On the Progressivity of Income Taxation

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Abstract

We study the progressivity of income taxation in an infinite-horizon Aiyagari-Bewley-Huggett economy with idiosyncratic risk in capital income as well as labor income. We investigate the tax incidence of increasing tax progressivity through perturbations on the system of the Hamilton-Jacobi-Bellman (HJB) and Kolmogorov Forward (KF) equations. Then, we adopt the variational approach to decompose the welfare effect of increasing tax progressivity into several components, showing that the response of the wealth distribution to taxation is important in the determination of optimal tax progressivity.

JEL classifications: E21, H21, H31

Keywords: progressive income taxation, idiosyncratic investment risk, optimal tax progressivity, perturbation method

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1 Introduction

This paper addresses the problem of the progressivity of personal income taxation, where income includes both capital and labor income. We assume that the tax scheme is imposed on the sum of different types of personal income and, specifically, it does not discriminate between capital and labor income.[1] Piketty et al. (2022) argued that the fuzziness of the capital vs. labor frontier is the simplest and the most compelling rationale for a comprehensive income tax which treats capital and labor income flows alike. Our tax scheme adopts a constant rate of progressivity (CRP) as popularized by Benabou (2002) and Heathcote et al. (2017). The later paper shows that the CRP tax scheme approximates the actual tax and transfer system of the U.S. economy pretty well. We explore the implications of imposing the CRP tax scheme for both positive and normative issues.

Our model builds on Achdou et al. (2022), which is a continuous-time version of the Aiyagari-Bewley-Huggett (ABH) model.[2] We extend the work of Achdou et al. (2022) to include the labor market in a general equilibrium, which has both idiosyncratic earnings risk and investment risk.[3] Following Angeletos (2007) and Panousi (2012), we introduce idiosyncratic investment risk into the model via the private firms.[4] Our infinite-horizon model, with the inclusion of idiosyncratic investment risk, generates a stationary wealth distribution in a general equilibrium.[5] We extend the mean field game to taxation analyses and study the

[1]This feature approximately holds in the real world. For example, the current U.S. personal income tax code is imposed on the sum of different types of personal incomes and does not distinguish between them in general (say, wages and salaries vs. interest and dividends) when computing tax liabilities. A notable exception with discrimination between labor and capital income is the so-called "dual income tax" (see Sørensen (1994) for details).

[2]During the past two decades, the ABH model has become a workhorse for policy evaluations in the current state-of-the-art macroeconomics that jointly addresses aggregate and inequality issues. For surveys of the literature, see Heathcote et al. (2009), Guvenen (2011), Quadrini and Ríos-Rull (2015), Krueger et al. (2016) and Chapter 18 of Ljungqvist and Sargent (2018).

[3]In their benchmark model, Achdou et al. (2022) solved a general equilibrium only with idiosyncratic earning risk. Without investment risk, their benchmark model does not intend to match the wealth distribution in the real world. In an extension in their online appendix, they introduced idiosyncratic investment risk into the model. However, they did not solve a general equilibrium there.

[4]In their online appendix (section G.4), Achdou et al. (2022) explicitly suggested that one particularly appealing interpretation of the risky asset is that the return of the risky asset is the return from owning and running a private firm as in Angeletos (2007).

[5]Angeletos (2007) and Panousi (2012) investigate impacts of idiosyncratic investment risk on the macroeconomy in general equilibrium models. However, their models do not have stationary wealth distributions or they need to use death rates to confine the spread of wealth distributions so that stationary distributions can arise in their models.

impact of progressive income taxes.[6] Taking derivatives of endogenous variables with respect to the progressivity of income taxation, we implement a perturbation analysis of the general equilibrium in a heterogeneous agents economy.

As in Achdou et al. (2022), individual households' consumption/saving decisions and the evolution of the wealth distribution are summarized by a Hamilton-Jacobi-Bellman (HJB) equation and a Kolmogorov Forward (KF) equation, respectively. We obtain the price vector and the stationary wealth distribution of the general equilibrium by solving the coupled HJB and KF equations. We use the KF equation to find the stationary wealth distribution in the economy. By comparing the stationary wealth distribution before the tax reform with that after the tax reform, we can exactly calculate the change in the stationary wealth distribution due to the tax reform. We numerically identify the response of the wealth distribution to tax reforms in the analysis of tax incidence and the determination of optimal tax progressivity.

We first implement calibration exercises using our model and numerically calculate the income and wealth distributions. Using the KF equation, we can find the stationary wealth distribution without drawing random numbers. The income and wealth distributions in our model display fat tails, and match the quantiles and top-percentage shares of the U.S. data.

Second, we investigate the tax incidence of increasing tax progressivity through perturbations on the system of the HJB and KF equations. We perform a nonlinear analysis of the consumption function and of the distribution function. We find that the economy-wide effect of progressivity changes depends on one elasticity, the elasticity of the saving rate to the progressivity. The nonlinear analysis permits us to investigate the detailed incidence of individuals at each percentile of the wealth distribution, from its bottom to the top.

Third, following the variational approach in the literature on optimal taxation, we decompose the welfare effect of increasing tax progressivity into several components: (i) the mechanical effect and behavioral response (Saez, 2001), (ii) pecuniary externalities (Dávila et al., 2012), (iii) private intermediation (Chang and Park, 2021), and (iv) the response of the wealth distribution to tax reforms (new to the optimal tax literature). We find that the response of the wealth distribution to tax reforms is an important channel in the analysis of the welfare effect and the determination of optimal tax progressivity.

Finally, we extend our model along two directions. The first one is to incorporate endogenous labor supply into the model. In the benchmark model, there is only the reaction of

[6] Panousi and Reis (2021) investigated optimal capital taxation in a heterogeneous agents model with idiosyncratic investment risk. Panousi and Reis (2022) found that the optimal capital tax rate is negative if idiosyncratic investment risk is low.

savings with respect to the tax progressivity. Now the tax progressivity also has distortions on the labor supply. In the second extension, we permit households to access a safe asset and to operate a private firm. Then households have a portfolio selection problem.

1.1 Related literature

Aiyagari (1994) and Huggett (1993) used additive shocks as idiosyncratic productivity risk, but they could not produce the fat tail of the wealth distribution.[7] Benhabib et al. (2011) used an overlapping generations model to theoretically show that multiplicative shocks can generate fat-tailed wealth distributions in heterogeneous-agent models. Achdou et al. (2022) featured a fat-tailed stationary wealth distribution by incorporating investment risk into the canonical ABH model. However, Benhabib et al. (2011, 2015) did not solve their problems in a general equilibrium setting. This paper finds the general equilibrium of a heterogeneous agent economy with idiosyncratic investment risk.[8]

Angeletos (2007) and Angeletos and Panousi (2009) examined heterogeneous agent models with idiosyncratic investment risk. Recent papers have found empirical evidence in support of heterogeneous investment returns. Smith et al. (2023) documented heterogeneity in rates of return in the U.S. Fagereng et al. (2020) found that individuals earn markedly different average returns on their net worth in Norway. Bach et al. (2020) found a large dispersion in wealth returns across Swedish households. Atkeson and Irie (2022) emphasized the role of family firms in generating high top wealth shares.

Heathcote et al. (2017) studied optimal tax progressivity under the CRP tax scheme for labor income. They investigated the optimal tax progressivity in an analytical framework. In the benchmark model they obtain an optimal progressivity which is lower than that in the U.S. data.[9] We obtained an optimal progressivity which was much higher than that in Heathcote et al. (2017). Our paper studies the tax scheme on the sum of capital and labor income. Hsu and Yang (2013) investigated optimal two-bracket income taxes in a heterogeneous agents economy with only idiosyncratic labor earnings risk.

Boar and Midrigan (2022) imposed the CRP tax scheme and characterized the optimal tax

[7]Stachurski and Toda (2019) demonstrated that canonical ABH models cannot explain the joint distribution of income and wealth; in particular, they cannot explain the empirical fact that wealth is heavier-tailed than income if (i) agents are infinitely-lived, (ii) saving is risk-free, and (iii) agents have constant discount factors.

[8]Benhabib et al. (2015) show that this mechanism can also generate the fat tail of the stationary wealth distribution in an infinite-horizon economy with idiosyncratic investment risk.

[9]After taking into account the planner's additional inequality aversion, Heathcote et al. (2017) found an optimal progressivity close to the U.S. data.

progressivity of income and wealth taxes in an ABH type of model.[10] Feenberg et al. (2017) documented a large and steady increase in tax progressivity in the U.S. during 1960-2008. They imposed the CRP tax scheme in a canonical ABH model and found that the optimal tax progressivity was very close to that measured in the data. While these two papers focus on the quantitative results of tax progressivity, our paper has theoretical as well as quantitative results.[11]

Ales and Sleet (2022) used a perturbation method to find an equation which determines optimal taxation in a discrete-choice model. They found that the optimal tax rate can be expressed as a convergent series of private elasticities. We use the perturbation method to find an equation that the elasticity of the distribution function with respect to the tax progressivity satisfies. We show that the elasticity of the distribution function only depends on the elasticity of savings.

Heathcote and Tsujiyama (2021) studied the optimal nonlinear income tax schedule in a model calibrated to match the U.S. economy. They found that the optimal marginal tax rates increased with income. Distributions of agent heterogeneity are typically exogenously given in the literature on optimal taxation. Chang and Park (2021), who extended the optimal income tax schedule of Diamond (1998) and Saez (2001) to incorporate the impact of private insurance, is an exception. They considered the design of a nonlinear labor income tax in the ABH type of models with endogenous wealth distributions. We complement their work by numerically identifying the role of endogenous wealth distributions.[12]

2 Basic model

Time is continuous, indexed by $t \in [0, \infty)$. The economy is populated by a continuum of infinitely-lived households of unit mass. The household is indexed by $i \in [0, 1]$. Each household consists of a worker and a producer ("entrepreneur"). The worker supplies one

[10] Boar and Midrigan (2022) showed that a uniform flat tax on capital and labor income combined with a lump-sum transfer is nearly optimal.

[11] Boar and Knowles (2022) found the formula for optimal tax rates in a heterogeneous agents economy with idiosyncratic investment risk, and they used death rates to generate the stationary wealth distribution in the model. Ge (2021) investigated the optimal top tax rate in a heterogeneous agents model with idiosyncratic investment risk.

[12] To find the stationary wealth distribution, Chang and Park (2021) used a simulation method. We use the KF equation to find the joint income and wealth distribution. The KF equation provides a technique to connect the macro description of the economy with the individual wealth accumulation process, in that the coefficient of the equation comes from the household's policy functions.

unit of labor inelastically to the labor market (we allow for endogenous labor supply in the extension). The entrepreneur runs a privately-held firm ("family business") by hiring labor from the labor market and accumulating capital within his own family business. Each household is atomistic and thus a price taker.

The evolution of capital k is given by the household budget,

$$dk(t) = d\pi(t) + [w(t)z(t) - c(t) - \delta k(t) - T(t)]\, dt, \tag{1}$$

where $d\pi(t)$ is the profit earned from running its own family business, $w(t)$ is the wage rate, $z(t)$ is the labor productivity shock, $c(t)$ is the household's consumption, δ is the depreciation rate of capital $k(t)$, and $T(t)$ is the income tax scheme imposed by the government. We let $z(t)$ follow a two-state Poisson process $z(t) \in \{z_1, z_2\}$ with $z_2 > z_1 > 0$. The process jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2.

Following Angeletos (2007) and Angeletos and Panousi (2009), we let

$$d\pi(t) = \left[Ak(t)^{\alpha} n(t)^{1-\alpha} - w(t)n(t)\right] dt + \sigma k(t) dB(t), \tag{2}$$

where A represents the total factor productivity (TFP), $n(t)$ is the amount of labor hired by the family business, and $B(t)$ is a standard Brownian motion. The scalar σ measures the undiversified idiosyncratic investment risk due to market incompleteness. The private firms use a neoclassical production technology and they hire labor in a competitive market. The total labor supply in the economy is exogenous and constant. Thus, the aggregate production in the stationary economy is constant.[13]

We assume that a Cobb-Douglas production technology is accessible to all entrepreneurs. Entrepreneurs choose $n(t)$ to achieve

$$\max_{n(t)} Ak(t)^{\alpha} n(t)^{1-\alpha} - w(t)n(t),$$

which yields the optimal labor hiring,

$$n(t) = \left(\frac{(1-\alpha)A}{w(t)}\right)^{\frac{1}{\alpha}} k(t). \tag{3}$$

[13] Jones and Kim (2018) investigate the income inequality in an endogenous growth economy, which displays an income distribution with a Pareto tail.

Substituting the optimal $n(t)$ into (2) we have

$$d\pi(t) = r(t)k(t)dt + \sigma k(t)dB(t), \tag{4}$$

where $r(t)$ represents the mean rate of return on capital with $r(t) = \alpha A^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{w(t)}\right)^{\frac{1}{\alpha}-1}$.

The income tax scheme $T(\cdot)$ is the same as in Benabou (2002) and Heathcote et al. (2017),

$$T(y) = y - \varphi y^{1-\zeta},\ \zeta < 1, \tag{5}$$

where $\varphi > 0$ and the parameter ζ determines the progressivity of income taxation. Since $\zeta = \frac{yT''(y)}{1-T'(y)}$ for all $y > 0$, the tax scheme is known as the CRP tax scheme and it is progressive, proportional, and regressive, depending on whether $\zeta > 0$, $\zeta = 0$, or $\zeta < 0$. $T(y(t))$ is imposed on the household's total income $y(t) = r(t)k(t) + w(t)z(t)$, which is in line with the tax code in the U.S. Because $T(y(t))$ could be negative if $y(t)$ is low, as noted by Heathcote et al. (2017), the tax function (5) is best seen as a tax and transfer scheme.

The government is required to balance its budget in each period,

$$g \int_0^1 y^i di = \int_0^1 T\left(y^i\right) di, \tag{6}$$

where g denotes the fraction of the aggregate output $Y(t) = \int_0^1 y^i(t)di$ that is devoted to the government consumption. In the numerical analysis, we let the government choose the pair (g, ζ), with φ being determined residually by the balanced budget (6). As in Judd (1985), Conesa et al. (2009), Hsu and Yang (2013), and Chang and Park (2021), we abstract from the issue of government bonds.

