposed by the government. We consider a fully non-linear capital tax system without assuming a functional form, we focus on a restrictive class of tax system. More precisely, we consider (1) a non-linear capital tax with a lump-sum transfer; (2) the tax is levied on the current period's income only (no history dependency); and (3) the tax function is time invariant. We have z_{t+1} which represents individual productivity, following an AR(1) process, $z_{t+1} = \rho z_t + \varepsilon_t$, and $\varepsilon_t \sim N(\mu, \sigma^2)$.

Firm profits are subject to undiversifiable idiosyncratic risk:

$$\pi_t(\theta_t, k_t, z_t) = \max_{n_t} \theta_t \exp(z_t) A k_t^\alpha n_t^{1-\alpha} - w_t n_t,$$

where A denotes the aggregate technology level, n_t represents the amount of labor that firms hire in the competitive labor market, and θ_t is an idiosyncratic productivity shock. We assume that the idiosyncratic productivity shock follows a lognormal distribution, $\log \theta_t \sim N(\mu_\theta, \sigma_\theta^2)$.

Households make current consumption decisions c_t and savings decisions k_{t+1} considering income y_t and the labor factor z_t , aiming to maximize the utility function $u(c_t)$ and the expected value function $\mathbb{E}(V(y_{t+1}, z_{t+1}))$ for the next period.

While solving for $\mathbb{E}(V(y_{t+1}, z_{t+1}))$, households cannot precisely determine the future productivity shock θ_{t+1} and labor efficiency z_{t+1} values. For each z_{t+1} , there are eleven possible values for θ , and there are seven possibilities for z_{t+1} . This implies there are 77

potential outcomes for y_{t+1} in the next period.

Considering the known transition probabilities multiplied by the 100 possible y_{t+1} values corresponding to z_{t+1} and accounting for the uniform distribution of θ_t , interpolation is used to compute the corresponding V values for these 77 y_{t+1} outcomes. Subsequently, the mean of these 77 V values is calculated to obtain $\mathbb{E}(V(y_{t+1}, z_{t+1}))$.

We can use the recursive relationship to express z_{t+1} in terms of z_t and the error term ε_t , specifically,

$$\begin{aligned} z_{t+1} &= \rho z_t + \varepsilon_t \\ \Rightarrow \quad \varepsilon_t &= z_{t+1} - \rho z_t. \end{aligned}$$

Since $\varepsilon_t \sim N(\mu, \sigma^2)$, we know that $z_{t+1} - \rho z_t \sim N(\mu, \sigma^2)$. Therefore,

$$Z_{t+1} \mid Z_t = z_t \sim N(\rho z_t + \mu, \sigma^2).$$

Hence, the distribution of z_{t+1} conditional on $Z_t = z_t$ is normal with parameters $\rho z_t + \mu$ and σ^2 ,

$$f_{Z_{t+1}|Z_t=z_t}(q) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(q - \rho z_t - \mu)^2}{2\sigma^2}\right).$$

2.2 Euler equation

We have Euler equation.

$$\begin{aligned} &\beta \mathbb{E}_t u'((r_{t+1} + 1 - \delta)k_{t+1} + w_{t+1} \exp(z_{t+1}) - T(k_{t+2}) - k_{t+2})(r_{t+1} + 1 - \delta) \\ &= u'((r_t + 1 - \delta)k_t + w_t \exp(z_t) - T(k_{t+1}) - k_{t+1})(1 + T'(k_{t+1})). \end{aligned} \tag{2}$$

2.3 Stationary distribution

Maximize company profits

$$\pi(k_t, \theta_t, z_t) = \max_{n_t} \theta_t \exp(z_t) A k_t^\alpha n_t^{1-\alpha} - w_t n_t.$$

The optimal labor demand is

$$n_t = \left[\frac{\theta_t \exp(z_t) A(1-\alpha)}{w_t} \right]^{\frac{1}{\alpha}} k_t.$$

Hence, the firm profit is

$$\pi(k_t, \theta_t, z_t) = \alpha(A\theta_t \exp(z_t))^{\frac{1}{\alpha}} \left(\frac{w_t}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}} k_t.$$

We have the income motion equation,

$$\begin{aligned} y_{t+1} &= \left[\alpha(A\theta_{t+1} \exp(z_{t+1}))^{\frac{1}{\alpha}} \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}} + 1 - \delta \right] (y_t - T(k_{t+1}) - c_t) + w_{t+1} \exp(z_{t+1}) \\ &= (r_{t+1} + 1 - \delta) (y_t - T(k_{t+1}) - c_t) + w_{t+1} \exp(z_{t+1}). \end{aligned}$$

We use r_{t+1} to denote the return on capital gains, which is calculated as $\alpha(A\theta_{t+1} \exp(z_{t+1}))^{\frac{1}{\alpha}} \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}$.

The cumulative distribution function of $F_{Y_{t+1}|Z_{t+1}=q, Z_t=z_t, Y_t=y_t}(p)$ is

$$\begin{aligned} &F_{Y_{t+1}|Z_{t+1}=q, Z_t=z_t, Y_t=y_t}(p) \\ &= P_r \left\{ \left[\alpha(A\theta_{t+1} \exp(q))^{\frac{1}{\alpha}} \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}} + 1 - \delta \right] (y_t - T(k_{t+1}) - c_t) + w_{t+1} \exp(q) \leq p \right\} \\ &= P_r \left\{ \theta_{t+1}^{\frac{1}{\alpha}} \leq \frac{\frac{p-w_{t+1} \exp(q)}{y_t-T(k_{t+1})-c_t} + \delta - 1}{\alpha(A \exp(q))^{\frac{1}{\alpha}} \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}} \right\} \\ &= P_r \left\{ \theta_{t+1} \leq \left[\frac{\frac{p-w_{t+1} \exp(q)}{y_t-T(k_{t+1})-c_t} + \delta - 1}{\alpha(A \exp(q))^{\frac{1}{\alpha}} \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}} \right]^{\alpha} \right\}, \end{aligned}$$

where $\log(\theta_{t+1}) \sim N(\mu_\theta, \sigma_\theta^2)$.

