Technical appendix for “On the Progressivity of Income Taxation”

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Theoretical results and proofs

1 Benchmark model

1.1 The proof of the HJB and KF equations

Individual households' consumption/saving decisions and the evolution of the wealth distribution are summarized by a Hamilton-Jacobi-Bellman (HJB) equation and a Kolmogorov Forward (KF) equation.

The valuation function $v(k)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho v_j(k) = \max_c u(c_j) + v_j'(k)s_j(k) + \frac{1}{2}v_j''(k)\sigma^2 k^2 + \lambda_j\left(v_{-j}(k) - v_j(k)\right),\ j = 1, 2, \tag{A.1}$$

where $s_j(k) = \varphi(rk + wz_j)^{1-\zeta} - \delta k - c_j(k)$. We adopt the convention that $-j = 2$ when $j = 1$, and $-j = 1$ when $j = 2$.

The stationary distributions $f_j(k)$, $j = 1, 2$, are governed by the KF equation,

$$0 = \frac{1}{2}\frac{d^2}{dk^2}\left[\sigma^2 k^2 f_j(k)\right] - \frac{d}{dk}\left[s_j(k)f_j(k)\right] - \lambda_j f_j(k) + \lambda_{-j}f_{-j}(k),\ j = 1, 2. \tag{A.2}$$

1.1.1 The proof of the HJB equation

We have the value function

$$v(k(t)) = \max_{\{c(\tau),k(\tau)\}_{\tau=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-\rho(\tau-t)} u(c(\tau)) d\tau,$$

$$s.t.\ dk(\tau) = \left[\varphi\left(rk(\tau) + wz(\tau)\right)^{1-\zeta} - \delta k(\tau) - c(\tau)\right] d\tau + \sigma k(\tau) dB(\tau).$$

Thus,

$$\begin{aligned} v(k(t)) &= \max_{\{c(\tau),k(\tau)\}_{\tau=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-\rho(\tau-t)} u(c(\tau)) d\tau \\ &= \max_{\{c(\tau),k(\tau)\}_{\tau=t}^{\infty}} \mathbb{E}_t \left\{ \int_t^{t+\Delta t} e^{-\rho(\tau-t)} u(c(\tau)) d\tau + \int_{t+\Delta t}^{\infty} e^{-\rho(\tau-t)} u(c(\tau)) d\tau \right\} \\ &= \max_{\{c(\tau),k(\tau)\}_{\tau=t}^{\infty}} \mathbb{E}_t \left\{ u(c(t))\Delta t + \int_{t+\Delta t}^{\infty} e^{-\rho(\tau-t)} u(c(\tau)) d\tau \right\} \\ &= \max_{\{c(\tau),k(\tau)\}_{\tau=t}^{\infty}} \left\{ u(c(t))\Delta t + \mathbb{E}_t \left[\max_{\{c(\tau),k(\tau)\}_{\tau=t+\Delta t}^{\infty}} \mathbb{E}_{t+\Delta t} \int_{t+\Delta t}^{\infty} e^{-\rho(\tau-t)} u(c(\tau)) d\tau \right] \right\} \\ &= \max_{\{c(\tau),k(\tau)\}_{\tau=t}^{\infty}} \left\{ u(c(t))\Delta t + (1-\rho\Delta t)\mathbb{E}_t v(k(t+\Delta t)) \right\}. \end{aligned} \tag{A.3}$$