The household suffers from idiosyncratic labor income risk and idiosyncratic investment risk. Both risks give rise to the precautionary saving motive. The household's utility maximization problem is

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c(t))dt,$$

where $\rho > 0$ is the time discount factor, $\mathbb{E}_0$ is the expectation operator conditional on the information set at time 0, and $u(c(t))$ is the instantaneous utility function. The household is subject to the constraint,

$$dk(t) = \left[\varphi(r(t)k(t) + w(t)z(t))^{1-\zeta} - \delta k(t) - c(t)\right] dt + \sigma k(t)dB(t), \tag{7}$$

plus $k(t) \geq 0$ for all $t \geq 0$. Equation (7) is obtained by combining equations (1), (4), and (5).

Let $v(k(t))$ be the value function of the household's problem,

$$v(k(t)) = \max_{\{c(\tau),k(\tau)\}_{\tau=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-\rho(\tau-t)} u(c(\tau)) d\tau,$$

where $\mathbb{E}_t$ is the expectation operator conditional on the information set at time t. The valuation function $v(k)$ satisfies the HJB equation,

$$\rho v_j(k) = \max_c u(c_j) + v_j'(k) s_j(k) + \frac{1}{2} v_j''(k) \sigma^2 k^2 + \lambda_j \left(v_{-j}(k) - v_j(k) \right), \; j = 1, 2, \tag{8}$$

where $s_j(k) = \varphi(rk + wz_j)^{1-\zeta} - \delta k - c_j(k)$. We adopt the convention that $-j = 2$ when $j = 1$, and $-j = 1$ when $j = 2$.[14] The household's problem is characterized by the HJB equation. We can obtain the consumption policy function $c_j(k)$ and saving policy function $s_j(k)$ by solving the HJB equation. The first-order condition is given by

$$v_j'(k) = u'(c_j(k)), \; j = 1, 2, \tag{9}$$

which gives $c_j(k) = (u')^{-1}(v_j'(k))$.

The cross-section capital distributions $f_j(k,t)$, $j = 1, 2$, are governed by the KF equation,[15]

$$\frac{\partial}{\partial t} f_j(k,t) = \frac{1}{2} \frac{\partial^2}{\partial k^2} \left[\sigma^2 k^2 f_j(k,t) \right] - \frac{\partial}{\partial k} \left[s_j(k) f_j(k,t) \right] - \lambda_j f_j(k,t) + \lambda_{-j} f_{-j}(k,t), \; j = 1, 2. \tag{10}$$

Letting $\frac{\partial}{\partial t} f_j(k,t) = 0$ in equation (10), we find that the stationary distributions $f_j(k)$, $j = 1, 2$, satisfy

$$0 = \frac{1}{2} \frac{d^2}{dk^2} \left[\sigma^2 k^2 f_j(k) \right] - \frac{d}{dk} \left[s_j(k) f_j(k) \right] - \lambda_j f_j(k) + \lambda_{-j} f_{-j}(k), \; j = 1, 2. \tag{11}$$

We have $m_1 + \int_0^{\infty} f_1(k) dk + \int_0^{\infty} f_2(k) dk = 1$, where m_1 represents a Dirac point mass at the lower bound $k = 0$. The stationary distribution could have a mass point since the wealth accumulation process may hit the lower bound $k = 0$ in a finite period of time with a positive probability. The savings rate of the low-type households is non-positive for high k. When the earnings risk is high and the redistribution is not high enough, low-type households have

[14] See online Technical Appendix 1.1.1 for the proof of the HJB equation.

[15] See online Technical Appendix 1.1.2 for the proof of the KF equation.

negative savings for all $k > 0$. If a household with $k > 0$ keeps drawing low earnings shocks, the wealth process eventually hits the lower bound within a finite period of time.[16]

We solve the consumption function from the HJB equation and then obtain the saving function. Besides the parameters, the saving function is the only coefficient of the KF equation. From the KF equation, we solve the wealth distribution function, which is the key element of aggregation in the model. The saving function is the only channel through which the individual behavior influences the aggregate variables of the economy.

We focus on a commonly used utility function.

Assumption 1 $u(c) = \frac{c^{1-\eta}}{1-\eta}$, $\eta > 1$.

The competitive equilibrium of the economy is standard.

Definition 1 *A stationary competitive equilibrium is defined as a pair of prices (r, w), government policy $T(y)$, and individual policy functions $c(k)$, $s(k)$, and $n(k)$, such that the following conditions hold:*

(i) given (r, w) and $T(y)$, the plans $c(k)$, $s(k)$, and $n(k)$ are optimal for the household;

(ii) the government budget is balanced as in equation (6);

(iii) the stationary distribution $f_j(k)$ for $j = 1, 2$ satisfies equation (11);

(iv) the labor market clears:

$$\sum_{j\in\{1,2\}} \int_0^\infty n(k) f_j(k) dk = N(t) \equiv \frac{z_1\lambda_2 + z_2\lambda_1}{\lambda_1 + \lambda_2}.$$

The wealth distribution $f_j(k)$ influences the aggregate economy through the labor market equilibrium. Even though labor supply is exogenously given, labor demand is determined by the wealth distribution. The wage rate is determined by the labor market equilibrium. On the other hand, the equilibrium wage rate and return on capital influence the saving function of households. The stationary wealth distribution is derived from the KF equation, which is determined by the saving function. Thus, $f_j(k)$ and the aggregate economy interact.

Proposition 1 *If $\zeta = 0$ and $\eta(\eta - 1)\sigma^2 < 2(\rho + \delta)$, there exists a general equilibrium.*

For $\zeta = 0$ we can show the existence of the general equilibrium in Proposition 1. Even though we can not show the existence result for $\zeta > 0$ analytically, we always find a general equilibrium in our calibration exercises numerically.

[16] Benhabib et al. (2015) shows that the wealth process hits the lower bound within a finite period of time with a positive probability in a discrete-time model.

3 Indirect diagnostics

We calibrate the parameters of the benchmark model. As an indirect diagnostic of our calibrated model, we quantitatively match the wealth inequality and, in particular, replicate the top wealth tail of the U.S. economy. Importantly, we quantitatively demonstrate that wealth is indeed heavier-tailed than income.

3.1 Calibration

We divide the parameters of the benchmark model into two groups. In the first group, parameter values are set from the existing literature. Given these set parameter values, we calibrate the parameters in the second group to match some key moments of the U.S. economy.

We take one moment in the model to be one calendar year in the data. We let $\alpha = 1/3$, which is standard. We set the coefficient of relative risk aversion $\eta = 1.1$, which is close to the logarithmic utility with $\eta = 1$. The depreciation rate $\delta = 0.03$. Bigio and Sannikov (2021) provides us with the value of the time discount factor $\rho = 0.04$. We set g (the government purchases to GDP ratio) equal to 0.189, following Heathcote et al. (2017). Heathcote et al. (2017) estimated $\zeta = 0.181$ for the U.S. economy. We adopt their estimation.[17]

Table 1: Calibration from the literature

Coefficient of relative risk aversion	$\eta = 1.1$
Time discount factor	$\rho = 0.04$
Depreciation rate	$\delta = 0.03$
Capital income share	$\alpha = 1/3$
Progressivity of income taxation	$\zeta = 0.181$
Government purchases to GDP ratio	$g = 0.189$

The parameters that remain to be determined in their values are σ (idiosyncratic investment risk), (λ_1, λ_2) (probabilities of transition for labor productivities), (z_1, z_2) (labor productivities), and TFP A. We calibrate these parameter values to match: (i) top 1% capital share, (ii) top 1% income share, (iii) 90-95% and 95-99% wealth shares, and (iv) interest

[17] To estimate the parameters of the CRP tax scheme, Heathcote et al. (2017) use the definition of income including both labor earnings and capital income ("labor earnings, self-employment income, private transfers (alimony, child support, help from relatives, miscellaneous transfers, private retirement income, annuities, and other retirement income), plus income from interest, dividends, and rents." (p. 1699)). Our setup for income taxation is consistent with the definition of income in their estimation.

rates. Tables 1 and 2 summarize the parameter values used in our quantitative study. Increasing σ can improve the wealth share of the top 1%. The proportion of low income households equals $\lambda_2/(\lambda_1+\lambda_2)$ and the proportion of the high income households equals $\lambda_1/(\lambda_1+\lambda_2)$. The 90-95%, 95-99%, and 99-100% income shares are affected by income transition probabilities. The capital-output ratio K/Y is 3.62. We refer to this value in Moll et al. (2022), which is 3. When $\zeta = 0.181$, the interest rate net of depreciation is 0.061.[18]

Table 2: Calibration from matching the targets

Parameter values	Targets
Volatility of Brownian motion $\sigma = 0.45$	Top 1% capital share.
Probability of transition for earnings $\{0.047,0.5\}$	Top 1% income share.
Labor productivities $\{0.15,3.5\}$	The 90-95 and 95-99 capital shares.
Total factor productivity $A = 0.6$	The capital-output ratio K/Y.

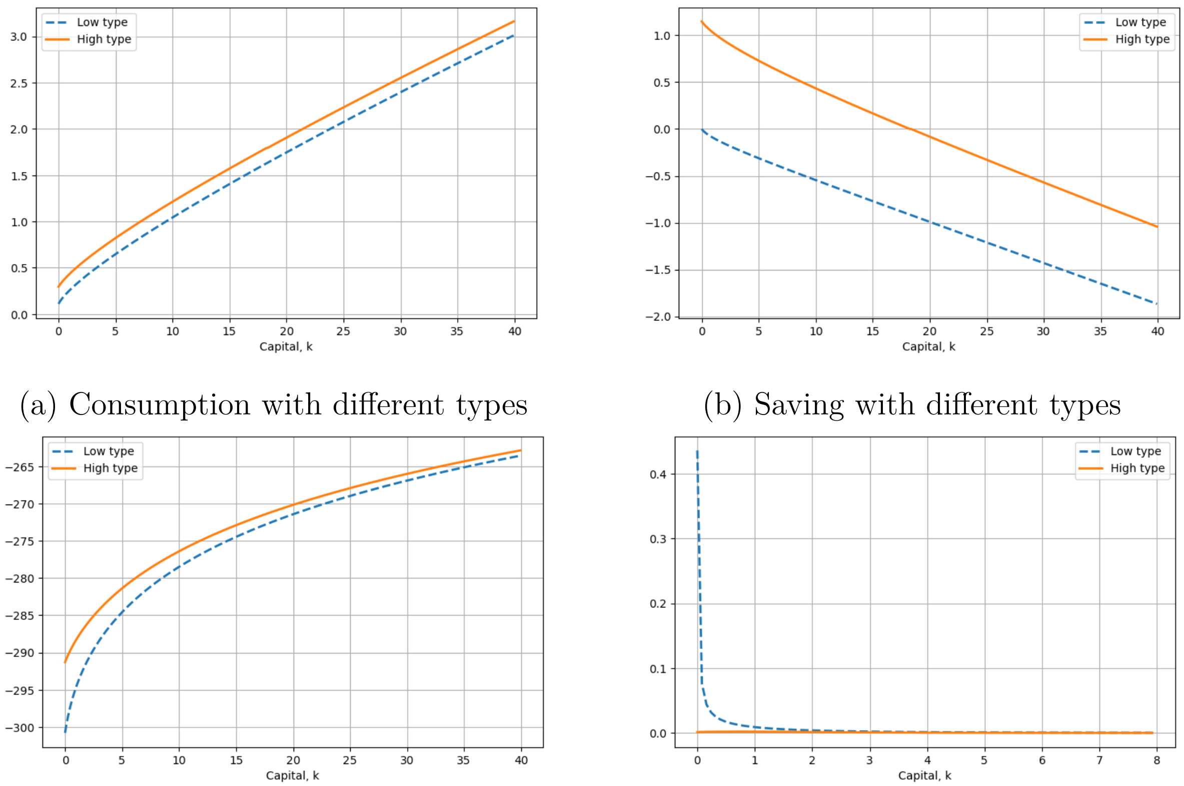
There are two sources causing precautionary saving, namely, idiosyncratic earnings risk and investment risk. Earnings risk, represented by the Poisson process, is prominent for the poor households. Due to the precautionary saving caused by earnings risk, the lower bound of the wealth space $k = 0$ acts as a reflecting barrier of the wealth accumulation process $\{k(t)\}_{t=0}^{\infty}$. Thus, the stochastic process $\{k(t)\}_{t=0}^{\infty}$ can not be stuck at zero and has a non-degenerating stationary wealth distribution. Idiosyncratic investment risk, represented by the volatility part of the Brownian motion, plays an important role in replicating the fat tail of the income and wealth distribution.

Using parameters in Tables 1 and 2, we numerically obtain the households' policy functions c_j and s_j at the tax progressivity $\zeta = 0.181$ as well as the value function v_j and the distribution function f_j for $j = 1, 2$.[19] Panel (a) of Figure 1 shows that as wealth increases, consumption increases monotonically for both low- and high-type households. Panel (b) of Figure 1 shows the saving functions. At $k(t) = 0$, the saving of the low type is zero while the saving of the high type is strictly positive. The positive saving of the high type at $k(t) = 0$ is due to the precautionary saving. The saving function for the low-earnings type is non-positive for all $k \geq 0$. The saving function for the high earnings type is first positive and then becomes negative as k increases. Panel (d) shows that a significant fraction of the low-type households

[18] When $\zeta = 0$, the equilibrium interest rate after depreciation is 0.042, under the parameter values of the benchmark calibration.

[19] We follow the numerical methods in Achdou et al. (2022) to solve the HJB and KF equations, and we carefully deal with the boundary conditions of the KFE.

are clustered at $k = 0$. Compared to $f_1(k)$, the size of $f_2(k)$ is rather insignificant.



(a) Consumption with different types

(b) Saving with different types

(c) Value function with different types

(d) Distribution with different types

Figure 1: Policy functions of households and the wealth distribution

3.2 Wealth and income distribution

We use the parameters in Tables 1 and 2 to numerically solve our model. We then calculate the quantiles of the distributions of income and wealth in the model and make a comparison with the U.S. data in Table 3, which were calculated by Díaz-Giménez et al. (2011) according to the data from the 2007 Survey of Consumer Finances (SCF).

Table 3: Wealth and income distribution

	Partition							
Percentile	0-20	20-40	40-60	60-80	80-90	90-95	95-99	99-100
Wealth share (data)	-0.002	0.011	0.045	0.112	0.120	0.111	0.267	0.336
Wealth share (model)	0	0	0.017	0.107	0.151	0.144	0.252	0.329
Income share (data)	0.028	0.067	0.113	0.183	0.138	0.102	0.159	0.210
Income share (model)	0.045	0.046	0.052	0.090	0.129	0.268	0.248	0.122

Our model produces a wealth distribution that matches that of the U.S. data quite well. The income distribution in the model, especially the top 1%, does not match the data very well, since there are only two states of labor earnings in the model.[20] For the U.S. data, the tail of the wealth distribution is fatter than that of the income distribution. Our model successfully produces this pattern: the top 1% share of the wealth distribution is higher than that of the income distribution.