Let $H(p)$ denotes $\left[\frac{\frac{p-w_{t+1} \exp(q)}{y_t-T(k_{t+1})-c_t} + \delta - 1}{\alpha(A \exp(q))^{\frac{1}{\alpha}} \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}} \right]^{\alpha}$, we have

$$f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_t, Z_t=z_t}(p) = \frac{1}{\sqrt{2\pi\sigma^2 H(p)}} \exp\left(-\frac{(\ln(H(p))-\mu)^2}{2\sigma^2}\right) \frac{\partial H(p)}{\partial p},$$

$$\frac{\partial}{\partial p} H(p) = \alpha \left[\frac{\frac{p-w_{t+1}\exp(q)}{y_t-T(k_{t+1})-c_t} + \delta - 1}{\alpha(A\exp(q))^{\frac{1}{\alpha}} \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}} \right]^{\alpha-1} \frac{1/(y_t-T(k_{t+1})-c_t)}{\alpha(A\exp(q))^{\frac{1}{\alpha}} \left(\frac{w_{t+1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}}, \text{ and } f_{Z_{t+1}|Z_t=z_t}(q) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(q-\rho z_t - \mu)^2}{2\sigma^2}\right).$$

Then, we obtain

$$f(p, q) = \int_0^\infty \int_0^\infty f_{Y_{t+1}, Z_{t+1}|Y_t=y, Z_t=z}(p, q) f(y, z) dy dz. \quad (3)$$

Assuming y has I grid points and z has J grid points, then (y, z) will have $I \times J$ combinations. We represent it as a $(I \times J, 1)$ matrix as follows.

$$\left[(y_1, z_1)(y_1, z_2) \cdots (y_1, z_J)(y_2, z_1)(y_2, z_2) \cdots (y_2, z_J) \cdots (y_I, z_1)(y_I, z_2) \cdots (y_I, z_J) \right]_{(I \times J, 1)}^T$$

Since $f_{Y_{t+1}, Z_{t+1}|Y_t=k, Z_t=z}(p, q) = f_{Y_{t+1}|Z_{t+1}=q, Y_t=k, Z_t=z}(p) f_{Z_{t+1}|Z_t=z}(q)$, then by fixing a combination of (p, q) , we can obtain

$$\begin{aligned} & \left[\begin{array}{cccc} f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_1, Z_t=z_1}(p) & f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_1, Z_t=z_2}(p) & \cdots & f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_1, Z_t=z_J}(p) \\ f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_2, Z_t=z_1}(p) & f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_2, Z_t=z_2}(p) & \cdots & f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_2, Z_t=z_J}(p) \\ \vdots & \vdots & \ddots & \vdots \\ f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_I, Z_t=z_1}(p) & f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_I, Z_t=z_2}(p) & \cdots & f_{Y_{t+1}|Z_{t+1}=q, Y_t=y_I, Z_t=z_J}(p) \end{array} \right]_{(I, J)} \\ & \odot \left[\begin{array}{cccc} f_{Z_{t+1}|Z_t=z_1}(q) & f_{Z_{t+1}|Z_t=z_2}(q) & \cdots & f_{Z_{t+1}|Z_t=z_J}(q) \\ f_{Z_{t+1}|Z_t=z_1}(q) & f_{Z_{t+1}|Z_t=z_2}(q) & \cdots & f_{Z_{t+1}|Z_t=z_J}(q) \\ \vdots & \vdots & \ddots & \vdots \\ f_{Z_{t+1}|Z_t=z_1}(q) & f_{Z_{t+1}|Z_t=z_2}(q) & \cdots & f_{Z_{t+1}|Z_t=z_J}(q) \end{array} \right]_{(I, J)} \\ & = \left[\begin{array}{cccc} G_d(p, q; y_1, z_1) & G_d(p, q; y_1, z_2) & \cdots & G_d(p, q; y_1, z_J) \\ G_d(p, q; y_2, z_1) & G_d(p, q; y_2, z_2) & \cdots & G_d(p, q; y_2, z_J) \\ \vdots & \vdots & \ddots & \vdots \\ G_d(p, q; y_I, z_1) & G_d(p, q; y_I, z_2) & \cdots & G_d(p, q; y_I, z_J) \end{array} \right]_{(I, J)} \end{aligned}$$

We need to reshape matrix G_d from (I, J) to $G_{(p,q)}(1, I \times J)$ as follows.

$$\left[G_d(p, q; y_1, z_1) G_d(p, q; y_1, z_2) \cdots G_d(p, q; y_1, z_J) \cdots G_d(p, q; y_I, z_1) G_d(p, q; y_I, z_2) \cdots G_d(p, q; y_I, z_J) \right]_{(1, I \times J)}$$

And the dimension of $G(p, q; y, z)$ is $(I \times J, I \times J)$ Then, we have

$$\begin{bmatrix} f(p_1, q_1) \\ f(p_1, q_2) \\ \vdots \\ f(p_1, q_j) \\ f(p_2, q_1) \\ f(p_2, q_2) \\ \vdots \\ f(p_2, q_j) \\ \vdots \\ f(p_i, q_1) \\ f(p_i, q_2) \\ \vdots \\ f(p_i, q_j) \end{bmatrix} = \begin{bmatrix} G(p_1, q_1; y_1, z_1) & \cdots & G(p_1, q_1; y_2, z_j) & \cdots & G(p_1, q_1; y_i, z_j) \\ G(p_1, q_2; y_1, z_1) & \cdots & G(p_1, q_2; y_2, z_j) & \cdots & G(p_1, q_2; y_i, z_j) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ G(p_1, q_j; y_1, z_1) & \cdots & G(p_1, q_j; y_2, z_j) & \cdots & G(p_1, q_j; y_i, z_j) \\ G(p_2, q_1; y_1, z_1) & \cdots & G(p_2, q_1; y_2, z_j) & \cdots & G(p_2, q_1; y_i, z_j) \\ G(p_2, q_2; y_1, z_1) & \cdots & G(p_2, q_2; y_2, z_j) & \cdots & G(p_2, q_2; y_i, z_j) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ G(p_2, q_j; y_1, z_1) & \cdots & G(p_2, q_j; y_2, z_j) & \cdots & G(p_2, q_j; y_i, z_j) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ G(p_i, q_1; y_1, z_1) & \cdots & G(p_i, q_1; y_2, z_j) & \cdots & G(p_i, q_1; y_i, z_j) \\ G(p_i, q_2; y_1, z_1) & \cdots & G(p_i, q_2; y_2, z_j) & \cdots & G(p_i, q_2; y_i, z_j) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ G(p_i, q_j; y_1, z_1) & \cdots & G(p_i, q_j; y_2, z_j) & \cdots & G(p_i, q_j; y_i, z_j) \end{bmatrix} \times \begin{bmatrix} f(y_1, z_1) \\ f(y_1, z_2) \\ \vdots \\ f(y_1, z_j) \\ f(y_2, z_1) \\ f(y_2, z_2) \\ \vdots \\ f(y_2, z_j) \\ \vdots \\ f(y_i, z_1) \\ f(y_i, z_2) \\ \vdots \\ f(y_i, z_j) \end{bmatrix} \odot \begin{bmatrix} \Delta y \end{bmatrix}$$

$(I \times J, 1)$

$(I \times J, I \times J)$

$(I \times J, 1) \quad (I \times J, 1)$

2.4 Stationary equilibrium

The competitive equilibrium of the economy is standard.

Definition 1 Given k_0 , a competitive equilibrium is defined as sequences of prices $\{w_t\}_{t=0}^{\infty}$, government polices $\{T_t\}_{t=0}^{\infty}$, aggregate allocations $\{C_t, N_t, K_t, Y_t\}_{t=0}^{\infty}$, and individual plans $\{c_t, n_t, k_t\}_{t=0}^{\infty}$, such that the following conditions hold:

(i) given $\{w_t\}_{t=0}^{\infty}$ and $\{T_t\}_{t=0}^{\infty}$, the plans $(\{c_t, n_t, k_t\}_{t=0}^{\infty})_{i \in [0,1]}$ are optimal for the household;

(ii) the labor market clears:

$$\int_0^1 n_t^i di = N_t \equiv \mathbb{E}(\exp(z_t)),$$

for all $t \geq 0$.