Hence, we obtain

$$\begin{aligned} & v_1(k(t)) \\ =& \max_{\{c(\tau),k(\tau)\}_{\tau=t}^{\infty}} \left\{ u(c_1(t))\Delta t + (1-\rho\Delta t)\mathbb{E}_t \left[(1-\lambda_1\Delta t) v_1(k(t+\Delta t)) + \lambda_1 \Delta t v_2(k(t+\Delta t)) \right] \right\} \\ =& \max_{c_1(t)} \left\{ u(c_1(t))\Delta t + (1-\rho\Delta t)\mathbb{E}_t \left[(1-\lambda_1\Delta t) v_1(k(t)+\Delta k) + \lambda_1 \Delta t v_2(k(t)+\Delta k) \right] \right\} \\ =& \max_{c_1(t)} \left\{ u(c_1(t))\Delta t + (1-\rho\Delta t)\mathbb{E}_t \left[\begin{array}{c} (1-\lambda_1\Delta t)\left\{ v_1(k(t)) + v_1'(k(t))\Delta k + \frac{1}{2} v_1''(k(t))(\Delta k)^2 \right\} \\ +\lambda_1 \Delta t \left\{ v_2(k(t)) + v_2'(k(t))\Delta k + \frac{1}{2} v_2''(k(t))(\Delta k)^2 \right\} \end{array} \right] \right\} \\ =& \max_{c_1(t)} \left\{ \begin{array}{c} u(c_1(t))\Delta t + (1-\rho\Delta t)\times \\ \mathbb{E}_t \left[\begin{array}{c} (1-\lambda_1\Delta t)\left\{ v_1(k(t)) + \left[v_1'(k(t)) s_1(k(t)) + \frac{1}{2} v_1''(k(t))\sigma^2 k(t)^2 \right]\Delta t + v_1'(k(t))\sigma k(t)\Delta B \right\} \\ +\lambda_1 \Delta t \left\{ v_2(k(t)) + \left[v_2'(k(t)) s_1(k(t)) + \frac{1}{2} v_2''(k(t))\sigma^2 k(t)^2 \right]\Delta t + v_2'(k(t))\sigma k(t)\Delta B \right\} \end{array} \right] \end{array} \right\} \\ =& \max_{c_1(t)} \left\{ \begin{array}{c} u(c_1(t))\Delta t + (1-\rho\Delta t)\times \\ \left[\begin{array}{c} (1-\lambda_1\Delta t)\left\{ v_1(k(t)) + \left[v_1'(k(t)) s_1(k(t)) + \frac{1}{2} v_1''(k(t))\sigma^2 k(t)^2 \right]\Delta t \right\} \\ +\lambda_1 \Delta t \left\{ v_2(k(t)) + \left[v_2'(k(t)) s_1(k(t)) + \frac{1}{2} v_2''(k(t))\sigma^2 k(t)^2 \right]\Delta t \right\} \end{array} \right] \end{array} \right\} \\ =& \max_{c_1(t)} \left\{ \begin{array}{c} u(c_1(t))\Delta t + v_1(k(t)) + \left[v_1'(k(t)) s_1(k(t)) + \frac{1}{2} v_1''(k(t))\sigma^2 k(t)^2 \right]\Delta t \\ -\lambda_1 v_1(k(t))\Delta t + \lambda_1 v_2(k(t))\Delta t - \rho v_1(k(t))\Delta t \end{array} \right\}. \end{aligned} \tag{A.4}$$

Thus, we have

$$0 = \max_{c_1(t)} \left\{ \begin{array}{c} u(c_1(t))\Delta t + \left[v_1'(k(t)) s_1(k(t)) + \frac{1}{2} v_1''(k(t))\sigma^2 k(t)^2 \right]\Delta t \\ -\lambda_1 v_1(k(t))\Delta t + \lambda_1 v_2(k(t))\Delta t - \rho v_1(k(t))\Delta t \end{array} \right\}.$$

Therefore, we have

$$\rho v_1(k) = \max_{c_1} \left\{ u(c_1) + v_1'(k)s_1(k) + \frac{1}{2}v_1''(k)\sigma^2 k^2 + \lambda_1(v_2(k) - v_1(k)) \right\}.$$

Similarly, we obtain

$$\rho v_2(k) = \max_{c_2} \left\{ u(c_2) + v_2'(k)s_2(k) + \frac{1}{2}v_2''(k)\sigma^2 k^2 + \lambda_2(v_1(k) - v_2(k)) \right\}.$$

1.1.2 The proof of the KF equation

If labor productivity $z_t = z_j$, we have

$$dk_t = s_j(k_t)dt + \sigma k_t dB_t,$$

where $s_j(k_t) = \varphi\left(rk_t + wz_j\right)^{1-\zeta} - \delta k_t - c_j(k_t)$, for $j = 1, 2$.