The top 1% group holds a large fraction of wealth in the wealth distribution. In a model with only idiosyncratic labor earnings shocks, the wealth distribution has an upper bound, if the earnings process is bounded. This property causes difficulties in generating large wealth fractions at the top without exaggerating the top shares of the income distribution. By incorporating investment risk, we are able to produce large fractions at the top in both the income and wealth distributions.

[20]Boar and Midrigan (2022) use a super-star state of the labor efficiency. They generate an income distribution which matches the data better than ours.

3.3 Pareto tail

Algoritm

1. We set $\bar{y}^* = 6$. The maximum value of y, denoted by $y_{\max}$, is 75.23. Running the program, we obtain $\int_{\bar{y}^*}^{y_{\max}} f(z)dz = M$.

2. Using the expression $\int_{\bar{y}^*}^{y_{\max}} Cz^{-\alpha-1}dz = M$, we derive $C = M\alpha(\bar{y}^*)^{\alpha}$, where $\alpha = 1.53$, according to Proposition 2, which states that $\alpha = 2 - \eta + \frac{2(\rho+\delta)}{\eta\sigma^2}$. Thus, we obtain the value of C.

3.We choose our $\bar{y}$ from 0.3 to y_{max}. If $\bar{y} < \bar{y}^*$, we need to calculate two brackets separately. The first bracket, $[\bar{y}, \bar{y}^*]$, is calculated using $y_1^m = \int_{\bar{y}}^{\bar{y}^*} Cz^{-\alpha}dz$ (which is represented as $np.sum(y * f(y))$ in the program) and $y_1 = \int_{\bar{y}}^{\bar{y}^*} Cz^{-\alpha-1}dz$ (which is represented as $np.sum(f(y))$ in the program). The second bracket, $[\bar{y}^*, y_{max}]$, is calculated using formula $y_2^m = \int_{\bar{y}^*}^{\infty} Cz^{-\alpha}dz = \frac{C}{\alpha-1}(\bar{y}^*)^{1-\alpha}$, and $y_2 = \int_{\bar{y}^*}^{\infty} Cz^{-\alpha-1}dz = \frac{C}{\alpha}(\bar{y}^*)^{-\alpha}$. We then calculate $\frac{y_1^m + y_2^m}{(y_1+y_2)\bar{y}}$. If $\bar{y} > \bar{y}^*$, we can use the formula $\frac{\alpha}{\alpha-1}$.

Figure 2 reports changes in the tail exponent of stationary wealth distributions as ζ varies. Increasing ζ from 0 to 0.181 leads to a higher Pareto index, implying a more equal distribution of wealth. This result is consistent with the prediction of Theorem **??** that Θ_a ($\zeta > 0$) is higher than Θ_b ($\zeta = 0$).

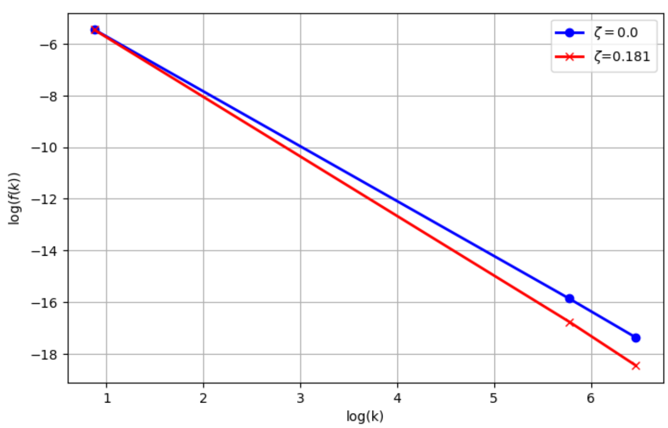


Figure 2: Pareto tail of the stationary distribution

The earnings distribution in Heathcote et al. (2017) displays a Pareto tail. Our model also generates a Pareto tail of the stationary wealth distribution. However, the mechanisms are different. In Heathcote et al. (2017), the log skill price is affine in learning abilities which are assumed to be exponentially distributed. The distribution of the skill prices has a Pareto tail since the exponential of an exponentially distributed random variable is Pareto distributed. We use a diffusion process with a reflecting barrier to produce the Pareto tail.[21] To obtain a stationary income distribution, Heathcote et al. (2017) need a death rate to confine the spread of the income process. We do not need this assumption, and our model is an infinite-horizon one.[22]

4 Incidence and welfare effect of tax reforms

This section first studies the incidence of tax reforms. It then studies the welfare effect of tax reforms. We obtain a characterization of optimal tax progressivity by imposing the condition that no tax reform has a positive impact on social welfare. The tax reform here is represented by varying tax progressivity at some ζ (say, $\zeta = 0.181$) by $\Delta\zeta = 0.01$, for a fixed level of g. We use a hat to represent the derivative with respect to the progressivity. Our computation of $\hat{m}$ is based on the numerical derivative method.[23]

4.1 Tax incidence

The household's consumption and saving behavior is influenced by the tax scheme through the HJB equation. A tax reform has impacts on the consumption and saving functions. It causes the distortion effects of taxes to change. Applying the KF equation enables us to numerically find the stationary wealth distribution in the economy. By comparing the stationary wealth distribution before the tax reform with that after the tax reform, we can exactly know $\hat{f}$ in response to a tax reform.

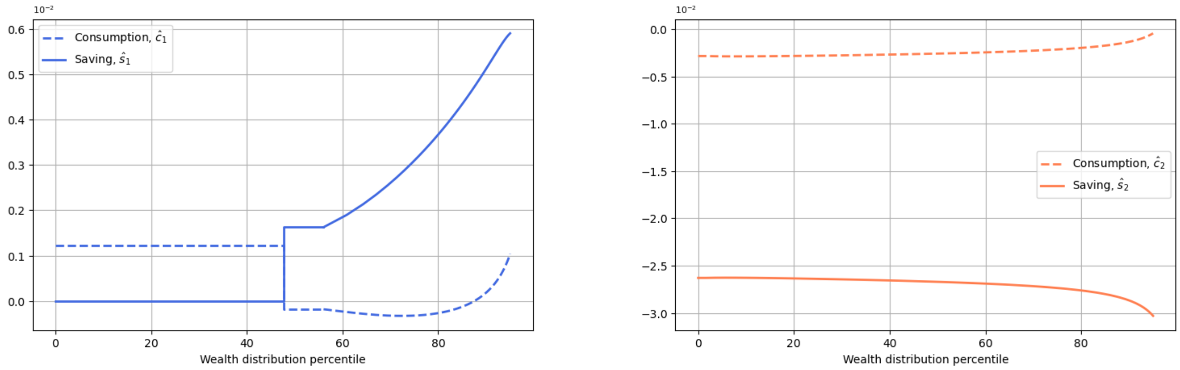
Figure 3 illustrates the effect of increasing tax progressivity on the policy functions, $\hat{c}_j$ and $\hat{s}_j$, $j = 1, 2$, at $\zeta = 0.181$.[24] The horizontal coordinate of the figure represents the wealth distribution percentile in the order of wealth k for each type of population.

[21]This diffusion process has two properties. The first property is that the drift term is asymptotically affine in the wealth level and is negative as the wealth approaches infinity. The second one is that the diffusion term is proportional to the wealth level. The investment risk is represented by a constant percentage change.

[22]Jones and Kim (2018) use a diffusion process similar to ours. However, the drift term in their model is positive and the destruction rate confines the spread of the productivity process.

[23]The computation includes two parts. Part 1: calculate $\hat{m}$. Part 2: calculate the resulting changes in social welfare associated with $\hat{m}$. The details of this method are in online Technical Appendices 5.2.1 and 5.3.1.

[24]The wealth distribution percentile in Figure 3 is only drawn to 95%, because the values of $\hat{c}_j$ and $\hat{s}_j$ for the group with a wealth ranking above 95% are too large relative to the group below this ranking. The detailed mechanism of how the progressive taxation affects consumption and savings is reported in online Technical Appendix 5.2.3.



(a) Low-type households (b) High-type households

Figure 3: $\hat{c}$ and $\hat{s}$ of households

The impact of changes in the progressivity on consumption comes from two channels: (i) Changes in the progressivity affect income and thus consumption. After-tax incomes are enhanced for low-wealth households with the low-earnings type, accounting for 99.63% of the low-type households. After-tax incomes are reduced for all high-type households.[25] The consumption is increasing in after-tax income and thus changes. (ii) Changes in the progressivity also affect consumption for a fixed after-tax income level. For low-type households, increasing progressive taxation causes people with low wealth to consume less. That causes all high-type households to increase consumption, since a higher tax progressivity gives rise to an insurance effect, which enables households to reduce both earnings and investment risk.[26]

Panel (a) of Figure 3 shows that, for the low-type households, their consumption/saving responses to increasing tax progressivity are quite different depending on their position in the wealth distribution. Those at the mass point increase their consumption and their wealth stays at zero. They account for 47.69% of the low-type population and 43.59% of the total population.[27] As such, they spend all of their increased after-tax incomes on consumption. Those with middle wealth (ranging from 47.69% to 87.19% of the low-type households), decrease their consumption and increase their saving. These households increase saving by more than their increased after-tax incomes. Due to earnings risk and investment risk, they

[25] See Figure A.8 of online Technical Appendix 5.2.3 for the details.

[26] We can redefine the consumption function as a function of the after-tax income to a separate channel (ii) from channel (i). See Figure A.5 (low type) and Figure A.7 (high type) in online Technical Appendix 5.2.3 for the details.

[27] Table 3 from 2007 SCF data shows that the share of wealth occupied by the bottom 40% of the whole population is only 0.009. This is approximately close to the calibration result obtained in our model where 43.59% of the overall population occupies a zero share of wealth.

are willing to provide high private insurance via saving in order to prevent themselves from moving into the trap where $k = 0$.

Panel (b) of Figure 3 shows that the high-type households respond to their decreased after-tax incomes by cutting down on saving. The higher the wealth a household has, the higher the reduction in saving is. With reductions in their after-tax incomes, households with higher wealth are better able to maintain consumption as before via dissaving. Another reason for the decline in saving is that higher progressivity reduces risk volatility, and the reduction in saving comes from the insurance effect. Even though saving for the high-type households decreases due to the tax reform, panel (a) of Figure 3 shows that the saving of low-type households could increase. The labor productivity follows a Poisson process. Thus, it is correlated over time. This property of labor earnings causes the correlation between the earnings distribution and the wealth distribution. In the stationary equilibrium, the high-type households not only have a higher labor income but also tend to have a higher capital income. As a result, the increasing progressivity of income taxation largely redistributes income from the high type to the low type.

Figure 4 shows the changes in the stationary wealth distribution, $\hat{f}_j$, $j = 1, 2$, after a tax reform at $\zeta = 0.181$.

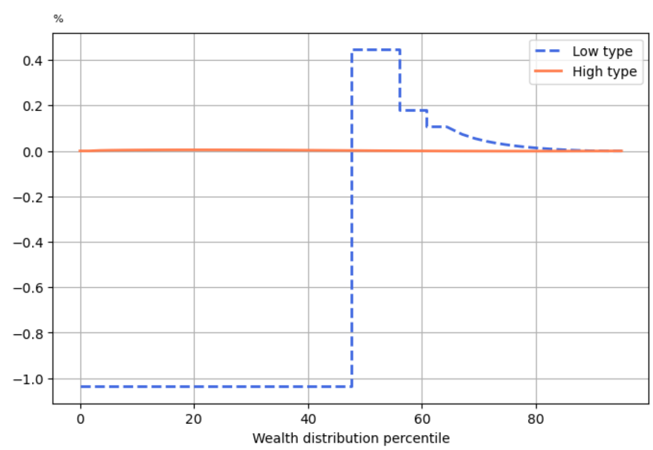


Figure 4: $\hat{f}$ with different types

Figure 4 shows that the change in the density function for the low-type households is much greater than that for the high-type households. Compared to $f_1(k)$, $f_2(k)$ is rather small. Given that $\lambda_1 = 0.047$ and $\lambda_2 = 0.5$, the fraction of the high-type households only equals $\frac{\lambda_1}{\lambda_1+\lambda_2} = 0.086$ in the stationary distribution. A rather small $f_2(k)$ relative to $f_1(k)$

naturally leads to the insignificance of $\hat{f}_2$ relative to $\hat{f}_1$. When the progressivity increases, the low-end households receive more transfers from the government. Thus, the population share at $k = 0$ decreases and the middle population increases. The change in the stationary distribution, $\hat{f}$, determines the change in the aggregate capital and the demand for labor. Thus, it influences the price vector of the economy and is an important determinant of the pecuniary externalities due to the tax reform. Letting $\hat{\mathbf{f}}(k) = (\hat{f}_1(k), \hat{f}_2(k))$, we have

Proposition 2 $\hat{\mathbf{f}}(\cdot)$ *satisfies*

$$\hat{\mathbf{f}} = \mathcal{A}_f \hat{\mathbf{f}} + \mathbf{Q}_f. \tag{12}$$

where $\mathcal{A}_f = \mathbf{X}_f^{-1}\mathbf{D}_f \frac{d}{dk} + \mathbf{X}_f^{-1}\mathbf{M}_f \frac{d^2}{dk^2}$ *and* $\mathbf{Q}_f = \mathbf{X}_f^{-1}\mathbf{Q}_b$.[28]

If $\| \mathcal{A}_f \| < 1$, $\hat{\mathbf{f}}(\cdot)$ *in (12) can be expressed as*

$$\hat{\mathbf{f}} = \mathbf{Q}_f + \sum_{n=1}^{\infty} \mathcal{A}_f^n \mathbf{Q}_f. \tag{13}$$

Equation (12) is obtained by performing a perturbation on the KF equation (11). Proposition 2 shows that $\hat{f}$ is determined by $\mathbf{Q}_f = \mathbf{X}_f^{-1}\mathbf{Q}_b$. From the definition of $\mathbf{Q}_b$ in Appendix A.1.2, we find that it is determined by $\hat{s}$ alone. Under the regular condition of $\| \mathcal{A}_f \| < 1$, $\hat{f}$ can be expressed as a Neumann series in terms of $\hat{s}$.