(iii) the aggregates are consistent with individual behavior, $C_t = \int_0^1 c_t^i di$, $K_t = \int_0^\infty k f(k) dk$, and $Y_t = \int_0^1 \theta_t \exp(z_t) A(k_t)^\alpha (n_t)^{1-\alpha} di$, for all $t \geq 0$.

The labor market clearing condition implies

$$\int_0^1 \left[\frac{\theta_t \exp(z_t) A(1-\alpha)}{w_t} \right]^{\frac{1}{\alpha}} k_t di = \mathbb{E}(\exp(z_t)),$$

which gives rise to the wage rate in general equilibrium

$$w_t = (1-\alpha) A \left(\frac{K_t}{N_t} \right)^\alpha \mathbb{E}(\theta_t), \quad (4)$$

and then the rate of return to capital in general equilibrium

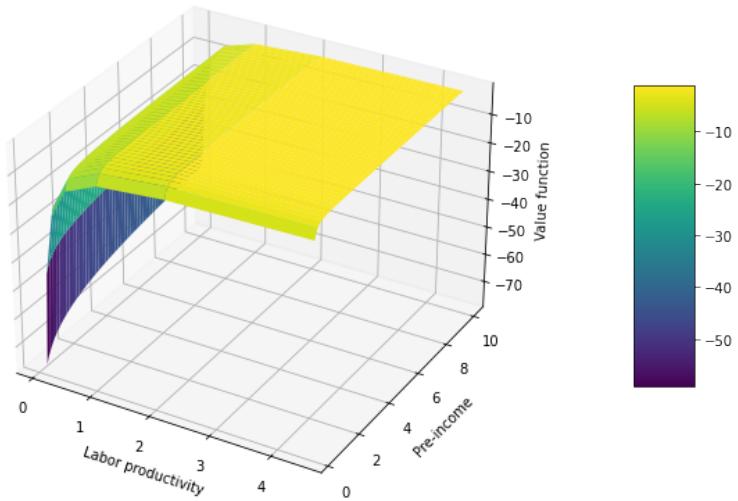
$$r_t = \alpha A \left(\frac{N_t}{K_t} \right)^{1-\alpha} \mathbb{E}(\theta_t). \quad (5)$$

The results of (4) and (5) are consistent with the aggregate production technology $Y_t = \mathbb{E}(\theta_t) A K_t^\alpha N_t^{1-\alpha}$.

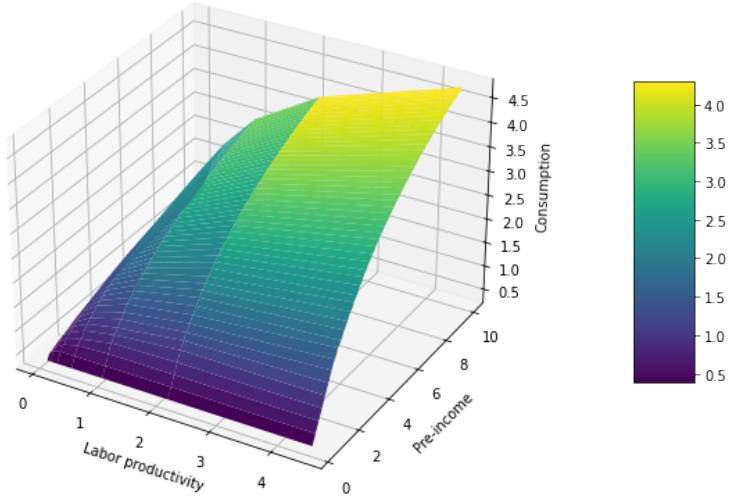
3 Calibration

Table 1: Calibration from literature

Coeffecient of relative risk aversion	$\eta = 3$
Time discount factor	$\beta = 0.95$
Depreciation rate	$\delta = 0.05$
Capital income share	$\alpha = 1/3$



(a) Value function



(b) Consumption

Figure 1: Value function and policy functions of households

4 Perturbation

4.1 Tax incidence (Gateaux derivative)

In deriving an optimal tax formula, we apply the variational approach (Piketty, 1997; Saez, 2001). Consider a perturbation (a small deviation) from a given tax schedule. If there is no welfare-improving perturbation within the class of tax system, the given tax schedule is optimal.

We start with the incidence of tax, the first-order effects of tax reforms. For a given tax schedule $T(k)$, suppose that the economy converges to a steady state where the distribution of state variables $\Phi(k_t)$, whose density is $\phi(k_t)$ is stationary. We assume that the economy starts from that steady state in period 0 .

Consider an arbitrary tax reform, which can be represented by a continuously differentiable function $\tau(\cdot)$ on \mathbb{R}_+ . Then, a perturbed tax schedule is $T(\cdot) + \mu\tau(\cdot)$, where $\mu \in \mathbb{R}$ parameterizes the size of the tax reform. As in Golosov et al. (2014) and Sachs et al. (2020), the first-order effects of this perturbation can be formally represented by the Gateaux derivative in the direction of τ . For example, the incidence on the labour supply is

$$dc(y, z) \equiv \lim_{\mu \rightarrow 0} \frac{1}{\mu} [c(y, z; T + \mu\tau) - c(y, z; T)].$$

We can define similar incidences for other variables such as the indirect utilities of individuals $V(k_0, z_0)$, government revenue R_t , and social welfare W .

From now on,We mostly focus on the elementary tax reforms represented by $\tau(k) = \frac{1}{1-F_K(k^*)} \mathbb{1}\{k \geq k^*\}$, where $\mathbb{1}$ is an indicator function for a given level of capital k^* . Under this tax reform, the tax payment of an individual with capital above k^* increases by a constant amount $\frac{1}{1-F_K(k^*)}$, and the marginal tax rate at capital level k^* is increased by $\frac{1}{1-F_K(k^*)}$ (which is obtained by the marginal perturbation: $\tau'(k) = \frac{1}{1-F_K(k^*)} \delta_{k^*}$, where δ_{k^*} is the Dirac delta function at k^*). Note that with this tax reform, the increased government revenue due to a mechanical increase in tax payment is equal to \$1. We can focus on this elementary tax reform without loss of generality, because any other perturbations can be expressed as a weighted sum of elementary tax reforms. See Sachs et al. (2020) for further details.