Let

$$\phi_1(k) = \phi(z_1, k),$$

and

$$\phi_2(k) = \phi(z_2, k),$$

for functions $\phi(z_1, k)$ and $\phi(z_2, k)$. We assume that $\phi_1(k)$ and $\phi_2(k)$ are twice continuously differentiable functions with compact support in $[0, \infty)$. For function $\phi(z, k)$, we have

$$E\left[\phi(z_t, k_t)\right] = \int_0^\infty \phi_1(k) f_1(k,t)dk + \phi_1(0)m_{1,t} + \int_0^\infty \phi_2(k) f_2(k,t)dk + \phi_2(0)m_{2,t},$$

where $f_j(k, t)$ is the density function of the wealth distribution for $z_t = z_j$. $m_{j,t}$ is the Dirac point masses at $k = 0$ corresponding to $z_t = z_j$. And we have

$$\begin{aligned} & E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t})\right] \\ = \; & \int_0^\infty \phi_1(k) f_1(k, t+\Delta t)dk + \phi_1(0)m_{1,t+\Delta t} + \int_0^\infty \phi_2(k) f_2(k, t+\Delta t)dk + \phi_2(0)m_{2,t+\Delta t}. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} & \frac{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t})\right] - E\left[\phi(z_t, k_t)\right]}{\Delta t} \\ = \; & \frac{1}{\Delta t}\left\{ \begin{array}{c} \int_0^\infty \phi_1(k)\left[f_1(k,t+\Delta t) - f_1(k,t)\right]dk + \phi_1(0)(m_{1,t+\Delta t} - m_{1,t}) \\ + \int_0^\infty \phi_2(k)\left[f_2(k,t+\Delta t) - f_2(k,t)\right]dk + \phi_2(0)(m_{2,t+\Delta t} - m_{2,t}) \end{array} \right\} \\ = \; & \int_0^\infty \phi_1(k)\frac{f_1(k,t+\Delta t) - f_1(k,t)}{\Delta t}dk + \phi_1(0)\frac{m_{1,t+\Delta t} - m_{1,t}}{\Delta t} \\ & + \int_0^\infty \phi_2(k)\frac{f_2(k,t+\Delta t) - f_2(k,t)}{\Delta t}dk + \phi_2(0)\frac{m_{2,t+\Delta t} - m_{2,t}}{\Delta t} \\ \approx \; & \int_0^\infty \phi_1(k)\frac{\partial}{\partial t}f_1(k,t)dk + \phi_1(0)\dot{m}_{1,t} + \int_0^\infty \phi_2(k)\frac{\partial}{\partial t}f_2(k,t)dk + \phi_2(0)\dot{m}_{2,t}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d\phi_j(k_t) &= \phi_j'(k_t)dk_t + \frac{1}{2}\phi_j''(k_t)\left(dk_t\right)^2 \\
&= \phi_j'(k_t)\left[s_j(k_t)dt + \sigma k_t dB_t\right] + \frac{1}{2}\phi_j''(k_t)\sigma^2 k_t^2 dt \\
&= \left[\phi_j'(k_t)s_j(k_t) + \frac{1}{2}\phi_j''(k_t)\sigma^2 k_t^2\right] dt + \phi_j'(k_t)\sigma k_t dB_t,
\end{aligned}$$