Perturbation is a powerful tool in nonlinear analysis. Ales and Sleet (2022) use a perturbation method to find an equation which determines optimal taxation. We also use a perturbation method here. Even though the KF equation is nonlinear, the perturbed equation is linear. $\hat{f}$ is determined by a linear operator. If the operator is compact, it is possible to write $\hat{f}$ as a convergent series. Using this method, we find that the economy-wide effect of progressivity changes depends on one elasticity, the elasticity of the saving rate to the progressivity.[29]

4.2 Welfare effect

Following Aiyagari and McGrattan (1998) and Conesa et al. (2009), the planner is to maximize the social welfare in stationary equilibrium. We adopt the utilitarian social welfare

[28] See Appendix A.1.2 for the definitions of $\mathbf{X}_f$, $\mathbf{D}_f$, $\mathbf{M}_f$, and $\mathbf{Q}_b$.

[29] The total measure of the population is 1. Thus, we have $\sum_{j\in\{1,2\}} \int_0^\infty \hat{f}_j(k)dk = 0$.

function,

$$W = \int_0^\infty \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c(k_t)) dt f(k_0) dk_0$$
$$= \frac{1}{\rho} \int_0^\infty u(c(k)) f(k) dk,$$

where we have used the stationary property of the wealth accumulation process.

On the basis of W, we use the variational method to obtain

$$\hat{W} = \frac{1}{\rho} \left[\int_0^\infty u'(c(k)) \hat{c}(k) f(k) dk + \int_0^\infty u(c(k)) \hat{f}(k) dk \right]$$
$$= \frac{1}{\rho} \sum_{j \in \{1,2\}} \left[\int_0^\infty u'(c_j(k)) \hat{c}_j(k) f_j(k) dk + \int_0^\infty u(c_j(k)) \hat{f}_j(k) dk \right]. \tag{14}$$

The above result shows that there are two separate channels through which the degree of tax progressivity exerts its impacts on the social welfare. The first channel is via consumption (i.e. $\hat{c}_j(k)$), and the second channel is via the stationary wealth distribution (i.e. $\hat{f}_j(k)$). In what follows we decompose $\hat{c}_j(k)$ into several components to evaluate the welfare effects of tax reform in detail.

4.2.1 Decomposition of the welfare effect

Let a variable with subscript b denote the variable before the tax reform and subscript a denote the variable after the tax reform. For example, c_{b1} represents the consumption of the low-type households before the tax reform and c_{a2} represents the consumption of the high-type households after the tax reform. The incidence on consumption is given by

$$\hat{c}_j(k) = c_{aj}(k) - c_{bj}(k), \ j = 1, 2.$$

With $T(y) = y - \varphi y^{1-\zeta}$, we define φ_m to satisfy the following equation

$$g \int_0^\infty h_b(y) dy = \int_0^\infty \left(y - \varphi_m y^{1-\zeta_a} \right) h_b(y) dy,$$

where $h(y)$ denotes the income distribution of the economy. That is, φ_m is chosen to meet the balanced budget constraint (6) with $\zeta = \zeta_a$ but $h = h_b$. According to the definition of consumption $c_j(k) = \varphi(rk + wz_j)^{1-\zeta} - s_j(k)$, we decompose $\hat{c}_j(k)$ as follows:

$$\begin{aligned}\hat{c}_j(k) =& \varphi_m(r_b k + w_b z_j)^{1-\zeta_a} - \varphi_b(r_b k + w_b z_j)^{1-\zeta_b} \\ & - \Delta T(y_j(k)) \\ & + r_a k + w_a z_j - (r_b k + w_b z_j) \\ & - (s_{aj}(k) - s_{bj}(k)),\end{aligned} \tag{15}$$

for $j = 1, 2$, where $\Delta T(y_j(k)) = y_a - \varphi_a y_a^{1-\zeta_a} - \left(y_b - \varphi_m y_b^{1-\zeta_a}\right)$.[30]

Let $\widehat{WP}$ denote $\hat{W}/|W|$.[31] Substituting the decomposition of $\hat{c}_j$ in (15) into (14) gives the decomposition of $\widehat{WP}$ as follows:

$$\widehat{WP} = \widehat{WP}_I + \widehat{WP}_{II} + \widehat{WP}_{III} + \widehat{WP}_{IV} - \widehat{WP}_V, \tag{16}$$

where

$$\begin{aligned}\widehat{WP}_I &= (\eta - 1) \sum_{j \in \{1,2\}} \int_0^\infty \frac{\varphi_m(r_b k + w_b z_j)^{1-\zeta_a} - \varphi_b(r_b k + w_b z_j)^{1-\zeta_b}}{c_{bj}(k)} \Gamma_{bj}(k) dk, \\ \widehat{WP}_{II} &= (1 - \eta) \sum_{j \in \{1,2\}} \int_0^\infty \frac{\Delta T(y_j(k))}{c_{bj}(k)} \Gamma_{bj}(k) dk, \\ \widehat{WP}_{III} &= (\eta - 1) \sum_{j \in \{1,2\}} \int_0^\infty \frac{r_a k + w_a z_j - (r_b k + w_b z_j)}{c_{bj}(k)} \Gamma_{bj}(k) dk, \\ \widehat{WP}_{IV} &= (1 - \eta) \sum_{j \in \{1,2\}} \int_0^\infty \frac{s_{aj}(k) - s_{bj}(k)}{c_{bj}(k)} \Gamma_{bj}(k) dk, \\ \widehat{WP}_V &= \sum_{j \in \{1,2\}} \int_0^\infty \frac{f_{aj}(k) - f_{bj}(k)}{f_{bj}(k)} \Gamma_{bj}(k) dk,\end{aligned}$$

with $\Gamma_{bj}(k) = \frac{u(c_{bj}(k)) f_{bj}(k)}{\sum_{j \in \{1,2\}} \int_0^\infty u(c_{bj}(k)) f_{bj}(k) dk}$ being a weight function.[32] Note that our weight function differs from the usually employed Negishi weights $\frac{u_c(c^i)}{\sum_{i=1}^N u_c(c^i)}$ (Saez (2001), Chang and Park (2021)). Recall that we impose Assumption 1 and choose $\eta = 1.1$.

[30] See Appendix A.4.1 for details of the decomposition of $\hat{c}_j$.

[31] We take the absolute value of W as the denominator, because W has a negative sign, which is caused by $\eta > 1$.

[32] As $u(c)$ is negative, we place a negative sign in front of $\widehat{WP}_V$ to express it as a percentage change in the distribution.

Here $\widehat{WP}$ implies the percentage change in the social welfare. $\widehat{WP}_I$ represents the welfare change in redistribution induced by the mechanical effect of a higher ζ (for the meaning of the mechanical effect, see Saez (2001)). $\widehat{WP}_{II}$ measures the efficiency cost of redistribution. As noted by Heathcote and Tsujiyama (2021), the efficiency cost can be interpreted as the fraction of hypothetical revenue that leaks away.[33] Note that the introduction of φ_m enables us to separate the efficiency cost from the mechanical effect. $\widehat{WP}_{III}$ represents the pecuniary externalities associated with changes in w and r in our general equilibrium framework. As in Dávila et al. (2012) and Chang and Park (2021), these changes in w and r generate a redistribution across households, but their effects are ignored by households in their decisions. As such, they need to be taken into account in our search for the optimal ζ. As in Chang and Park (2021), $s(k)$ represents the private intermediation or insurance against risk since consumption smoothing can be achieved by savings. As to $\widehat{WP}_{IV}$, it captures how a higher ζ affects the private insurance and its subsequent welfare effects. These four effects decompose the first part of equation (14) associated with $\hat{c}$. The last one $\widehat{WP}_V$ represents the second part of equation (14) associated with $\hat{f}$.

Let $\widehat{WP}_\Sigma \equiv \widehat{WP}_I + \widehat{WP}_{II} + \widehat{WP}_{III} + \widehat{WP}_{IV}$ in equation (16). Table 4 reports the resulting welfare changes and their decomposition from increasing tax progressivity at different values of ζ.

Table 4: Channel decomposition of social welfare effects (percentage)

	Progressivity of income taxation					
Channel	0	0.181	0.31	0.38*	0.43	0.46
$\widehat{WP}_I$	16.73	11.88	9.53	7.68	6.9	6.49
$\widehat{WP}_{II}$	5.79	2.17	-2.53	-3.62	-6.1	-6.54
$\widehat{WP}_{III}$	-14.48	-7.9	-4.91	-3.36	-2.67	-2.35
$\widehat{WP}_{IV}$	-2	-1.76	-2.19	1.36	4.4	5.36
$\widehat{WP}_\Sigma$	6.04	4.39	-0.1	2.06	2.53	2.96
$\widehat{WP}_V$	-2.35	-1.88	-2.33	2.45	6.51	7.7
$\widehat{WP} = \widehat{WP}_\Sigma - \widehat{WP}_V$	8.39	6.27	2.23	-0.39	-3.98	-4.74

Table 4 shows that the welfare changes of redistribution induced by the mechanical effect are always positive, while those associated with pecuniary externalities are always negative.

[33]This effect is also referred to as fiscal externalities in Chang and Park (2021), since the change in income here is caused by that of the price vector in the equilibrium.

The former result is intuitive, since there is basically no cost involved in implementing redistribution if redistribution only involves the mechanical effects. Chang and Park (2021) found that the presence of pecuniary externalities prevents the optimal tax schedule from being highly progressive. Increasing tax progressivity lowers the size of the capital stock, which raises the interest rate but cuts the wage rate. The main source of income for the low-wealth group is labor income, while that for the high-wealth group is capital income. The result of a higher interest rate but a lower wage rate is favorable to the high-wealth group but unfavorable to the low-wealth group. This explains the negative pecuniary externalities in Table 4. A higher progressivity reduces savings through the insurance effect. Thus, consumption rises and social welfare increases. When ζ is high, the insurance effect is strong and hence $\widehat{WP}_{IV}$ is more likely to be positive.

Table 4 demonstrates the importance of $\widehat{WP}_V$ relative to $\widehat{WP}_\Sigma$ in the determination of $\widehat{WP}$. It shows that the response of the wealth distribution to taxation is no less important in magnitude in terms of social welfare than the sum of all the other components.

Bhandari et al. (2023) decompose the welfare effects into three components: the aggregate efficiency effect, redistribution effect, and insurance effect. We use a different way of decomposition, which emphasizes the distribution effect. The distribution effect, represented by $\hat{f}$, reflects the change in population size for a fixed wealth level k. We use a perturbation method on the KF equation to calculate it.

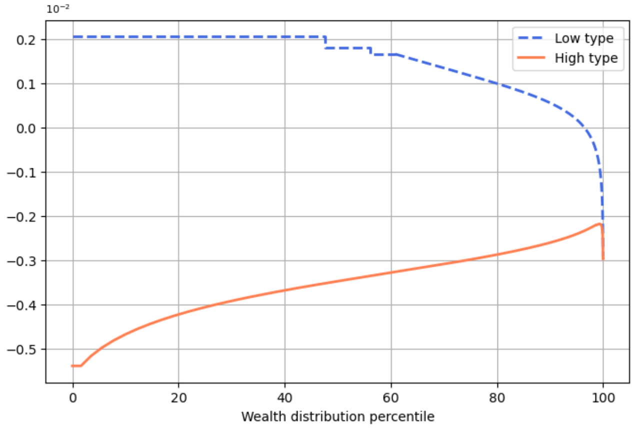
Figure 5 plots the contribution of different $k's$ to $\widehat{WP}_I(\zeta)$, $\widehat{WP}_{II}(\zeta)$, $\widehat{WP}_{III}(\zeta)$, $\widehat{WP}_{IV}(\zeta)$, and $\widehat{WP}_V(\zeta)$ at $\zeta = 0.181$. This presentation is inspired by the distributional national account of Piketty et al. (2018), which takes into consideration the distribution of the aggregate national income across households. By analogy, we take into account the distribution of the aggregate welfare change (reported in Table 4) across households. Note from equation (16) that each component of $\widehat{WP}$ is weighted by the same $\Gamma_{bj}(k)$. The horizontal axis of the figure represents the wealth distribution percentile in the order of wealth k in each type of population.

Panel (a) of Figure 5 displays the curve for $(\eta - 1)\frac{\varphi_m(r_b k + w_b z_j)^{1-\zeta_a} - \varphi_b(r_b k + w_b z_j)^{1-\zeta_b}}{c_{bj}(k)}$ associated with $\widehat{WP}_I$, indicating that the mechanical effect leads to an increase in welfare for the low-type households whose wealth ranks below 96.48%. The reason for this result is that these households receive redistribution from the tax reform. The mechanical effect leads to a decrease in welfare in the group of the high type. For the low-type households, the benefits are greatest for the low-wealth group. For the high-type households, the low-wealth group has a greater reduction in social welfare through this channel.

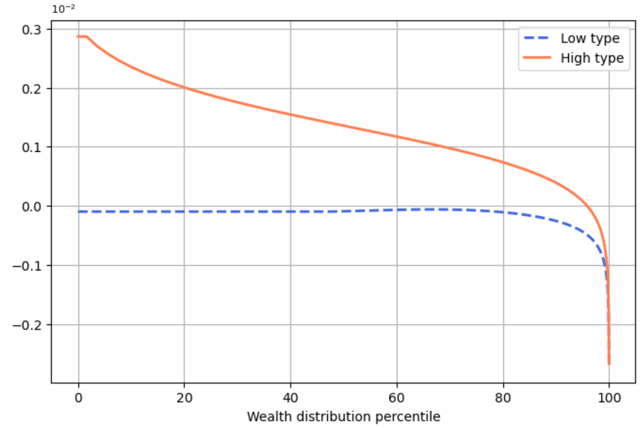
In panel (b) of Figure 5, we present the curve for $(1-\eta)\frac{\Delta T(y_j(k))}{c_{bj}(k)}$ associated with $\widehat{WP}_{II}$. The fraction of tax revenue that leaks away is small and close to zero for most of the low-type households. This is not true for most of the high-type households.

Panel (c) of Figure 5 illustrates the curve for $(\eta-1)\frac{r_a k+w_a z_j-(r_b k+w_b z_j)}{c_{bj}(k)}$ associated with $\widehat{WP}_{III}$. It shows that the social welfare from the pecuniary externality channel is increasing in the wealth distribution percentile for both the low- and high-type households. It starts from a larger negative value for the high-type than the low-type households. Higher progressivity via the reduction in the capital stock leads to a higher interest rate but a lower wage rate. The main source of income for the low- and middle-wealth group is labor income, while the main source of income for the high-wealth group is capital income. This explains the pattern of pecuniary externalities shown in the figure.