Euler equation. Performing perturbation on the Euler equation, we obtain

$$(\mathcal{B} - \mathcal{D}) \hat{s}(y_t) = \mathcal{G}\tau'(s(y_t)) + \mathcal{J} - \mathcal{A}, \quad (6)$$

where

$$\begin{aligned} \mathcal{A} &= \beta \mathbb{E}_t [u''(c(y_{t+1})) (\hat{y}_{t+1} - T'(s(s(y_t))) \hat{s}(s(y_t)) \\ &\quad - \int_0^{s(s(y_t))} \tau'(m) dm - \hat{T}(0) - \hat{s}(s(y_t))] (r_{t+1} + 1 - \delta)] + \beta \mathbb{E}_t u'(c(y_{t+1})) \hat{r}_{t+1}, \end{aligned} \quad (6 \text{ a})$$

$$\begin{aligned} \mathcal{B} &= \beta \mathbb{E}_t [u''(c(y_{t+1})) ((r_{t+1} + 1 - \delta) - T'(s(s(y_t))) s'(s(y_t)) - s'(s(y_t))) \\ &\quad \times (r_{t+1} + 1 - \delta)], \end{aligned} \quad (6 \text{ b})$$

$$\mathcal{D} = u'(c(y_t)) T''(s(y_t)) - u''(c(y_t)) (1 + T'(s(y_t)))^2, \quad (6 \text{ c})$$

$$\mathcal{G} = u'(c(y_t)), \quad (6 \text{ d})$$

$$\mathcal{J} = u''(c(y_t)) \left(\hat{y}_t - \int_0^{s(y_t)} \tau'(m) dm - \hat{T}(0) \right) (1 + T'(s(y_t))). \quad (6 \text{ e})$$

Incidence of tax on consumption.

We know $c_t = y_t - T(k_{t+1}) - k_{t+1}$.

$$\begin{aligned} &c_t + \mu \hat{c}_t \\ &= y_t - \int_0^{s(y_t) + \mu \hat{s}(y_t)} T'(m) dm + \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) - s(y_t) - \mu \hat{s}(y_t) \\ &= y_t - \int_0^{s(y_t) + \mu \hat{s}(y_t)} T'(m) dm - \int_0^{s(y_t) + \mu \hat{s}(y_t)} \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) - s(y_t) - \mu \hat{s}(y_t) \\ &= y_t - \int_0^{s(y_t)} T'(m) dm - T(0) - s(y_t) - T'(s(y_t)) \mu \hat{s}(y_t) - \int_0^{s(y_t)} \mu \tau'(m) dm - \mu \hat{T}(0) - \mu \hat{s}(y_t). \end{aligned} \quad (7)$$

Then,

$$\hat{c}_t = - (1 + T'(s(y_t))) \hat{s}(y_t) - \int_0^{s(y_t)} \tau'(m) dm - \hat{T}(0).$$

Incidence of tax on government revenue. As any government revenue, denoted as R , is subsequently redistributed to households in the form of lump-sum transfers. Subsequently, we obtain the following relationship:

$$\begin{aligned} 0 &= \int_0^\infty T(s(y))f(y)dy \\ &= \int_0^\infty \left(T(0) + \int_0^{s(y)} T'(m)dm \right) f(y)dy \\ &= T(0) + \int_0^\infty \int_0^{s(y)} T'(m)dm f(y)dy. \end{aligned}$$

Then, we have

$$T(0) = - \int_0^\infty \int_0^{s(y)} T'(m)dm f(y)dy.$$

Therefore,

$$R = -T(0) = \int_0^\infty \int_0^{s(y)} T'(m)dm f(y)dy.$$

and

$$\hat{R} = -\hat{T}(0).$$

We know

$$\begin{aligned} R + \mu\hat{R} &= \int_0^\infty \int_0^{s(y)+\mu\hat{s}(y)} (T'(m) + \mu\tau'(m)) dm (f(y) + \mu\hat{f}(y)) dy \\ &= \int_0^\infty \int_0^{s(y)} T'(m)dm f(y)dy + \int_0^\infty T'(s(y))\mu\hat{s}(y)f(y)dy \\ &\quad + \int_0^\infty \int_0^{s(y)} T'(m)dm \mu\hat{f}(y)dk + \int_0^\infty \int_0^{s(y)} \mu\tau'(m)dm f(y)dy. \end{aligned} \tag{8}$$

Hence, we obtain

$$\hat{R} = \int_0^\infty T'(s(y))\hat{s}(y)f(y)dy + \int_0^\infty \int_0^{s(y)} T'(m)dm \hat{f}(y)dy + 1. \tag{9}$$

5 Optimal non-linear estate taxation formula

We have social welfare $W = \sum_{t=0}^{\infty} \beta^t \int_0^{\infty} u(c)f(y)dy = \frac{1}{1-\beta} \int_0^{\infty} u(c)f(y)dy$, Then, we perform perturbation on social welfare,

$$\begin{aligned}
\hat{W} &= \frac{1}{1-\beta} \int_0^{\infty} \hat{u}(c)f(y)dy + \frac{1}{1-\beta} \int_0^{\infty} u(c)\hat{f}(y)dy \\
&= \frac{1}{1-\beta} \int_0^{\infty} u'(c)\hat{c}(y)f(y)dy + \frac{1}{1-\beta} \int_0^{\infty} u(c)\hat{f}(y)dy \\
&= \frac{1}{1-\beta} \int_0^{\infty} u'(c) \left[k_t \hat{r}_t + \exp(z_t) \hat{w}_t - T'(s(k_t)) \hat{s}(k) - \int_0^{s(k_t)} \tau'(m)dm \right] f(k)dk \\
&\quad + T'(s^*(k)) \frac{\mathcal{G}(s^*(k))}{\mathcal{B}(s^*(k)) - \mathcal{D}(s^*(k))} \frac{f(s^*(k))}{1 - F(s^*(k))} \frac{1}{1-\beta} \int_0^{\infty} u'(c)f(k)dk \\
&\quad + \frac{1}{1-\beta} \int_0^{\infty} u'(c) \left(\int_0^{\infty} T'(s(k)) \frac{\mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(k)dk \right) f(k)dk \\
&\quad + \frac{1}{1-\beta} \int_0^{\infty} u'(c) \left(\int_0^{\infty} \int_0^{s(k)} T'(m)dm \hat{f}(k)dk + 1 \right) f(k)dk + \frac{1}{1-\beta} \int_0^{\infty} u(c)\hat{f}(k)dk.
\end{aligned} \tag{10}$$

We need $\hat{W} = 0$ to obtain the optimal social welfare,

$$T'(k^*) = \frac{1 - F(k^*)}{k^* f(k^*)} \lambda(k^*) [\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E} + \mathcal{F}],$$

where

$$\begin{aligned}
\lambda(k^*) &= \frac{[\mathcal{B}(k^*) - \mathcal{D}(k^*)] k^*}{\mathcal{G}(k^*)}, \\
\mathcal{A} &= \int_{k^*}^{\infty} \left(1 - \frac{u'(c)}{\varphi}\right) \frac{f(y)}{1 - F(k^*)} dy, \\
\mathcal{B} &= \frac{1}{\varphi} \int_0^{\infty} u'(c) (k\hat{r} + \exp(z)\hat{w}) f(y) dy, \\
\mathcal{C} &= \int_0^{\infty} T'(s(y)) \frac{\mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(y) dy, \\
\mathcal{D} &= \int_0^{\infty} \int_0^{s(y)} T'(m) dm \hat{f}(y) dy, \\
\mathcal{E} &= -\frac{1}{\varphi} \int_0^{\infty} u'(c) T'(s(y)) \hat{s}(y) f(y) dy, \\
\mathcal{F} &= \frac{1}{\varphi} \int_0^{\infty} u(c) \hat{f}(y) dy, \\
\varphi &= \int_0^{\infty} u'(c) f(y) dy.
\end{aligned}$$