for $j = 1, 2$, by Itô's lemma. Thus, we have

$$\begin{aligned}
& \frac{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t})\right] - E\left[\phi(z_t, k_t)\right]}{\Delta t} \\
= \;& \frac{1}{\Delta t} E\left\{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_t, k_t\right] - \phi(z_t, k_t)\right\} \\
= \;& \frac{1}{\Delta t}\left\{\begin{array}{c}
\int_0^\infty \left\{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_1, k\right] - \phi_1(k)\right\} f_1(k,t)dk \\
+ \left\{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_1, 0\right] - \phi_1(0)\right\} m_{1,t} \\
+ \int_0^\infty \left\{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_2, k\right] - \phi_2(k)\right\} f_2(k,t)dk \\
+ \left\{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_2, 0\right] - \phi_2(0)\right\} m_{2,t}
\end{array}\right\} \\
= \;& \int_0^\infty \frac{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_1, k\right] - \phi_1(k)}{\Delta t} f_1(k,t)dk + \frac{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_1, 0\right] - \phi_1(0)}{\Delta t} m_{1,t} \\
& + \int_0^\infty \frac{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_2, k\right] - \phi_2(k)}{\Delta t} f_2(k,t)dk + \frac{E\left[\phi(z_{t+\Delta t}, k_{t+\Delta t}) | z_2, 0\right] - \phi_2(0)}{\Delta t} m_{2,t} \\
\approx \;& \int_0^\infty \left[\phi_1'(k)s_1(k) + \frac{1}{2}\phi_1''(k)\sigma^2 k^2 - \lambda_1\phi_1(k) + \lambda_1\phi_2(k)\right] f_1(k,t)dk \\
& + \left[\phi_1'(0)s_1(0) - \lambda_1\phi_1(0) + \lambda_1\phi_2(0)\right] m_{1,t} \\
& + \int_0^\infty \left[\phi_2'(k)s_2(k) + \frac{1}{2}\phi_2''(k)\sigma^2 k^2 - \lambda_2\phi_2(k) + \lambda_2\phi_1(k)\right] f_2(k,t)dk \\
& + \left[\phi_2'(0)s_2(0) - \lambda_2\phi_2(0) + \lambda_2\phi_1(0)\right] m_{2,t}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \int_0^\infty \phi_1(k)\frac{\partial}{\partial t}f_1(k,t)dk + \phi_1(0)\dot{m}_{1,t} + \int_0^\infty \phi_2(k)\frac{\partial}{\partial t}f_2(k,t)dk + \phi_2(0)\dot{m}_{2,t} \\
= \;& \int_0^\infty \left[\phi_1'(k)s_1(k) + \frac{1}{2}\phi_1''(k)\sigma^2 k^2 - \lambda_1\phi_1(k) + \lambda_1\phi_2(k)\right] f_1(k,t)dk \\
& + \left[\phi_1'(0)s_1(0) - \lambda_1\phi_1(0) + \lambda_1\phi_2(0)\right] m_{1,t} \\
& + \int_0^\infty \left[\phi_2'(k)s_2(k) + \frac{1}{2}\phi_2''(k)\sigma^2 k^2 - \lambda_2\phi_2(k) + \lambda_2\phi_1(k)\right] f_2(k,t)dk \\
& + \left[\phi_2'(0)s_2(0) - \lambda_2\phi_2(0) + \lambda_2\phi_1(0)\right] m_{2,t}.
\end{aligned}$$

Letting $\phi_1(0) = \phi_1'(0) = 0$ and $\phi_2(0) = \phi_2'(0) = 0$, we have

$$\begin{aligned}
& \int_0^\infty \phi_1(k) \frac{\partial}{\partial t} f_1(k,t)dk + \int_0^\infty \phi_2(k) \frac{\partial}{\partial t} f_2(k,t)dk \\
= & \int_0^\infty \left[\phi_1'(k)s_1(k) + \frac{1}{2}\phi_1''(k)\sigma^2 k^2 - \lambda_1\phi_1(k) + \lambda_1\phi_2(k) \right] f_1(k,t)dk \\
& + \int_0^\infty \left[\phi_2'(k)s_2(k) + \frac{1}{2}\phi_2''(k)\sigma^2 k^2 - \lambda_2\phi_2(k) + \lambda_2\phi_1(k) \right] f_2(k,t)dk. \qquad \text{(A.5)}
\end{aligned}$$

If we pick $\phi_2(k) = 0$ for all $k \geq 0$, we have

$$\begin{aligned}
& \int_0^\infty \phi_1(k) \frac{\partial}{\partial t} f_1(k,t)dk \\
= & \int_0^\infty \left[\phi_1'(k)s_1(k) + \frac{1}{2}\phi_1''(k)\sigma^2 k^2 - \lambda_1\phi_1(k) \right] f_1(k,t)dk + \int_0^\infty \lambda_2\phi_1(k)f_2(k,t)dk,
\end{aligned}$$

from equation (A.5). Thus, we have

$$\begin{aligned}
& \int_0^\infty \phi_1(k) \frac{\partial}{\partial t} f_1(k,t)dk \\
= & \int_0^\infty \phi_1'(k)s_1(k)f_1(k,t)dk + \frac{1}{2}\int_0^\infty \phi_1''(k)\sigma^2 k^2 f_1(k,t)dk \\
& - \int_0^\infty \lambda_1\phi_1(k)f_1(k,t)dk + \int_0^\infty \lambda_2\phi_1(k)f_2(k,t)dk \\
= & -\int_0^\infty \phi_1(k)\frac{\partial}{\partial k}\left[s_1(k)f_1(k,t)\right]dk + \frac{1}{2}\int_0^\infty \phi_1(k)\frac{\partial^2}{\partial k^2}\left[\sigma^2 k^2 f_1(k,t)\right]dk \\
& - \int_0^\infty \lambda_1\phi_1(k)f_1(k,t)dk + \int_0^\infty \lambda_2\phi_1(k)f_2(k,t)dk \\
= & \int_0^\infty \phi_1(k)\left\{ \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma^2 k^2 f_1(k,t)\right] - \frac{\partial}{\partial k}\left[s_1(k)f_1(k,t)\right] - \lambda_1 f_1(k,t) + \lambda_2 f_2(k,t) \right\} dk.
\end{aligned}$$