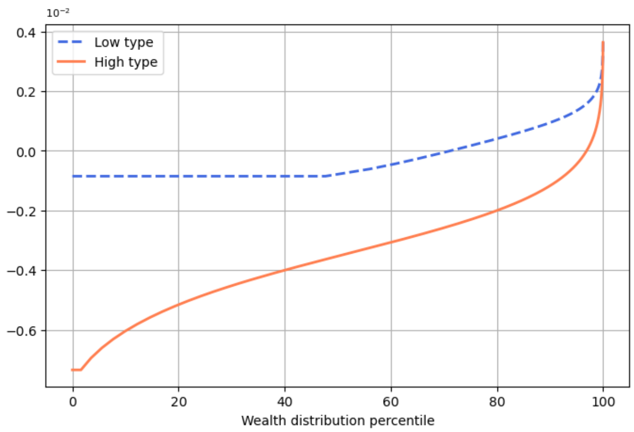
Panel (d) in Figure 5 depicts the curve for $(1-\eta)\frac{s_{aj}(k)-s_{bj}(k)}{c_{bj}(k)}$ associated with $\widehat{WP}_{IV}$, manifesting that all of the high-type households benefit from this channel. Since the saving of the low-wealth population in the low-type households remains at zero, the private insurance of this group is zero. Of the low-type households whose wealth ranks from 47.69% to 99.21%, their social welfare is reduced through this channel.



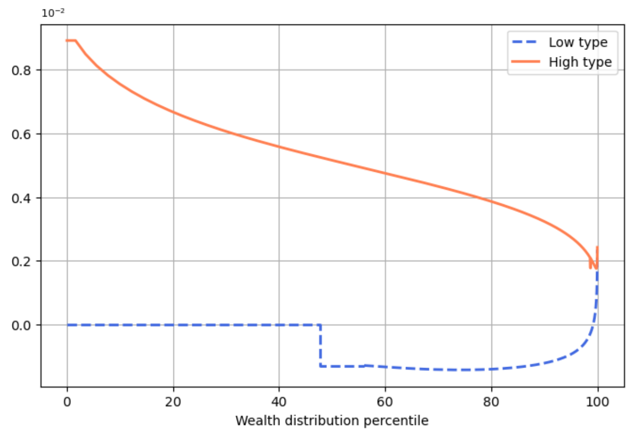
(a) Mechanical effect



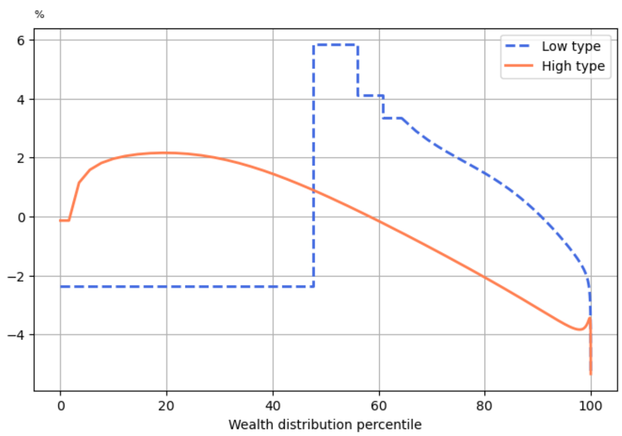
(b) Efficiency cost



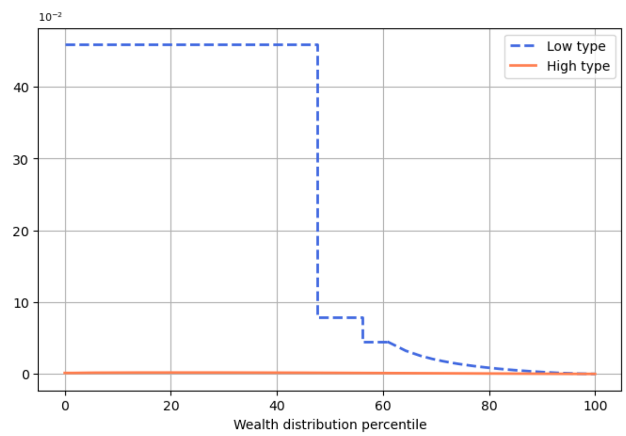
(c) Pecuniary externalities



(d) Private insurance



(e) Distribution effect



(f) Weight function

Figure 5: Welfare effect decomposition at $\zeta = 0.181$

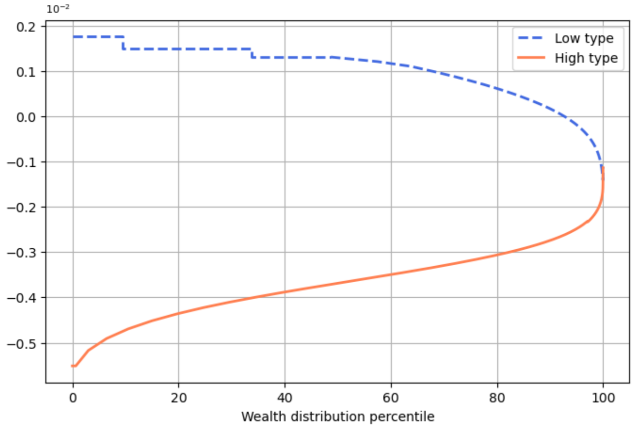
Panel (e) of Figure 5 shows the curve for $\frac{f_{aj}(k)-f_{bj}(k)}{f_{bj}(k)}$ associated with $\widehat{WP}_V$. It represents the percentage change in the population due to the tax reform at each wealth distribution percentile. It is seen from the scale of the vertical axis of Figure 5 that the distribution effect in panel (e) exerts a much larger effect on social welfare than the other components in panels (a)-(d).[34] Increasing tax progressivity causes larger percentage changes in population for the low-type than for the high-type households. In particular, more than 2% of households escape from the trap of $k = 0$ due to the tax reform.

Panel (f) of Figure 5 displays the curve for the weight function $\Gamma_{bj}(k)$. The welfare weights of the high-type households are rather small compared to those of the low-type households. The population of the low-wealth group has a large proportion, which leads to their dominance in the welfare weights.

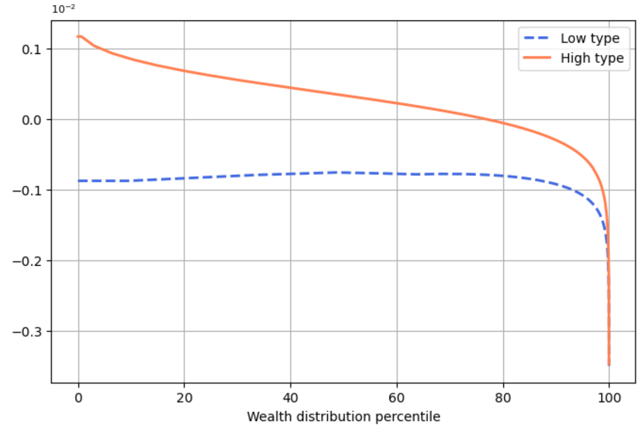
Figure 5 shows the results of the tax reform at $\zeta = 0.181$, while Figure 6 shows the results of the tax reform at $\zeta = 0.43$. It demonstrates that the distribution effect is again larger than the other components in panels (a)-(d). From panel (f) of Figure 6, we find that the welfare weights of the middle-wealth group play a dominant role compared to the other groups in the population. This pattern is different from that of the tax reform at $\zeta = 0.181$. One reason is that the population size at $k = 0$ is 9.53% of the low-type population and 8.71% of the total population at $\zeta = 0.43$, which is much smaller than that at $\zeta = 0.181$.[35] Increasing income tax progressivity causes larger effects on social welfare for low- and middle-wealth households.

[34] Although one is expressed in % while the rest are expressed as 10^{-2}, they are comparable in their values.

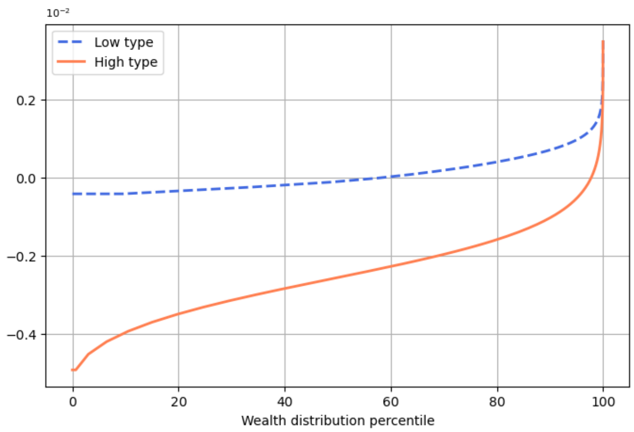
[35] This brings us to the intuition of the progressive taxation, which is mainly concerned with the welfare of the bottom section of the population. The progressivity of $\zeta = 0.43$ is already too high.



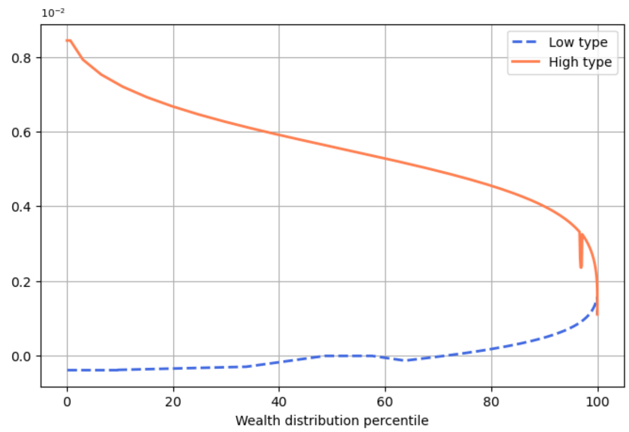
(a) Mechanical effect



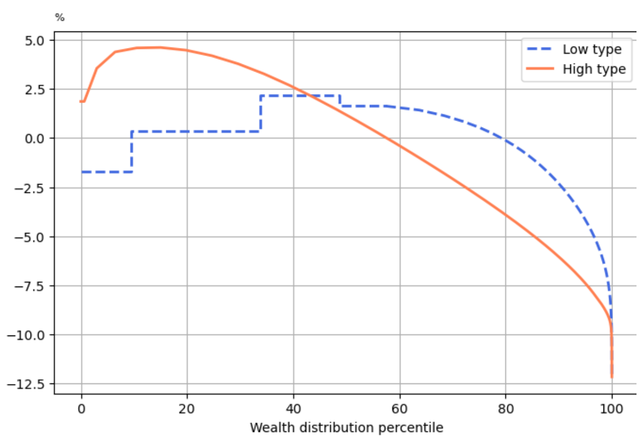
(b) Efficiency cost



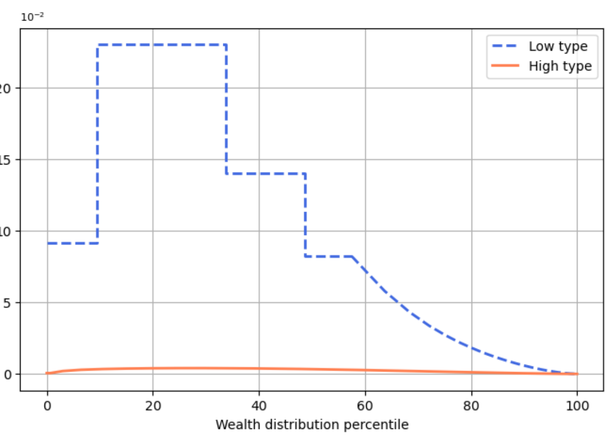
(c) Pecuniary externalities



(d) Private insurance



(e) Distribution effect



(f) Weight function

Figure 6: Welfare effect decomposition at $\zeta = 0.43$

4.2.2 Optimal tax progressivity

Let ζ^* denote the optimal tax progressivity. Letting $\widehat{WP} = 0$ in equation (16), we obtain a formula for ζ^*,

$$\zeta^* = 1 - \frac{\widehat{WP}_d(\zeta^*)}{\widehat{WP}_I(\zeta^*) + \widehat{WP}_{II}(\zeta^*) + \widehat{WP}_{III}(\zeta^*) - \widehat{WP}_V(\zeta^*)}, \tag{17}$$

where $\widehat{WP}_d(\zeta^*) = (1-\eta)\sum_{j\in\{1,2\}}\int_0^\infty \Upsilon(k)\frac{s_{bj}(k)}{c_{bj}(k)}\Gamma_{bj}(k)dk$, and $\Upsilon(k) = -\frac{1-\zeta^*}{\Delta\zeta}\frac{s_{aj}(k)-s_{bj}(k)}{s_{bj}(k)}$ is the elasticity of $s_j(k)$ with respect to $1-\zeta$.[36] The last row of Table 4 shows that $\widehat{WP}$ is closest to zero when tax progressivity reaches 0.38, indicating that raising tax progressivity can not bring about any more social welfare improvement. Thus, $\zeta^* = 0.38$ is the optimal progressivity of income taxation.

From the column of $\zeta^* = 0.38$ in Table 4, we can find that the denominator of the fraction in equation (17) is negative, and the numerator is negative ($\eta > 1$). The greater the difference between $s_{aj}(k)$ and $s_{bj}(k)$ indicates the stronger the private insurance effect. Private insurance serves as a substitute for public insurance (represented by ζ), thus preventing the optimal tax schedule from being highly progressive.

At the optimal level of progressivity, the negative value of $\widehat{WP}_{II}(\zeta^*)$ indicates an increase in tax collection for society, which leads to a decrease in consumption and a subsequent increase in social welfare. $\widehat{WP}_{IV}(\zeta^*)$ is positive, indicating that private insurance has a significant impact when tax progressivity is at 0.38, leading to increased consumption and welfare. Moreover, an increase in progressivity leads to an expansion of the middle-wealth group and a reduction in the number of people in the low-wealth group. As a result of the significant role played by the middle-wealth group, the societal welfare increases, resulting in a positive effect represented by $\widehat{WP}_V(\zeta^*)$. In this regard, increasing the progressivity of the tax system can help promote the formation of an "olive-shaped" society that features a relatively small number of extremely wealthy individuals, a large middle class, and a small number of low-income earners, contributing to a more equitable and sustainable society.[37]

We decompose the effect of a tax reform into several components and obtain a characterization of optimal taxation (ζ^* in our case) by imposing the condition that no tax reform has a positive impact on social welfare. This approach to optimal taxation is known as the variational approach in the literature. Piketty (1997) and Saez (2001) pioneered the approach. It

[36] As demonstrated by Heathcote et al. (2017), $1-\zeta$ measures the elasticity of after-tax to pre-tax income.

[37] See online Technical Appendix 5.3.2 for the welfare effect decomposition at the optimal progressivity.

has been extended and applied to different environments of taxation; see, e.g. Kleven et al. (2009), Golosov et al. (2014), Sachs et al. (2020), and Chang and Park (2021).