6 Numerical experiment

7 Conclusion

Appendix A Derivations and proofs

A.1 Proof of tax incidence on consumption

Based on equation (1),

$$V(k_t, z_t) = u((r_t + 1 - \delta)k_t + w_t \exp(z_t) - T(k_{t+1}) - k_{t+1}) + \beta \mathbb{E}_t V(k_{t+1}, z_{t+1}) \quad (11)$$

Taking the derivative of equation (11) with respect to k_{t+1} yields

$$0 = -u'((r_t + 1 - \delta)k_t + w_t \exp(z_t) - T(k_{t+1}) - k_{t+1})(1 + T'(k_{t+1})) + \beta \mathbb{E}_t V'_k(k_{t+1}, z_{t+1}). \quad (12)$$

Deriving equation (11) with respect to k_t , we have

$$V'_k(k_t, z_t) = u'((r_t + 1 - \delta)k_t + w_t \exp(z_t) - T(k_{t+1}) - k_{t+1})(r_t + 1 - \delta).$$

Hence,

$$V'_k(k_{t+1}, z_{t+1}) = u'((r_{t+1} + 1 - \delta)k_{t+1} + w_{t+1} \exp(z_{t+1}) - T(k_{t+2}) - k_{t+2})(r_{t+1} + 1 - \delta). \quad (13)$$

Combining equations (12) and (13), we obtain

$$\begin{aligned} & \beta \mathbb{E}_t u'((r_{t+1} + 1 - \delta)k_{t+1} + w_{t+1} \exp(z_{t+1}) - T(k_{t+2}) - k_{t+2})(r_{t+1} + 1 - \delta) \\ &= u'((r_t + 1 - \delta)k_t + w_t \exp(z_t) - T(k_{t+1}) - k_{t+1})(1 + T'(k_{t+1})). \end{aligned} \quad (14)$$

This is Euler equation.

We denote k_{t+1} as s_t , and $y_t = (r_t + 1 - \delta)k_t + w_t \exp(z_t)$. Fully differentiating

equation (14), we obtain

$$\begin{aligned}
& \beta \mathbb{E}_t u'((r_{t+1} + \mu \hat{r}_{t+1} + 1 - \delta)(s_t + \mu \hat{s}_t) + (w_{t+1} + \mu \hat{w}_{t+1}) \exp(z_{t+1})) \\
& - \int_0^{s_{t+1}(s_t(k) + \mu \hat{s}_t(k)) + \mu \hat{s}_{t+1}(s_t(k))} T'(m) + \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) - s_{t+1}(s_t(k) + \mu \hat{s}_t(k)) - \mu \hat{s}_{t+1}(s_t(k)) \\
& \times (r_{t+1} + \mu \hat{r}_{t+1} + 1 - \delta) \\
& = u' \left((r_t + \mu \hat{r}_t + 1 - \delta) k_t + (w_t + \mu \hat{w}_t) \exp(z_t) - \int_0^{s_t(k) + \mu \hat{s}_t(k)} T'(m) + \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) \right. \\
& \left. - s_t(k) - \mu \hat{s}_t(k) \right) (1 + T'(s_t(k) + \mu \hat{s}_t(k)) + \mu \tau'(s_t(k))) .
\end{aligned} \tag{15}$$

We first focus on the first row of equation (15),

$$\begin{aligned}
& (r_{t+1} + \mu \hat{r}_{t+1} + 1 - \delta)(s_t + \mu \hat{s}_t) + (w_{t+1} + \mu \hat{w}_{t+1}) \exp(z_{t+1}) \\
& = (r_{t+1} + 1 - \delta)s_t + w_{t+1} \exp(z_{t+1}) + \mu \hat{r}_{t+1}s_t + \mu \hat{s}_t(r_{t+1} + 1 - \delta) + \mu \hat{w}_{t+1} \exp(z_{t+1}) \\
& = y_{t+1} + \mu \hat{r}_{t+1}s_t + \mu \hat{s}_t(r_{t+1} + 1 - \delta) + \mu \hat{w}_{t+1} \exp(z_{t+1}) .
\end{aligned} \tag{16}$$

The first part of second row in equation (15),

$$\begin{aligned}
& - \int_0^{s_{t+1}(s_t(k)) + \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))} T'(m) + \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) \\
&= - \int_0^{s_{t+1}(s_t(k)) + \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))} T'(m) dm - \int_0^{s_{t+1}(s_t(k)) + \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))} \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) \\
&= - \int_0^{s_{t+1}(s_t(k)) + s'_{t+1}(s_t(k)) \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))} T'(m) dm - \int_0^{s_{t+1}(s_t(k)) + s'_{t+1}(s_t(k)) \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))} \mu \tau'(m) dm \\
&\quad - T(0) - \mu \hat{T}(0) \\
&= - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - T'(s_{t+1}(s_t(k))) [s'_{t+1}(s_t(k)) \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))] \\
&\quad - \int_0^{s_{t+1}(s_t(k))} \mu \tau'(m) dm - \mu \tau'(s_{t+1}(s_t(k))) [s'_{t+1}(s_t(k)) \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))] - \mu \hat{T}(0) \\
&= - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - T'(s_{t+1}(s_t(k))) [s'_{t+1}(s_t(k)) \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))] \\
&\quad - \int_0^{s_{t+1}(s_t(k))} \mu \tau'(m) dm - \mu \hat{T}(0).
\end{aligned} \tag{17}$$

The second part of the second row,

$$\begin{aligned}
& - s_{t+1}(s_t(k)) + \mu \hat{s}_t(k) - \mu \hat{s}_{t+1}(s_t(k)) \\
&= - s_{t+1}(s_t(k)) - s'_{t+1}(s_t(k)) \mu \hat{s}_t(k) - \mu \hat{s}_{t+1}(s_t(k)).
\end{aligned} \tag{18}$$

The first part in fourth row,

$$\begin{aligned}
& (r_t + \mu \hat{r}_t + 1 - \delta) k_t + (w_t + \mu \hat{w}_t) \exp(z_t) \\
&= y_t + \mu \hat{r}_t k_t + \mu \hat{w}_t \exp(z_t).
\end{aligned} \tag{19}$$

The second part in fourth row,

$$\begin{aligned}
& - \int_0^{s_t(k) + \mu \hat{s}_t(k)} T'(m) + \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) \\
& = - \int_0^{s_t(k) + \mu \hat{s}_t(k)} T'(m) dm - \int_0^{s_t(k) + \mu \hat{s}_t(k)} \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) \\
& . = - \int_0^{s_t(k)} T'(m) dm - T'(s_t(k)) \mu \hat{s}_t(k) - \int_0^{s_t(k)} \mu \tau'(m) dm - \mu \tau'(s_t(k)) \mu \hat{s}_t(k) - T(0) - \mu \hat{T}(0) \\
& = - \int_0^{s_t(k)} T'(m) dm - T(0) - T'(s_t(k)) \mu \hat{s}_t(k) - \int_0^{s_t(k)} \mu \tau'(m) dm - \mu \hat{T}(0).
\end{aligned} \tag{20}$$