Since $\phi_1(k)$ is arbitrary, we have

$$\frac{\partial}{\partial t} f_1(k,t) = \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma^2 k^2 f_1(k,t)\right] - \frac{\partial}{\partial k}\left[s_1(k)f_1(k,t)\right] - \lambda_1 f_1(k,t) + \lambda_2 f_2(k,t).$$

If we pick $\phi_1(k) = 0$ for all $k \geq 0$, we have

$$\begin{aligned}
& \int_0^\infty \phi_2(k) \frac{\partial}{\partial t} f_2(k,t)dk \\
= & \int_0^\infty \lambda_1\phi_2(k)f_1(k,t)dk + \int_0^\infty \left[\phi_2'(k)s_2(k) + \frac{1}{2}\phi_2''(k)\sigma^2 k^2 - \lambda_2\phi_2(k) \right] f_2(k,t)dk,
\end{aligned}$$

from equation (A.5). Therefore, we have

$$\frac{\partial}{\partial t} f_2(k,t) = \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[\sigma^2 k^2 f_2(k,t)\right] - \frac{\partial}{\partial k}\left[s_2(k)f_2(k,t)\right] - \lambda_2 f_2(k,t) + \lambda_1 f_1(k,t).$$

1.2 The derivative of consumption

Here we report a theoretic result, showing that $\hat{c}_j(k)$ can be expressed by Neumann series, like $\hat{f}_j(k)$. Letting $\hat{\mathbf{c}}(\mathbf{k}) \equiv (\hat{c}_1(k), \hat{c}_2(k))$, we have

Proposition 5 $\hat{\mathbf{c}}(\cdot)$ *satisfies*

$$\hat{\mathbf{c}} = \mathcal{A}_c \hat{\mathbf{c}} + \mathbf{Q}_c, \tag{A.6}$$

where $\mathcal{A}_c = \mathbf{X}_c^{-1}\mathbf{D}_c \frac{d}{dk} + \mathbf{X}_c^{-1}\mathbf{M}_c \frac{d^2}{dk^2}$ *and* $\mathbf{Q}_c = \mathbf{X}_c^{-1}\mathbf{Q}_a$.[1]

If $|| \mathcal{A}_c ||< 1$, $\hat{\mathbf{c}}(\cdot)$ *in (A.6) can be expressed as*

$$\hat{\mathbf{c}}(\mathbf{k}) = \mathbf{Q}_c + \sum_{n=1}^{\infty} \mathcal{A}_c^n \mathbf{Q}_c. \tag{A.7}$$

We implement the perturbation analysis on the Euler equation to obtain equation (A.6) that the derivative of consumption functions satisfies.

Proof of Proposition 5: First, we have an Euler equation as follows.

Lemma 8 *The consumption and saving functions,* $c_j(k)$ *and* $s_j(k)$, *for* $j = 1, 2$ *satisfy*

$$\begin{aligned}\left[\rho + \delta - \varphi r(1-\zeta)(wz_j + rk)^{-\zeta}\right] u'(c_j(k)) = {} & u''(c_j(k))c_j'(k)\left[s_j(k) + \sigma^2 k\right] \\ & + \frac{1}{2}\left[u'''(c_j(k))c_j'(k)^2 + u''(c_j(k))c_j''(k)\right]\sigma^2 k^2 \\ & + \lambda_j\left(u'(c_{-j}(k)) - u'(c_j(k))\right).\end{aligned} \tag{A.8}$$

Proof of Lemma 8: Differentiating the HJB equation (A.1) with respect to k and using $v_j'(k) = u_j'(c_j(k))$, $v_j''(k) = u_j''(c_j(k))c_j'(k)$, and $v_j'''(k) = u_j'''(c_j(k))(c_j'(k))^2 + u_j''(c_j(k))c_j''(k)$, we have the Euler equation (A.8). □

The Euler equation (A.8) characterizes the household's optimal intertemporal choice in the household problem. The terms in the bracket of its left-hand side reflect the relative forces of time discounting and the after-tax return to capital The return takes into account the progressivity of income taxation, which differs from that in Achdou et al. (2022).