Figure 7 plots the resulting social welfare values at each progressivity from 0 to 0.5. It shows that the social welfare reaches its highest level at the progressivity where $\zeta^* = 0.38$. In the face of both earnings and investment risk, our policy prescription is that the value of ζ should be much higher than the status quo where $\zeta = 0.181$. Heathcote et al. (2017) estimated $\zeta = 0.181$ for the U.S. economy, which we indicate in Figure 7 with a black dashed line.

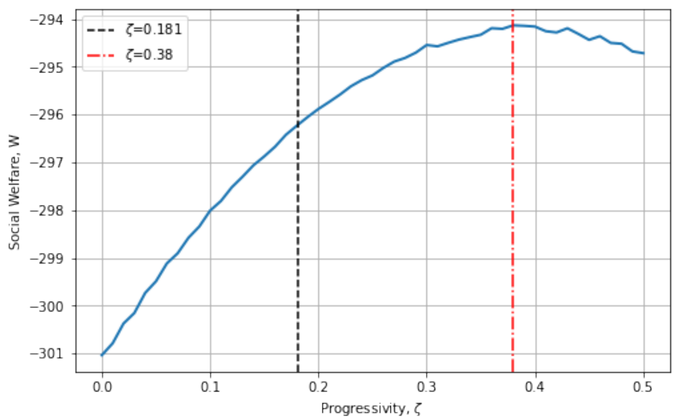


Figure 7: Social welfare

As the progressivity of income taxation increases, the efficiency of the society is decreasing while the social wealth is more equitable. Thus, the hump shape of the welfare function in Figure 7 appears, enabling us to obtain the optimal progressivity of income taxation. We obtain a relatively high level of the optimal progressivity in our calibration exercise. One reason is that the progressive income taxation influences the social welfare mainly through the poor and middle-wealth households. The elasticity of the savings with respect to the progressivity approaches zero as the wealth level goes to infinity.

We also compare our results with those of Heathcote and Tsujiyama (2021). $\overline{T'}$ is the average income-weighted marginal tax rate in percent. Tr is transfers defined as the sum of taxes for households whose taxes are negative (i.e. subsidies). Tr/Y is transfers as a percentage of average income. The first two rows of Table 5 show the status quo $\zeta = 0.181$, while the last two rows compare the results under the optimal progressivity.

Table 5: Comparison between Heathcote and Tsujiyama (2021) and our model

System	Parameters		$\overline{T'}(\%)$	$Tr/Y(\%)$
Heathcote (U.S.)	$\varphi = 0.84$	$\zeta = 0.181$	33.5	2.3
Our model (U.S.)	$\varphi = 0.795$	$\zeta = 0.181$	33.5	4
Heathcote (opt.)	$\varphi = 0.817$	$\zeta = 0.331$	46.6	6.4
Our model (opt.)	$\varphi = 0.654$	$\zeta = 0.38$	49.7	13.2

5 Two Extensions

This section considers two extensions of the benchmark model: (i) allowing for endogenous labor supply, and (ii) including a safe asset. We focus on reporting the theoretical results.[38]

5.1 Endogenous labor supply

Let $\ell(t)$ denote the household's "raw" labor supply (hours worked) at time t. We extend Assumption 1 to incorporate $\ell(t)$. More specifically, we consider the GHH preferences (Greenwood et al. (1988)).

Assumption 2 $u(c,\ell) = \frac{[c-\gamma(\ell)]^{1-\eta}}{1-\eta}$, $\eta > 1$, $\gamma'(\ell) > 0$, $\gamma''(\ell) > 0$.

The objective of the household is

$$\max_{\{c(t),\ell(t),k(t)\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \frac{(c(t) - \gamma(\ell(t)))^{1-\eta}}{1-\eta} dt.$$

Everything else remains the same as in the benchmark model, except that labor income $w(t)z_j$ in the benchmark model is replaced by $w(t)z_j\ell_j(t)$.

We define $\tilde{c}(k) = c(k) - \gamma(\ell(k))$. The corresponding HJB equation becomes

$$\rho v_j(k) = \max_{\tilde{c}_j(k)} u(\tilde{c}_j(k)) + v_j'(k)s_j(k) + \frac{1}{2}v_j''(k)\sigma^2k^2 + \lambda_j\left(v_{-j}(k) - v_j(k)\right), \ j = 1, 2, \tag{18}$$

[38] We report some quantitative results in online Technical Appendices 6 and 7, including (i) the households' policy functions with endogenous labor supply, social welfare and the Gini coefficient of wealth under different progressivities, welfare effect decomposition at the optimal progressivity, the quantiles of wealth and income distributions for a progressivity of 0.181, (ii) the wealth and income distributions at a progressivity of 0.181 with the portfolio problem, and a comparison of policy functions at different progressivities.

where $s_j(k) = \varphi(rk + wz_j\ell_j(k))^{1-\zeta} - \delta k - \tilde{c}_j(k) - \gamma(\ell_j(k))$. The first-order conditions are

$$u'(\tilde{c}_j(k)) = v_j'(k), \tag{19}$$

and

$$\gamma'(\ell) = \varphi(1-\zeta)(rk + wz_j\ell_j)^{-\zeta} wz_j. \tag{20}$$

The corresponding KF equation remains unchanged. We show in the online Technical Appendix that both Proposition **??** and Theorem **??** are robust with respect to the replacement of Assumption 1 with Assumption 2.[39] As before, labor income plays no role in the determination of the asymptotic results, despite labor being endogenous rather than exogenous in the extension.

Proposition 3 *Under Assumption 2, we have $\ell_1 < \ell_2$. For $\zeta > 0$, we have*

$$\frac{\partial \ell_j(k)}{\partial k} < 0,$$

for $j = 1, 2$. For $\zeta = 0$, we have

$$\frac{\partial \ell_j(k)}{\partial k} = 0,$$

for $j = 1, 2$.

For $\zeta = 0$, ℓ_j does not change with k. This is the traditional result of $\partial \ell_j(k)/\partial k = 0$ for the GHH utility. For $\zeta > 0$, the endogenous labor supply decreases in k. This new result is caused by the tax scheme, which is imposed on the labor income and the capital income.

The total labor supply in the economy becomes

$$N(t) = \int_0^\infty z_1\ell_1(k,t)f_1(k)dk + \int_0^\infty z_2\ell_2(k,t)f_2(k)dk.$$

5.1.1 Welfare analysis

We adopt the utilitarian social welfare function,

$$W = \frac{1}{\rho}\int_0^\infty u(\tilde{c}(k))f(k)dk.$$

[39] See online Technical Appendix 2 for these theoretical results and related proofs.

Taking the derivative of W with respect to ζ, we have the decomposition of the welfare effect,

$$\begin{aligned}\hat{W} =& \frac{1}{\rho}\left[\int_0^\infty u'(\tilde{c}(k))\hat{\tilde{c}}(k)f(k)dk + \int_0^\infty u(\tilde{c}(k))\hat{f}(k)dk\right] \\ =& \frac{1}{\rho}\sum_{j\in\{1,2\}}\left[\int_0^\infty u'(\tilde{c}_j(k))\hat{\tilde{c}}_j(k)f_j(k)dk + \int_0^\infty u(\tilde{c}_j(k))\hat{f}_j(k)dk\right].\end{aligned}$$

As in the benchmark model, we also explore the impact of the progressivity changes on the policy functions. However the additional consideration here is the effect of the progressivity changes on the endogenous labor supply.[40] Taking the derivative of $\tilde{c}(k)$ with respect to ζ, we have

$$\begin{aligned}\hat{\tilde{c}}_j(k) &= \tilde{c}_{aj}(k) - \tilde{c}_{bj}(k) \\ &= \varphi_m(r_b k + w_b z_j \ell_{bj}(k))^{1-\zeta_a} - \varphi_b(r_b k + w_b z_j \ell_{bj}(k))^{1-\zeta_b} \\ &- \Delta T_\ell(y_j(k)) \\ &+ r_a k + w_a z_j \ell_{aj}(k) - (r_b k + w_b z_j \ell_{bj}(k)) \\ &- (s_{aj}(k) - s_{bj}(k))\end{aligned}$$

for $j = 1, 2$, and

$$\Delta T_\ell(y_j(k)) = y_a - \varphi_a y_a^{1-\zeta_a} - \left(y_b - \varphi_m y_b^{1-\zeta_a}\right).$$

As before φ_m is defined by

$$g\int_0^\infty h_b(y)dy = \int_0^\infty \left(y - \varphi_m y^{1-\zeta_a}\right) h_b(y)dy.$$

The difference from the benchmark model is that we have $y_j(k) = rk + wz_j\ell_j(k)$ rather than $y_j(k) = rk + wz_j$.

Like the benchmark model, the decomposition on social welfare is as follows,

$$\widehat{WP}_L = \widehat{WP}_{LI} + \widehat{WP}_{LII} + \widehat{WP}_{LIII} + \widehat{WP}_{LIV} - \widehat{WP}_{LV},$$

[40] We use the first-order condition to cancel $\hat{\ell}_j(k)$ caused by the progressivity variation. See online Technical Appendix 2 for details on the treatment of $\hat{\ell}_j(k)$.

where

$$\begin{aligned}
\widehat{WP}_{LI} =&(\eta-1)\sum_{j\in\{1,2\}}\int_0^\infty \frac{\varphi_m(r_bk+w_bz_j\ell_{bj}(k))^{1-\zeta_a}-\varphi_b(r_bk+w_bz_j\ell_{bj}(k))^{1-\zeta_b}}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,\\
\widehat{WP}_{LII} =&(1-\eta)\sum_{j\in\{1,2\}}\int_0^\infty \frac{\Delta T_\ell(y_j(k))}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,\\
\widehat{WP}_{LIII} =&(\eta-1)\sum_{j\in\{1,2\}}\int_0^\infty \frac{r_ak+w_az_j\ell_{aj}(k)-(r_bk+w_bz_j\ell_{bj}(k))}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,\\
\widehat{WP}_{LIV} =&(1-\eta)\sum_{j\in\{1,2\}}\int_0^\infty \frac{s_{aj}(k)-s_{bj}(k)}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,\\
\widehat{WP}_{LV} =&\sum_{j\in\{1,2\}}\int_0^\infty \frac{f_{aj}(k)-f_{bj}(k)}{f_{bj}(k)}\Gamma_{bj}(k)dk,
\end{aligned}$$

and $\Gamma_{bj}(k)=\frac{u(\tilde{c}_{bj}(k))f_{bj}(k)}{\sum_{j\in\{1,2\}}\int_0^\infty u(\tilde{c}_{bj}(k))f_{bj}(k)dk}$ is the weight function.

The interpretation of the decomposition for each channel is consistent with the benchmark model. We find that the optimal progressivity $\zeta^*=0.31$ in the extension, which is lower than $\zeta^*=0.38$ in the basic model. The optimal progressivity is now lower since (i) there is an additional tax distortion margin with endogenous labor supply and hence an additional cost of raising ζ, and (ii) adjusting the endogenous labor supply provides households with a margin to insure against idiosyncratic shocks and hence there is no need for a higher ζ to insure against idiosyncratic shocks.

5.2 Inclusion of a safe asset

In the canonical ABH model, households face idiosyncratic labor income risk and "self-insure" against risk through the accumulation of safe or risk-free assets. Here we extend the benchmark model to include a safe asset. Households access a safe asset and operate a private firm. Thus they have a portfolio selection problem.

Following Angeletos (2007), we introduce a second sector, called "public equity." The crucial difference between private equity and public equity is the risk. The private equity faces idiosyncratic investment risk, while the return on the public equity is deterministic. Thus, the private equity has a risk premium. The aggregate production technology of the public equity sector is given by $Y_g(t)=A_gB(t)^\alpha L(t)^{1-\alpha}$, where A_g denotes TFP, and $B(t)$ and $L(t)$ are the total capital and labor allocated to the public-equity sector. Clearly, in equilibrium, the wage rate $w(t)$ and the risk-free rate $r_b(t)$ must satisfy $w(t) = A_g(1-\alpha)(B(t)/L(t))^{\alpha}$ and $r_b(t) = A_g\alpha(L(t)/B(t))^{1-\alpha}$. The public and private equity sectors share the same wage rate in equilibrium.

The main departure from the benchmark model is that households now have access to the risk-free public equity $b(t)$, in addition to the risky equity $k(t)$. As such, households in the extended model hold different portfolios of risky and risk-free assets as in the real world. We let $\delta = 0$ for simplicity.

As of 2012, 73 percent of all businesses in the U.S. were sole proprietorships owned by individuals or married couples (Slemrod and Bakija, 2017, p. 39). Presumably, they correspond to the "family business" described in the benchmark model and represent the private equity sector in our extended model.[41] As to the public equity sector in our extended model, it corresponds to the standard setup in the canonical ABH model.

With the availability of $b(t)$, instead of (7), the households are subject to the constraint in terms of wealth $a(t) = b(t) + k(t)$,

$$da(t) = \left\{\varphi\left[w(t)z(t) + r_b(t)b(t) + \phi r(t)k(t)\right]^{1-\zeta} + (1-\phi)r(t)k(t) - c(t)\right\}dt + \sigma k(t)dB(t), \tag{21}$$

where $\phi < 1$ denotes the fraction of $r(t)k(t)$ subject to income taxation and $a(t) \geq 0$, $k(t) \geq 0$, and $b(t) \geq 0$ for all $t \geq 0$.

We treat the taxation of $r_b(t)b(t)$ and $r(t)k(t)$ asymmetrically, in that while the whole of $r_b(t)b(t)$ is subject to income tax $T(t)$, only the $\phi < 1$ fraction of $r(t)k(t)$ is subject to income tax $T(t)$. This asymmetric tax treatment has its de jure and de facto reasons.

De jure: It is known that capital gains are treated leniently in income taxation compared to other types of income. First, accrued but unrealized capital gains are not accounted for as taxable income and are exempted from taxation. Second, even if realized, capital gains are subject to lower statutory tax rates than other types of income if the sale of an asset is held more than a year after it is purchased. Table 2.4 of Slemrod and Bakija (2017, p. 45) shows that, for the period 1987-2013, the fraction of capital gains (less losses) included on personal income tax returns is significantly lower than the fractions of other types of income.