The last row,

$$\begin{aligned}
& 1 + T'(s_t(k) + \mu \hat{s}_t(k)) + \mu \tau'(s_t(k)) \\
& = 1 + T'(s_t(k)) + T''(s_t(k)) \mu \hat{s}_t(k) + \mu \tau'(s_t(k)).
\end{aligned} \tag{21}$$

Then, we have

$$\begin{aligned}
& \beta \mathbb{E}_t \left[\left(u' \left(y_{t+1} - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - s_{t+1}(s_t(k)) \right) \right. \right. \\
& + u'' \left(y_{t+1} - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - s_{t+1}(s_t(k)) \right) (\mu \hat{r}_{t+1} s_t + \mu \hat{s}_t(r_{t+1} + 1 - \delta) + \mu \hat{w}_{t+1} \exp(z_{t+1})) \\
& - T'(s_{t+1}(s_t(k))) [s'_{t+1}(s_t(k)) \mu \hat{s}_t(k) + \mu \hat{s}_{t+1}(s_t(k))] - \int_0^{s_{t+1}(s_t(k))} \mu \tau'(m) dm - \mu \hat{T}(0) - s'_{t+1}(s_t(k)) \mu \hat{s}_t(k) \\
& \left. \left. - \mu \hat{s}_{t+1}(s_t(k))) (r_{t+1} + \mu \hat{r}_{t+1} + 1 - \delta) \right] \right. \\
& = \left[u' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) + u'' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) \right. \\
& \times \left. \left(\mu \hat{r}_t k_t + \mu \hat{w}_t \exp(z_t) - T'(s_t(k)) \mu \hat{s}_t(k) - \int_0^{s_t(k)} \mu \tau'(m) dm - \mu \hat{T}(0) - \mu \hat{s}_t(k) \right) \right] \\
& \times (1 + T'(s_t(k)) + T''(s_t(k)) \mu \hat{s}_t(k) + \mu \tau'(s_t(k))). \tag{22}
\end{aligned}$$

Futhurmore,

$$\begin{aligned}
& \beta \mathbb{E}_t \left[\left(u' \left(y_{t+1} - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - s_{t+1}(s_t(k)) \right) \right. \right. \\
& + u'' \left(y_{t+1} - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - s_{t+1}(s_t(k)) \right) (\mu \hat{r}_{t+1} s_t + \mu \hat{w}_{t+1} \exp(z_{t+1}) \\
& - T'(s_{t+1}(s_t(k))) \mu \hat{s}_{t+1}(s_t(k)) - \int_0^{s_{t+1}(s_t(k))} \mu \tau'(m) dm - \mu \hat{T}(0) - \mu \hat{s}_{t+1}(s_t(k))) (r_{t+1} + \mu \hat{r}_{t+1} + 1 - \delta) \Big] \\
& + \mu \hat{s}_t(k) \beta \mathbb{E}_t \left[u'' \left(y_{t+1} - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - s_{t+1}(s_t(k)) \right) \right. \\
& \times ((r_{t+1} + 1 - \delta) - T'(s_{t+1}(s_t(k))) s'_{t+1}(s_t(k)) - s'_{t+1}(s_t(k))) (r_{t+1} + \mu \hat{r}_{t+1} + 1 - \delta) \Big] \\
& = u' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) (1 + T'(s_t(k))) \\
& + u' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) T''(s_t(k)) \mu \hat{s}_t(k) + u' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) \mu \tau'(s_t(k)) \\
& + u'' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) \left(\mu \hat{r}_t k_t + \mu \hat{w}_t \exp(z_t) - \int_0^{s_t(k)} \mu \tau'(m) dm - \mu \hat{T}(0) \right) (1 + T'(s_t(k))) \\
& + u'' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) \left(\mu \hat{r}_t k_t + \mu \hat{w}_t \exp(z_t) - \int_0^{s_t(k)} \mu \tau'(m) dm - \mu \hat{T}(0) \right) T''(s_t(k)) \mu \hat{s}_t(k) \\
& + u'' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) \left(\mu \hat{r}_t k_t + \mu \hat{w}_t \exp(z_t) - \int_0^{s_t(k)} \mu \tau'(m) dm - \mu \hat{T}(0) \right) \mu \tau'(s_t(k)) \\
& - u'' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) (1 + T'(s_t(k)))^2 \mu \hat{s}_t(k). \tag{23}
\end{aligned}$$

Eliminating the terms that are equal on the left and right sides, we have

$$\begin{aligned}
& \beta \mathbb{E}_t \left[u'' \left(y_{t+1} - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - s_{t+1}(s_t(k)) \right) (\mu \hat{r}_{t+1} s_t + \mu \hat{w}_{t+1} \exp(z_{t+1}) \right. \\
& \quad \left. - T'(s_{t+1}(s_t(k))) \mu \hat{s}_{t+1}(s_t(k)) - \int_0^{s_{t+1}(s_t(k))} \mu \tau'(m) dm - \mu \hat{T}(0) - \mu \hat{s}_{t+1}(s_t(k)) \right) (r_{t+1} + 1 - \delta) \right] \\
& + \beta \mathbb{E}_t u' \left(y_{t+1} - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - s_{t+1}(s_t(k)) \right) \mu \hat{r}_{t+1} \\
& + \mu \hat{s}_t(k) \beta \mathbb{E}_t \left[u'' \left(y_{t+1} - \int_0^{s_{t+1}(s_t(k))} T'(m) dm - T(0) - s_{t+1}(s_t(k)) \right) \right. \\
& \quad \times \left. ((r_{t+1} + 1 - \delta) - T'(s_{t+1}(s_t(k))) s'_{t+1}(s_t(k)) - s'_{t+1}(s_t(k))) (r_{t+1} + 1 - \delta) \right] \\
& = u' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) T''(s_t(k)) \mu \hat{s}_t(k) + u' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) \mu \tau'(s_t(k)) \\
& + u'' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) \left(\mu \hat{r}_t k_t + \mu \hat{w}_t \exp(z_t) - \int_0^{s_t(k)} \mu \tau'(m) dm - \mu \hat{T}(0) \right) (1 + T'(s_t(k))) \\
& - u'' \left(y_t - \int_0^{s_t(k)} T'(m) dm - T(0) - s_t \right) (1 + T'(s_t(k)))^2 \mu \hat{s}_t(k). \tag{24}
\end{aligned}$$