Rewrite equation (A.8) as

$$\begin{aligned} & \left[\rho + \delta - \varphi r(1-\zeta)(wz_j + rk)^{-\zeta} + \lambda_j\right] u'(c_j(k)) - \lambda_j u'(c_{-j}(k)) \\ = & u''(c_j(k))c_j'(k)\left[s_j(k) + \sigma^2 k\right] \\ + & \frac{1}{2}\left[u'''(c_j(k))c_j'(k)^2 + u''(c_j(k))c_j''(k)\right]\sigma^2 k^2. \end{aligned} \tag{A.9}$$

[1] See proof for the definitions of $\mathbf{X}_c, \mathbf{D}_c, \mathbf{M}_c$, and $\mathbf{Q}_a$.

Under Assumption 1, $u_j'(k) = c_j(k)^{-\eta}, u_j''(k) = -\eta c_j(k)^{-\eta-1}$, and $u_j'''(k) = \eta(\eta+1)c_j(k)^{-\eta-2}$. Substituting them into equation (A.9), we obtain

$$\begin{aligned}
&\left[\rho + \delta - \varphi r(1-\zeta)(wz_j + rk)^{-\zeta} + \lambda_j\right] c_j(k)^{-\eta} - \lambda_j c_{-j}(k)^{-\eta} \\
=& - \eta c_j(k)^{-\eta-1} c_j'(k) \left[s_j(k) + \sigma^2 k\right] \\
&+ \frac{1}{2}\sigma^2 k^2 \eta(\eta+1) c_j(k)^{-\eta-2} c_j'(k)^2 - \frac{1}{2}\sigma^2 k^2 \eta c_j(k)^{-\eta-1} c_j''(k).
\end{aligned} \tag{A.10}$$

Differentiating equation (A.10) with respect to ζ, we have

$$\begin{aligned}
&\hat{p}_j c_j(k)^{-\eta} - p_j \eta c_j(k)^{-\eta-1}\hat{c}_j(k) + \lambda_j \eta c_{-j}(k)^{-\eta-1}\hat{c}_{-j}(k) \\
=&\eta(\eta+1)c_j(k)^{-\eta-2}\hat{c}_j(k)c_j'(k)\left[s_j(k)+\sigma^2 k\right] - \eta c_j(k)^{-\eta-1}\hat{c}_j(k)'\left[s_j(k)+\sigma^2 k\right] \\
&-\eta c_j(k)^{-\eta-1}c_j'(k)\hat{s}_j(k) - \frac{1}{2}\sigma^2 k^2 \eta(\eta+1)(\eta+2)c_j(k)^{-\eta-3}\hat{c}_j(k)c_j'(k)^2 \\
&+\sigma^2 k^2 \eta(\eta+1)c_j(k)^{-\eta-2}\hat{c}_j(k)' + \frac{1}{2}\sigma^2 k^2 \eta(\eta+1)c_j(k)^{-\eta-2}\hat{c}_j(k)c_j''(k) - \frac{1}{2}\sigma^2 k^2 \eta c_j(k)^{-\eta-1}\hat{c}_j(k)'',
\end{aligned}$$

where $y_j(k) = wz_j + rk$, $p_j = \rho + \delta - \varphi r(1-\zeta)y_j(k)^{-\zeta} + \lambda_j$, $\hat{s}_j(k) = \hat{\varphi} y_j^{1-\zeta} + \varphi[-\ln y_j + (1-\zeta)\hat{y}_j/y_j]y_j^{1-\zeta} - \hat{c}_j(k)$, and $\hat{p}_j = -\varphi r(1-\zeta)y_j(k)^{-\zeta}\left[\hat{\varphi}/\varphi + \hat{r}/r - 1/(1-\zeta) - \ln y_j(k) - \zeta(\hat{r}k + \hat{w}z)/y_j(k)\right]$.