De facto: The enforcement and administration of modern tax systems crucially rely on third-party information by employers, which report taxable income on behalf of their

[41] In 2012, "pass-through-entities" (sole proprietorships, partnerships, S corporations) accounted for 95 percent of all businesses; see Slemrod and Bakija (2017, p. 38). They may be understood as representing the private equity sector in a broader sense.

employees directly to the government. Kleven et al. (2011) and Kleven et al. (2016) showed theoretically and empirically that tax enforcement is successful if and only if third-party information is in effect. According to Table 5.1 in Slemrod and Bakija (2017, p. 257), 99 percent of wages and salaries and 93 percent of interest and dividend income are subject to substantial third-party information reporting, whereas proprietor income is subject to little or no third-party information reporting. As a result, the compliance rate (the percentage of true income that is reported to the tax authorities) for wages and salaries is 99% and for interest and dividend income is 96%, whereas that for nonfarm proprietor income is only 43%.[42,43]

The setup of $\phi < 1$ rather than $\phi = 1$ in equation (21) is to account for both the de jure and de facto features described above.

The HJB equation (8) and the KF equation (11) in the benchmark model now become

$$\begin{aligned}
\rho v_j(a) = & \max_{c, 0 \leq k \leq a} u(c) + v_j'(a) \left(\varphi \left[w z_j + r_b (a-k) + \phi r k \right]^{1-\zeta} + (1-\phi) r k - c \right) \\
& + \frac{1}{2} v_j''(a) \sigma^2 k^2 + \lambda_j \left(v_{-j}(a) - v_j(a) \right), \ j = 1, 2, \qquad (22) \\
0 = & -\frac{d}{da} \left[s_j(a) f_j(a) \right] + \frac{1}{2} \frac{d^2}{da^2} \left[\sigma^2 k_j(a)^2 f_j(a) \right] - \lambda_j f_j(a) + \lambda_{-j} f_{-j}(a), \ j = 1, 2. \qquad (23)
\end{aligned}$$

The first-order condition with respect to k gives

$$-v_j''(a) \sigma^2 k = v_j'(a) \left[(1-\zeta) \varphi (w z_j + r_b (a-k) + \phi r k)^{-\zeta} (\phi r - r_b) + (1-\phi) r \right], \ j = 1, 2.$$

Using the bisection method we obtain the policy functions. As before, $s_j(a)$ is the optimal saving policy function and $k_j(a)$ is the optimal choice of the risky asset. It can be seen that (22) is an optimal portfolio allocation problem as in Merton (1969) and $k_j(a)/a$ is the share of the household's portfolio invested in the risky asset.

[42] 81 percent of partnership and S corporation income and capital gains is subject to some, not substantial, third-party information reporting. The compliance rate for partnership and S corporation income is 82% and that for capital gains is 88%.

[43] Using leaked and amnesty data, Alstadsaeter, Johannesen and Zucman (2019) found that the probability of hiding assets offshore rises sharply and significantly with wealth in Norway, Sweden, and Denmark. In particular, they found that the top 0.01 percent evades about 25 percent of its tax liability by concealing assets and investment income abroad. They noted that Scandinavian economies rank among the countries with the strongest respect for the rule of law and highest "tax morale," suggesting that evasion among the wealthy may be even higher elsewhere.

General equilibrium The labor market clearing condition is

$$N(t) + L(t) = \frac{z_1\lambda_2 + z_2\lambda_1}{\lambda_1 + \lambda_2},$$

and the capital market clearing condition is

$$r(t) = A_g^{-1/\alpha} A^{1/\alpha} r_b(t),$$

where $A > A_g$ such that $r_b(t) < r(t)$.[44]

We obtain that the policy function involves risky assets of households and the theorem about the stationary wealth distribution in online Technical Appendix 3. We also use the calibrated parameters to calculate the quantiles of the distributions of income and wealth in the model and compare them with the U.S. data. We compare the policy functions (saving, private equity, public equity) and the stationary distribution with a flat tax ($\zeta = 0$), and also progressive taxes ($\zeta = 0.181$ and $\zeta = 0.25$). These numerical results can be found in online Technical Appendix 7.

6 Conclusion

Our model economy builds on Achdou et al. (2022), which is a continuous-time version of the Aiyagari-Bewley-Huggett heterogeneous-agent model. We introduce the idiosyncratic investment risk to the heterogeneous-agent continuous-time model and solve the problem in a general equilibrium setting with the labor market. The feature of the general equilibrium enables us to investigate the pecuniary externalities of income taxation in a heterogeneous-agent model with investment risk.

First, we theoretically show how the progressivity of income taxation alters the policy function of households. Furthermore we find that the Pareto exponent of the wealth distribution does not depend on the progressivity explicitly, but it does depend on the progressivity implicitly. In the two cases of a progressive tax and a flat tax, we obtain different Pareto exponents. Progressive taxation makes the Pareto index increase compared to flat taxation, which means that the distribution of wealth will be more equal, although it is worth noting that progressivity does not affect the Pareto index of the wealth distribution under a progressive tax system.

[44] See online Technical Appendix 3 for the proof of the capital market clearing condition.

Second, we use the calibrated parameters to calculate the quantiles of the distributions of income and wealth in the model and compare them with the U.S. data. The income and wealth distributions of our calibrated model match the target moments reasonably well. From the numerical results, we also find that the Pareto index under progressive taxation is larger than that under flat taxation.

Finally, we implement a perturbation analysis on the social welfare function. We adopt the variational approach to investigate the impacts of increasing progressivity on the economy, and propose a welfare decomposition method. Applying this new method, we decompose the welfare effects into different components. Changes in the population size are comparable to the sum of the effects of all other channels. Increasing the progressivity of income taxation causes larger effects on social welfare for low- and middle-wealth households.

We also consider two extensions of the benchmark model: (i) allowing for endogenous labor supply, and (ii) including a safe asset. We find that introducing an endogenous labor supply reduces the optimal progressivity tax rate. We find for the first time a general equilibrium with portfolio selection.

A Appendix

A.1 Proof of Propositions and Theorems

A.1.1 Proof of Proposition 1

We need the following technical assumption.

Assumption 3 z_2/z_1 *is sufficiently high.*

Assumption 3 implies that the earnings process is sufficiently volatile. Thus, the precautionary saving motive is strong enough. The precautionary saving caused by labor efficiency shocks prevents the wealth accumulation process from being absorbed into the degenerating state of zero.

For $\zeta = 0$, we have $f(k) \sim \xi k^{-\Theta_b - 1}$ and $\Theta_b = 2 - \eta - \frac{2(\varphi r - \rho - \delta)}{\eta \sigma^2}$(similar to Yang et al., 2023). Let $\bar{r} = \frac{2(\rho+\delta) - \eta(\eta-1)\sigma^2}{2\varphi}$. Thus, $r \geq \bar{r}$ implies $\Theta_b \leq 1$.

There exists a $\bar{k}$ such that

$$f(k) > \frac{\xi}{2} k^{-\Theta_b - 1},$$

for $k > \bar{k}$. Then, we have

$$\int_0^\infty k f(k) dk > \frac{\xi}{2} \int_{\bar{k}}^\infty k^{-\Theta_b} dk = \infty.$$

for $\Theta_b \leq 1$.

For capital supply $K_s = \int_0^\infty k f(k) dk$, we have

$$\lim_{r \uparrow \bar{r}} K_s(r) = \infty, \tag{24}$$

since $r \to \bar{r}$ implies $\Theta_b \to 1$.

For capital demand, $K_d = \left(\frac{r}{\alpha A}\right)^{\frac{1}{\alpha - 1}} N$, we have

$$\lim_{r \downarrow 0} K_d(r) = \infty. \tag{25}$$

Continuity of $K_s(r)$. The optimal saving policy function $s(k, z; r)$ is continuous in r because it is the policy function of the HJB equation which depends on r in a continuous fashion.

We next show that this implies that the stationary wealth distribution is continuous in r. For a given trajectory of earnings and investment return realizations $\omega = \{(z(t), B(t))\}_{t\geq 0}$, denote by $k(t)^\omega(r)$ the solution to $dk(t) = s\,(k(t), z(t); r)\, dt + \sigma k(t) dB(t)$ for a fixed k_0. Given that the saving policy function s is continuous in r, then so is the wealth trajectory $k(t)^\omega(r)$ for any given income trajectory ω and at all times $t \geq 0$. Denote by $F(k, z; r)$ the CDF of the stationary joint distribution of income and wealth. Further denote by $F(k; r)$ the stationary wealth distribution, $F(k; r) = \sum_{j\in\{1,2\}} F(k, j; r)$. By its definition, the stationary wealth distribution is the stationary distribution of the process $k(t)^\omega(r)$. Because $k(t)^\omega(r)$ is continuous in r for all ω and t so is F (see also Theorem 7 of Zhu (2020)).

Finally, given that the stationary wealth distribution F is continuous in r so is the aggregate capital $K_s(r)$ in the stationary distribution (the first moment of that distribution).

We know that $K_s(r)$ is continuous in r and satisfies equation (24). $K_d(r)$ is continuous in r and satisfies equation (25). There must exist $r^* \in (0, \bar{r})$ such that $K_s(r^*) = K_d(r*)$. Thus, there exists a stationary equilibrium. ■

A.1.2 Proof of Proposition 2

Differentiating the KF equation (11) with respect to ζ, we obtain

$$\begin{aligned} 0 =& \sigma^2 \hat{f}_j(k) + 2\sigma^2 k \hat{f}_j'(k) + \frac{\sigma^2}{2} k^2 \hat{f}_j''(k) - \hat{s}_j'(k) f_j(k) - s_j'(k)\hat{f}_j(k) - \hat{s}_j(k) f_j'(k) - s_j(k)\hat{f}_j'(k) \\ & - \lambda_j \hat{f}_j(k) + \lambda_{-j}\hat{f}_{-j}(k), \end{aligned} \tag{26}$$

for $j = 1, 2$. Rearranging equation (26), we have

$$\left[s_1'(k) - \sigma^2 + \lambda_1\right] \hat{f}_1(k) - \lambda_2 \hat{f}_2(k) = \frac{\sigma^2}{2} k^2 \hat{f}_1''(k) + (2\sigma^2 k - s_1(k))\hat{f}_1'(k) + \left[-\hat{s}_1'(k) f_1(k) - \hat{s}_1(k) f_1'(k)\right], \tag{27}$$

and

$$\left[s_2'(k) - \sigma^2 + \lambda_2\right] \hat{f}_2(k) - \lambda_1 \hat{f}_1(k) = \frac{\sigma^2}{2} k^2 \hat{f}_2''(k) + (2\sigma^2 k - s_2(k))\hat{f}_2'(k) + \left[-\hat{s}_2'(k) f_2(k) - \hat{s}_2(k) f_2'(k)\right], \tag{28}$$

respectively.

We can write equations (27) and (28) as

$$
\begin{pmatrix} s_1'(k)-\sigma^2+\lambda_1 & -\lambda_2 \\ -\lambda_1 & s_2'(k)-\sigma^2+\lambda_2 \end{pmatrix}\begin{pmatrix} \hat{f}_1(k) \\ \hat{f}_2(k) \end{pmatrix}
$$

$$
=\begin{pmatrix} 2\sigma^2k-s_1(k) & 0 \\ 0 & 2\sigma^2k-s_2(k) \end{pmatrix}\begin{pmatrix} \left(\hat{f}_1(k)\right)' \\ \left(\hat{f}_2(k)\right)' \end{pmatrix}+\begin{pmatrix} \sigma^2k^2/2 & 0 \\ 0 & \sigma^2k^2/2 \end{pmatrix}\begin{pmatrix} \left(\hat{f}_1(k)\right)'' \\ \left(\hat{f}_2(k)\right)'' \end{pmatrix}
$$

$$
+\begin{pmatrix} -\hat{s}_1'(k)f_1(k)-\hat{s}_1(k)f_1'(k) \\ -\hat{s}_2'(k)f_2(k)-\hat{s}_2(k)f_2'(k) \end{pmatrix}. \tag{29}
$$

Since $\hat{\mathbf{f}}(k)=(\hat{f}_1(k),\hat{f}_2(k))$, we can write equation (29) in the following form,

$$
\mathbf{X}_f\hat{\mathbf{f}}=\mathbf{D}_f\hat{\mathbf{f}}'+\mathbf{M}_f\hat{\mathbf{f}}''+\mathbf{Q}_b, \tag{30}
$$

where

$$
\mathbf{X}_f=\begin{pmatrix} s_1'(k)-\sigma^2+\lambda_1 & -\lambda_2 \\ -\lambda_1 & s_2'(k)-\sigma^2+\lambda_2 \end{pmatrix},
$$

$$
\mathbf{D}_f=\begin{pmatrix} 2\sigma^2k-s_1(k) & 0 \\ 0 & 2\sigma^2k-s_2(k) \end{pmatrix},
$$

$$
\mathbf{M}_f=\begin{pmatrix} \sigma^2k^2/2 & 0 \\ 0 & \sigma^2k^2/2 \end{pmatrix},
$$

$$
\mathbf{Q}_b=\begin{pmatrix} -\hat{s}_1'(k)f_1(k)-\hat{s}_1(k)f_1'(k) \\ -\hat{s}_2'(k)f_2(k)-\hat{s}_2(k)f_2'(k) \end{pmatrix}.
$$

Thus, we have

$$
\hat{\mathbf{f}}=\mathbf{X}_f^{-1}\mathbf{D}_f\hat{\mathbf{f}}'+\mathbf{X}_f^{-1}\mathbf{M}_f\hat{\mathbf{f}}''+\mathbf{X}_f^{-1}\mathbf{Q}_b. \tag{31}
$$

We can express equation (31) in the following form

$$
\hat{\mathbf{f}}=\mathcal{A}_f\hat{\mathbf{f}}+\mathbf{Q}_f. \tag{32}
$$

where $\mathcal{A}_f=\mathbf{X}_f^{-1}\mathbf{D}_f\frac{d}{dk}+\mathbf{X}_f^{-1}\mathbf{M}_f\frac{d^2}{dk^2}$, and $\mathbf{Q}_f=\mathbf{X}_f^{-1}\mathbf{Q}_b$.