Then, we have

$$(\mathcal{B} - \mathcal{D}) \hat{s}(k_t) = \mathcal{G} \tau'(s(k_t)) + \mathcal{J} - \mathcal{A}, \tag{25}$$

where

$$\begin{aligned} \mathcal{A} &= \beta \mathbb{E}_t \left[u'' \left(y_{t+1} - \int_0^{s(s(k_t))} T'(m) dm - T(0) - s(s(k_t)) \right) (\hat{r}_{t+1} s_t + \hat{w}_{t+1} \exp(z_{t+1}) \right. \\ &\quad \left. - T'(s(s(k_t))) \hat{s}(s(k_t)) - \int_0^{s(s(k_t))} \tau'(m) dm - \hat{T}(0) - \hat{s}(s(k_t)) \right) (r_{t+1} + 1 - \delta) \right] \\ &\quad + \beta \mathbb{E}_t u' \left(y_{t+1} - \int_0^{s(s(k_t))} T'(m) dm - T(0) - s(s(k_t)) \right) \hat{r}_{t+1}, \end{aligned} \quad (25 \text{ a})$$

$$\begin{aligned} \mathcal{B} &= \beta \mathbb{E}_t \left[u'' \left(y_{t+1} - \int_0^{s(s(k_t))} T'(m) dm - T(0) - s(s(k_t)) \right) \right. \\ &\quad \times ((r_{t+1} + 1 - \delta) - T'(s(s(k_t))) s'(s(k_t)) - s'(s(k_t))) (r_{t+1} + 1 - \delta) \left. \right], \end{aligned} \quad (25 \text{ b})$$

$$\begin{aligned} \mathcal{D} &= u' \left(y_t - \int_0^{s(k_t)} T'(m) dm - T(0) - s(k_t) \right) T''(s(k_t)) \\ &\quad - u'' \left(y_t - \int_0^{s(k_t)} T'(m) dm - T(0) - s(k_t) \right) (1 + T'(s(k_t)))^2, \end{aligned} \quad (25 \text{ c})$$

$$\mathcal{G} = u' \left(y_t - \int_0^{s(k_t)} T'(m) dm - T(0) - s(k_t) \right), \quad (25 \text{ d})$$

$$\begin{aligned} \mathcal{J} &= u'' \left(y_t - \int_0^{s(k_t)} T'(m) dm - T(0) - s(k_t) \right) \left(\hat{r}_t k_t + \hat{w}_t \exp(z_t) - \int_0^{s(k_t)} \tau'(m) dm - \hat{T}(0) \right) \\ &\quad \times (1 + T'(s(k_t))). \end{aligned} \quad (25 \text{ e})$$

We have

$$\int_0^{s(k)} T'(m) dm = T(s(k)) - T(0). \quad (26)$$

Then,

$$\int_0^{s(k_t)} T'(m) dm + T(0) = T(s(k_t)) \quad (27)$$

We also know $c_t = (r_t + 1 - \delta)k_t + w_t \exp(z_t) - T(k_{t+1}) - k_{t+1}$.

$$\begin{aligned}
& c_t + \mu \hat{c}_t \\
&= (r_t + \mu \hat{r}_t + 1 - \delta)k_t + (w_t + \mu \hat{w}_t) \exp(z_t) - \int_0^{s(k_t) + \mu \hat{s}(k_t)} T'(m) + \mu \tau'(m) dm - T(0) - \mu \hat{T}(0) - k_{t+1} \\
&= (r_t + 1 - \delta)k_t + \mu \hat{r}_t k_t + w_t \exp(z_t) + \mu \hat{w}_t \exp(z_t) - \int_0^{s(k_t) + \mu \hat{s}(k_t)} T'(m) dm - \int_0^{s(k_t) + \mu \hat{s}(k_t)} \mu \tau'(m) dm \\
&\quad - T(0) - \mu \hat{T}(0) - k_{t+1} \\
&= (r_t + 1 - \delta)k_t + w_t \exp(z_t) - \int_0^{s(k_t)} T'(m) dm - T(0) - k_{t+1} + \mu \hat{r}_t k_t + \mu \hat{w}_t \exp(z_t) - T'(s(k_t)) \mu \hat{s}(k) \\
&\quad - \int_0^{s(k_t)} \mu \tau'(m) dm - \mu \hat{T}(0).
\end{aligned} \tag{28}$$

Then,

$$\hat{c}_t = k_t \hat{r}_t + \exp(z_t) \hat{w}_t - T'(s(k_t)) \hat{s}(k) - \int_0^{s(k_t)} \tau'(m) dm - \hat{T}(0).$$

A.2 Proof of tax incidence on government revenue

As any government revenue, denoted as R , is subsequently redistributed to households in the form of lump-sum transfers. Subsequently, we obtain the following relationship:

$$\begin{aligned}
0 &= \int_0^\infty T(s(k)) f(k) dk \\
&= \int_0^\infty \left(T(0) + \int_0^{s(k)} T'(m) dm \right) f(k) dk \\
&= T(0) + \int_0^\infty \int_0^{s(k)} T'(m) dm f(k) dk
\end{aligned}$$

Then, we have

$$T(0) = - \int_0^\infty \int_0^{s(k)} T'(m) dm f(k) dk.$$

Therefore,

$$R = -T(0) = \int_0^\infty \int_0^{s(k)} T'(m) dm f(k) dk.$$

and

$$\hat{R} = -\hat{T}(0).$$

We know

$$\begin{aligned}
& R + \mu \hat{R} \\
&= \int_0^\infty \int_0^{s(k)+\mu\hat{s}(k)} (T'(m) + \mu\tau'(m)) dm (f(k) + \mu\hat{f}(k)) dk \\
&= \int_0^\infty \int_0^{s(k)+\mu\hat{s}(k)} T'(m) dm (f(k) + \mu\hat{f}(k)) dk + \int_0^\infty \int_0^{s(k)+\mu\hat{s}(k)} \mu\tau'(m) dk (f(k) + \mu\hat{f}(k)) dk \\
&= \int_0^\infty \int_0^{s(k)+\mu\hat{s}(k)} T'(m) dm f(k) dk + \int_0^\infty \int_0^{s(k)+\mu\hat{s}(k)} T'(m) dm \mu\hat{f}(k) dk \\
&\quad + \int_0^\infty \int_0^{s(k)+\mu\hat{s}(k)} \mu\tau'(m) dm f(k) dk \\
&= \int_0^\infty \int_0^{s(k)} T'(m) dm + T'(s(k)) \mu\hat{s}(k) f(k) dk + \int_0^\infty \int_0^{s(k)} T'(m) dm \mu\hat{f}(k) dk \\
&\quad + \int_0^\infty \int_0^{s(k)} \mu\tau'(m) dm f(k) dk \\
&= \int_0^\infty \int_0^{s(k)} T'(m) dm f(k) dk + \int_0^\infty T'(s(k)) \mu\hat{s}(k) f(k) dk + \int_0^\infty \int_0^{s(k)} T'(m) dm \mu\hat{f}(k) dk \\
&\quad + \int_0^\infty \int_0^{s(k)} \mu\tau'(m) dm f(k) dk.
\end{aligned} \tag{29}$$

Hence, we obtain

$$\begin{aligned}
\hat{R} &= \int_0^\infty T'(s(k)) \hat{s}(k) f(k) dk + \int_0^\infty \int_0^{s(k)} T'(m) dm \hat{f}(k) dk + \int_0^\infty \int_0^{s(k)} \tau'(m) dm f(k) dk \\
&= \int_0^\infty T'(s(k)) \hat{s}(k) f(k) dk + \int_0^\infty \int_0^{s(k)} T'(m) dm \hat{f}(k) dk + 1.
\end{aligned} \tag{30}$$