From equation (32), we have

$$
(\mathbf{I}-\mathcal{A}_f)\hat{\mathbf{f}}=\mathbf{Q}_f,
$$

where $\mathbf{I}$ is the identity operator: $\mathbf{I}\cdot\hat{\mathbf{f}}=\hat{\mathbf{f}}$. $\mathcal{A}_f$ is a linear operator. If $\|\mathcal{A}_f\|<1$, $\mathbf{I}-\mathcal{A}_f$ has a unique bounded linear inverse $(\mathbf{I}-\mathcal{A}_f)^{-1}$ which is a Neumann series,

$$\begin{aligned}\hat{\mathbf{f}} =&(\mathbf{I}-\mathcal{A}_f)^{-1}\mathbf{Q}_f \\ =&\sum_{n=0}^{\infty}\mathcal{A}_f^n\mathbf{Q}_f \\ =&\mathbf{Q}_f+\sum_{n=1}^{\infty}\mathcal{A}_f^n\mathbf{Q}_f,\end{aligned} \tag{33}$$

by Theorem 2 in Chapter II of Yosida (1995). ■

A.1.3 Proof of Proposition 3

For $z_1 < z_2$, suppose that $\ell_1(z_1) \geq \ell_2(z_2)$. By equation (20), we have

$$\gamma'(\ell_1) = \varphi(1-\zeta)(rk + wz_1\ell_1)^{-\zeta}wz_1, \tag{34}$$

and

$$\gamma'(\ell_2) = \varphi(1-\zeta)(rk + wz_2\ell_2)^{-\zeta}wz_2. \tag{35}$$

Thus, we obtain

$$\gamma'(\ell_1) - \gamma'(\ell_2) = \varphi(1-\zeta)\left[(rk + wz_1\ell_1)^{-\zeta}wz_1 - (rk + wz_2\ell_2)^{-\zeta}wz_2\right]. \tag{36}$$

The LHS of equation (36) is non-negative. And the RHS of equation (36) is negative, since

$$(rk + wz_1\ell_1)^{-\zeta}wz_1 < (rk + wz_2\ell_1)^{-\zeta}wz_2 \leq (rk + wz_2\ell_2)^{-\zeta}wz_2.$$

We have a contradiction.

Next we explore $\partial\ell_j(k)/\partial k$ for cases of $\zeta > 0$ and $\zeta = 0$.

Taking the derivative of both sides of equation (20) with respect to k, we have

$$\ell_j'(k) = -\frac{\gamma''(\ell_j(k))}{\varphi\zeta(1-\zeta)(rk + wz_j\ell_j(k))^{-\zeta-1}w^2z_j^2} - \frac{r}{wz_j}.$$

Under Assumption 2, we know that $\ell_j'(k) < 0$ for $\zeta > 0$.

For $\zeta = 0$, we have

$$\gamma'(\ell_j) = \varphi wz_j.$$

from equation (20). Thus, we know that ℓ_j does not depend on k. ■

A.2 Computation algorithm

We numerically solve the equilibrium of our model. Following Achdou et al. (2022) we use a finite difference (FD) method to solve the HJB equation (8) and the KF equation (11).

Our computing algorithm runs as follows,

1. Set the lower bound r_1 and upper bound r_2 of the interest rate, i.e. $r \in [r_1, r_2]$. Calculate r^n by $r^n = (r_1 + r_2)/2$. Hence we obtain $w^n = (1-\alpha)\left(\frac{r^n}{\alpha A^{1/\alpha}}\right)^{\alpha/(\alpha-1)}$. In the same way, we have $\varphi \in [\varphi_1, \varphi_2]$. Use $\varphi^n = (\varphi_1 + \varphi_2)/2$ to obtain φ^n.

2. Approximate both v_1, v_2 at I discrete points in the space dimension, $k_i, i = 1, \ldots, I$. We consider an equispaced grid with distance Δk and choose $I = 10000$.

3. Solve the HJB equation (8) and calculate $s_j(k)$. For the calculation of $v_j'(k_i)$, use a forward difference approximation whenever the drift of the state variable (here, saving $s_j(k) = \varphi^n(r^n k + w^n z_j)^{1-\zeta} - \delta k - c_j(k))$ is positive and a backward difference whenever it is negative.[45]

4. Solve the KF equation (11) for $f_j(k; r^n)$ by a FD method.

5. Obtain the interest rate r^{n+1} and φ^{n+1} from the endogenous wealth distribution $f_j(k; r^n)$. The aggregate capital is determined by

$$K^{n+1} = \int_0^\infty k f_j(k; r^n) dk.$$

With $N = \frac{z_1\lambda_2 + z_2\lambda_1}{\lambda_1 + \lambda_2}$ and K^{n+1} in hand, we obtain

$$r^{n+1} = \alpha A \left(\frac{K^{n+1}}{N}\right)^{\alpha - 1}.$$

[45] A forward and a backward difference approximation are, respectively,

$$v_j'(k_i) \approx \frac{v_j(k_{i+1}) - v_j(k_i)}{\Delta k},$$

and

$$v_j'(k_i) \approx \frac{v_j(k_i) - v_j(k_{i-1})}{\Delta k}.$$

And we have

$$\varphi^{n+1} = \frac{(1-g)\int_0^\infty y f_j(y;r^n)dy}{\int_0^\infty y^{1-\zeta} f_j(y;r^n)dy}$$

from equation (6), where $f_j(y;r^n)$ is determined by $f_j(k;r^n)$ and $y_j = r^n k + w^n z_j$.

6. If φ^n is greater than φ^{n+1}, assign φ^n to φ_2 and vice versa to φ_1, and calculate whether the difference between φ_2 and φ_1 is less than 10^{-3}. If it is greater than 10^{-3}, go back to step 2; if it is less than 10^{-3}, determine whether r^n is greater than r^{n+1}; if r^n is greater than r^{n+1}, assign r^n to r_2, and vice versa r^n to r_1. Furthermore, calculate whether the difference between r_2 and r_1 is less than 10^{-3}. If it is greater than 10^{-3}, then go back to step 2; if it is less than 10^{-3}, we reach the stationary equilibrium. We have the equilibrium interest rate $r^* = r^n$, $\varphi^* = \varphi^n$.

A.3 Social welfare function

We adopt the utilitarian social welfare function,

$$\begin{aligned}
W &= \int_0^\infty \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c(k_t)) dt f(k_0) dk_0 \\
&= \int_0^\infty e^{-\rho t} \mathbb{E}_0 \int_0^\infty u(c(k_t)) f(k_0) dk_0 dt \\
&= \int_0^\infty e^{-\rho t} \int_0^\infty \mathbb{E}_0 u(c(k_t)) f(k_0) dk_0 dt \\
&= \int_0^\infty e^{-\rho t} \int_0^\infty u(c(k)) f(k) dk dt \\
&= \frac{1}{\rho} \int_0^\infty u(c(k)) f(k) dk.
\end{aligned}$$

where the fourth line uses the stationary property of the wealth accumulation process.

A.4 Tax reform analysis

A.4.1 Decomposition of welfare effect

In order to separate the detailed channels of the tax impacts, we propose a new decomposition method for the welfare effects. First, we describe a decomposition of $\hat{c}_j(k)$ at different progressivities. It can be used to evaluate the welfare effects of tax reform starting from any progressivity, even if it is non-optimal.

Consider the household's consumption change at ζ,

$$\hat{c}_j(k) = c_{aj}(k) - c_{bj}(k).$$

Using the definition of $c_j(k) = rk + wz_j - \left[rk + wz_j - \varphi(rk + wz_j)^{1-\zeta}\right] - s_j(k)$, we obtain

$$\begin{aligned}\hat{c}_j(k) =& r_a k + w_a z_j - \left[r_a k + w_a z_j - \varphi_a(r_a k + w_a z_j)^{1-\zeta_a}\right] - s_{aj}(k) \\ & - \left\{r_b k + w_b z_j - \left[r_b k + w_b z_j - \varphi_b(r_b k + w_b z_j)^{1-\zeta_b}\right] - s_{bj}(k)\right\}.\end{aligned} \tag{37}$$

Rewriting equation (37), we have

$$\begin{aligned}\hat{c}_j(k) =& r_a k + w_a z_j - (r_b k + w_b z_j) \\ & - \left[r_a k + w_a z_j - \varphi_a(r_a k + w_a z_j)^{1-\zeta_a}\right] \\ & + \left[r_b k + w_b z_j - \varphi_b(r_b k + w_b z_j)^{1-\zeta_b}\right] \\ & - \left(s_{aj}(k) - s_{bj}(k)\right).\end{aligned}$$

Introducing φ_m into the above equation, we obtain

$$\begin{aligned}\hat{c}_j(k) =& \varphi_m(r_b k + w_b z_j)^{1-\zeta_a} - \varphi_b(r_b k + w_b z_j)^{1-\zeta_b} \\ & - \left[r_a k + w_a z_j - \varphi_a(r_a k + w_a z_j)^{1-\zeta_a}\right] \\ & + \left[r_b k + w_b z_j - \varphi_m(r_b k + w_b z_j)^{1-\zeta_a}\right] \\ & + r_a k + w_a z_j - (r_b k + w_b z_j) \\ & - \left(s_{aj}(k) - s_{bj}(k)\right).\end{aligned} \tag{38}$$

The first line in equation (38) denotes the mechanical effect (see Saez (2001)). Before the tax reform, the balanced budget constraint of government satisfies

$$\varphi_b \int_0^\infty y_b^{1-\zeta_b} h_b(y)dy + g\int_0^\infty y_b h_b(y)dy = \int_0^\infty y_b h_b(y)dy, \tag{39}$$

where $h(y)$ denotes the income distribution of the economy. After the tax reform, we have

$$\varphi_m \int_0^\infty y_b^{1-\zeta_a} h_b(y)dy + g\int_0^\infty y_b h_b(y)dy = \int_0^\infty y_b h_b(y)dy, \tag{40}$$

if there was no behavioral response. Subtracting equation (39) from equation (40), we obtain

$$\varphi_m \int_0^\infty y_b^{1-\zeta_a} h_b(y)dy - \varphi_b \int_0^\infty y_b^{1-\zeta_b} h_b(y)dy = 0. \tag{41}$$

From equation (41), we find that the total social output does not change. By utilizing the concept of φ_m in the construction process, the initial line in equation (38) represents the welfare effects of redistribution induced by the mechanical effect of a higher ζ.

Following Heathcote and Tsujiyama (2021), we define

$$\begin{aligned}\Delta T(y_j(k)) =& y_a - \varphi_a y_a^{1-\zeta_a} - \left(y_b - \varphi_m y_b^{1-\zeta_a}\right) \\ &+ \left[r_a k + w_a z_j - \varphi_a (r_a k + w_a z_j)^{1-\zeta_a}\right] \\ &- \left[r_b k + w_b z_j - \varphi_m (r_b k + w_b z_j)^{1-\zeta_a}\right].\end{aligned}$$

We interpret the efficiency cost of the reform due to behavioral responses to be the revenue that would be collected from increasing the progressivity of income taxation in the absence of a behavioral response (i.e., $y_b - \varphi_m y_b^{1-\zeta_a}$), minus the actual extra transfers that can be funded in equilibrium, which we denote by $y_a - \varphi_a y_a^{1-\zeta_a}$. Then we have

$$\Delta T(y_j(k)) = \int_0^\infty \left(y - \varphi_a y^{1-\zeta_a}\right) h_a(y)dy - \int_0^\infty \left(y - \varphi_m y^{1-\zeta_a}\right) h_b(y)dy.$$

Therefore, $\Delta T(y_j)$ denotes the efficiency cost.

The fourth line in equation (38) represents the change in the price vector caused by the change in the progressivity, which affects the pre-tax income, and then affects consumption. This effect is pecuniary externalities in Chang and Park (2021). Furthermore, the last line shows the change in saving behavior caused by the tax reform, which is denoted by private insurance as in Chang and Park (2021).

We use $\widehat{WP}$ to denote $\hat{W}/|W|$. Substituting the decomposition of $\hat{c}_j$ above into (14), we have

$$\widehat{WP} \equiv \frac{\hat{W}}{|W|} = \frac{\sum_{j\in\{1,2\}} \left[\int_0^\infty u'(c_j(k))\hat{c}_j(k) f_j(k)dk + \int_0^\infty u(c_j(k))\hat{f}_j(k)dk\right]}{\left|\sum_{j\in\{1,2\}} \int_0^\infty u(c_j(k)) f_j(k)dk\right|}.$$

Therefore, we can decompose $\widehat{WP}$ as follows,

$$\begin{aligned}\widehat{WP} =&(\eta-1)\sum_{j\in\{1,2\}}\int_0^\infty \frac{\varphi_m(r_bk+w_bz_j)^{1-\zeta_a}-\varphi_b(r_bk+w_bz_j)^{1-\zeta_b}}{c_{bj}(k)}\Gamma_{bj}(k)dk\\&+(1-\eta)\sum_{j\in\{1,2\}}\int_0^\infty \frac{\Delta T(y_j(k))}{c_{bj}(k)}\Gamma_{bj}(k)dk\\&+(\eta-1)\sum_{j\in\{1,2\}}\int_0^\infty \frac{r_ak+w_az_j-(r_bk+w_bz_j)}{c_{bj}(k)}\Gamma_{bj}(k)dk\\&+(1-\eta)\sum_{j\in\{1,2\}}\int_0^\infty \frac{s_{aj}(k)-s_{bj}(k)}{c_{bj}(k)}\Gamma_{bj}(k)dk\\&-\sum_{j\in\{1,2\}}\int_0^\infty \frac{f_{aj}(k)-f_{bj}(k)}{f_{bj}(k)}\Gamma_{bj}(k)dk,\end{aligned}$$

and $\Gamma_{bj}(k)=\frac{u(c_{bj}(k))f_{bj}(k)}{\sum_{j\in\{1,2\}}\int_0^\infty u(c_{bj}(k))f_{bj}(k)dk}$ is the weight function. Here we use $u'(c_{bj})c_{bj}/u(c_{bj})=1-\eta$.

A.4.2 Optimal progressivity formula

Recalling equation (16), we obtain the optimal progressive income taxation ζ by making $\widehat{WP}$ equal to zero,

$$0=\widehat{WP}_I(\zeta^*)+\widehat{WP}_{II}(\zeta^*)+\widehat{WP}_{III}(\zeta^*)+\widehat{WP}_{IV}(\zeta^*)-\widehat{WP}_V(\zeta^*). \tag{42}$$

We denote

$$\widehat{WP}_d(\zeta^*)=(1-\eta)\sum_{j\in\{1,2\}}\int_0^\infty \Upsilon(k)\frac{s_{bj}(k)}{c_{bj}(k)}\Gamma_{bj}(k)dk,$$

where $\Upsilon(k)=-\frac{1-\zeta^*}{\Delta\zeta}\frac{s_{aj}(k)-s_{bj}(k)}{s_{bj}(k)}$ is the elasticity of $s_j(k)$ with respect to $1-\zeta$. Substituting $\widehat{WP}_d$ into equation (42), we have

$$\zeta^*=1-\frac{\widehat{WP}_d(\zeta^*)}{\widehat{WP}_I(\zeta^*)+\widehat{WP}_{II}(\zeta^*)+\widehat{WP}_{III}(\zeta^*)-\widehat{WP}_V(\zeta^*)},$$

Hence ζ^* is the optimal progressivity of income taxation.

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