From equation (25), $\hat{s}_t(k) = \frac{\mathcal{G}\tau'(s(k)) + \mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}}$, substituting it into above equation, we obtain

$$\begin{aligned}\hat{R} &= \int_0^\infty T'(s(k)) \frac{\mathcal{G}\tau'(s(k)) + \mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(k) dk + \int_0^\infty \int_0^{s(k)} T'(m) dm \hat{f}(k) dk + 1 \\ &= T'(s^*(k)) \frac{\mathcal{G}(s^*(k))}{\mathcal{B}(s^*(k)) - \mathcal{D}(s^*(k))} \frac{f(s^*(k))}{1 - F(s^*(k))} + \int_0^\infty T'(s(k)) \frac{\mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(k) dk \quad (31) \\ &\quad + \int_0^\infty \int_0^{s(k)} T'(m) dm \hat{f}(k) dk + 1.\end{aligned}$$

From equation (6), $\hat{s}_t(y) = \frac{\mathcal{G}\tau'(s(y)) + \mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}}$, we obtain

$$\hat{c} = -(1 + T'(s(y))) \hat{s}(y) - \int_0^{s(y)} \tau'(m) dm - \hat{T}(0) \quad (32)$$

$$\begin{aligned}&= -(1 + T'(s(y))) \hat{s}(y) + T'(k^*) \frac{\mathcal{G}(k^*)}{\mathcal{B}(k^*) - \mathcal{D}(k^*)} \frac{f(k^*)}{1 - F(k^*)} + \int_0^\infty T'(s(y)) \frac{\mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(y) dy \\ &\quad \quad \quad (33)\end{aligned}$$

$$+ \int_0^\infty \int_0^{s(y)} T'(m) dm \hat{f}(y) dy + 1 - \int_0^{s(y)} \tau'(m) dm. \quad (34)$$

A.3 Incidence on social welfare

We have social welfare $W = \sum_{t=0}^{\infty} \beta^t \int_0^{\infty} u(c) f(k) dk = \frac{1}{1-\beta} \int_0^{\infty} u(c) f(k) dk$, Then, we perform perturbation on social welfare,

$$\begin{aligned}
\hat{W} &= \frac{1}{1-\beta} \int_0^{\infty} \hat{u}(c) f(k) dk + \frac{1}{1-\beta} \int_0^{\infty} u(c) \hat{f}(k) dk \\
&= \frac{1}{1-\beta} \int_0^{\infty} u'(c) \hat{c}(k) f(k) dk + \frac{1}{1-\beta} \int_0^{\infty} u(c) \hat{f}(k) dk \\
&= \frac{1}{1-\beta} \int_0^{\infty} u'(c) \left[-T'(s(k_t)) \hat{s}(k) - \int_0^{s(k_t)} \tau'(m) dm \right] f(k) dk \\
&\quad + T'(s^*(k)) \frac{\mathcal{G}(s^*(k))}{\mathcal{B}(s^*(k)) - \mathcal{D}(s^*(k))} \frac{f(s^*(k))}{1 - F(s^*(k))} \frac{1}{1-\beta} \int_0^{\infty} u'(c) f(k) dk \\
&\quad + \frac{1}{1-\beta} \int_0^{\infty} u'(c) \left(\int_0^{\infty} T'(s(k)) \frac{\mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(k) dk \right) f(k) dk \\
&\quad + \frac{1}{1-\beta} \int_0^{\infty} u'(c) \left(\int_0^{\infty} \int_0^{s(k)} T'(m) dm \hat{f}(k) dk + 1 \right) f(k) dk + \frac{1}{1-\beta} \int_0^{\infty} u(c) \hat{f}(k) dk.
\end{aligned} \tag{35}$$

We need $\hat{W} = 0$ to obtain the optimal social welfare,

$$\begin{aligned}
&T'(s^*(k)) \frac{\mathcal{G}(s^*(k))}{\mathcal{B}(s^*(k)) - \mathcal{D}(s^*(k))} \frac{f(s^*(k))}{1 - F(s^*(k))} \int_0^{\infty} u'(c) f(k) dk \\
&= \int_0^{\infty} u'(c) \left[k_t \hat{r}_t + \exp(z_t) \hat{w}_t - T'(s(k_t)) \hat{s}(k) - \int_0^{s(k_t)} \tau'(m) dm \right] f(k) dk \\
&\quad + \int_0^{\infty} u'(c) \left(\int_0^{\infty} T'(s(k)) \frac{\mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(k) dk \right) f(k) dk \\
&\quad + \int_0^{\infty} u'(c) \left(\int_0^{\infty} \int_0^{s(k)} T'(m) dm \hat{f}(k) dk + 1 \right) f(k) dk + \int_0^{\infty} u(c) \hat{f}(k) dk.
\end{aligned} \tag{36}$$

We use h to denote $\frac{\mathcal{G}(s^*(k))}{\mathcal{B}(s^*(k)) - \mathcal{D}(s^*(k))} \frac{f(s^*(k))}{1 - F(s^*(k))} \int_0^\infty u'(c) f(k) dk$. We obtain

$$\begin{aligned} T'(s^*(k)) &= \frac{1}{h} \int_0^\infty u'(c) \left[k\hat{r} + \exp(z)\hat{w} - T'(s(k))\hat{s}(k) - \int_0^{s(k)} \tau'(m) dm \right] f(k) dk \\ &\quad + \frac{1}{h} \int_0^\infty u'(c) \left(\int_0^\infty T'(s(k)) \frac{\mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(k) dk \right) f(k) dk \\ &\quad + \frac{1}{h} \int_0^\infty u'(c) \left(\int_0^\infty \int_0^{s(k)} T'(m) dm \hat{f}(k) dk + 1 \right) f(k) dk + \frac{1}{h} \int_0^\infty u(c) \hat{f}(k) dk. \end{aligned} \tag{37}$$

Rearranging above equation, we obtain We need $\hat{W} = 0$ to obtain the optimal non-linear taxation,

$$T'(k^*) = \frac{1 - F(k^*)}{k^* f(k^*)} \lambda(k^*) [\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E} + \mathcal{F}],$$

where

$$\begin{aligned} \lambda(k^*) &= \frac{[\mathcal{B}(k^*) - \mathcal{D}(k^*)] k^*}{\mathcal{G}(k^*)}, \\ \mathcal{A} &= \int_{k^*}^\infty \left(1 - \frac{u'(c)}{\varphi} \right) \frac{f(y)}{1 - F(k^*)} dy, \\ \mathcal{B} &= \frac{1}{\varphi} \int_0^\infty u'(c) (k\hat{r} + \exp(z)\hat{w}) f(y) dy, \\ \mathcal{C} &= \int_0^\infty T'(s(y)) \frac{\mathcal{J} - \mathcal{A}}{\mathcal{B} - \mathcal{D}} f(y) dy, \\ \mathcal{D} &= \int_0^\infty \int_0^{s(y)} T'(m) dm \hat{f}(y) dy, \\ \mathcal{E} &= -\frac{1}{\varphi} \int_0^\infty u'(c) T'(s(y)) \hat{s}(y) f(y) dy, \\ \mathcal{F} &= \frac{1}{\varphi} \int_0^\infty u(c) \hat{f}(y) dy, \\ \varphi &= \int_0^\infty u'(c) f(y) dy. \end{aligned}$$

Appendix B Computation algorithm

References

BALL L, MANKIW N.G. Market Power in Neoclassical Growth Models[J]. The Review of the Economic Studies. 2022.