After combining similar terms, we obtain

$$\begin{aligned}
&\left\{ \begin{array}{l} -p_j \eta c_j(k)^{-\eta-1} - \eta(\eta+1)c_j(k)^{-\eta-2}c_j'(k)\left[s_j(k)+\sigma^2 k\right] - \eta c_j(k)^{-\eta-1}c_j'(k) \\ +\frac{1}{2}\sigma^2 k^2 \eta(\eta+1)c_j(k)^{-\eta-3}\left[(\eta+2)c_j'(k)^2 - c_j(k)c_j''(k)\right] \end{array} \right\} \hat{c}_j(k) + \\
&\lambda_j \eta c_{-j}(k)^{-\eta-1}\hat{c}_{-j}(k) \\
=&\eta c_j(k)^{-\eta-2}\left[(\eta+1)\sigma^2 k^2 - c_j(k)(s_j(k)+\sigma^2 k)\right]\hat{c}_j(k)' - \frac{1}{2}\sigma^2 k^2 \eta c_j(k)^{-\eta-1}\hat{c}_j(k)'' \\
&-c_j(k)^{-\eta-1}\left[\eta c_j'(k)\left(\hat{\varphi} y_j^{1-\zeta} + \varphi[-\ln y_j + (1-\zeta)(\hat{r}k + \hat{w}z_j)/y_j]y_j^{1-\zeta}\right) + \hat{p}_j c_j(k)\right].
\end{aligned} \tag{A.11}$$

Rewrite equation (A.11) as

$$\begin{aligned}
\begin{pmatrix} A_1 & X_1 \\ A_2 & X_2 \end{pmatrix} \begin{pmatrix} \hat{c}_1(k) \\ \hat{c}_2(k) \end{pmatrix} =& \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} (\hat{c}_1(k))' \\ (\hat{c}_2(k))' \end{pmatrix} \\
&+ \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} (\hat{c}_1(k))'' \\ (\hat{c}_2(k))'' \end{pmatrix} \\
&+ \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},
\end{aligned} \tag{A.12}$$

where

$$
\begin{aligned}
A_1 &= -p_1\eta c_1(k)^{-\eta-1} - \eta(\eta+1)c_1(k)^{-\eta-2}c_1'(k)\left[s_1(k)+\sigma^2 k\right] - \eta c_1(k)^{-\eta-1}c_1'(k) \\
&\quad + \frac{1}{2}\sigma^2 k^2 \eta(\eta+1)c_1(k)^{-\eta-3}\left[(\eta+2)c_1'(k)^2 - c_1(k)c_1''(k)\right], \\
A_2 &= \lambda_2\eta c_1(k)^{-\eta-1}, \\
X_1 &= \lambda_1\eta c_2(k)^{-\eta-1}, \\
X_2 &= -p_1\eta c_2(k)^{-\eta-1} - \eta(\eta+1)c_2(k)^{-\eta-2}c_2'(k)\left[s_2(k)+\sigma^2 k\right] - \eta c_2(k)^{-\eta-1}c_2'(k) \\
&\quad + \frac{1}{2}\sigma^2 k^2 \eta(\eta+1)c_2(k)^{-\eta-3}\left[(\eta+2)c_2'(k)^2 - c_2(k)c_2''(k)\right], \\
D_1 &= \eta c_1(k)^{-\eta-2}\left[(\eta+1)\sigma^2 k^2 - c_1(k)(s_1(k)+\sigma^2 k)\right], \\
D_2 &= \eta c_2(k)^{-\eta-2}\left[(\eta+1)\sigma^2 k^2 - c_2(k)(s_2(k)+\sigma^2 k)\right], \\
M_1 &= -\frac{1}{2}\sigma^2 k^2 \eta c_1(k)^{-\eta-1}, \\
M_2 &= -\frac{1}{2}\sigma^2 k^2 \eta c_2(k)^{-\eta-1}, \\
Q_1 &= -\eta c_1(k)^{-\eta-1}c_1'(k)\left(\hat{\varphi} y_1^{1-\zeta} + \varphi[-\ln y_1 + (1-\zeta)(\hat{r}k + \hat{w}z_1)/y_1]y_1^{1-\zeta}\right), \\
Q_2 &= -\eta c_2(k)^{-\eta-1}c_2'(k)\left(\hat{\varphi} y_2^{1-\zeta} + \varphi[-\ln y_2 + (1-\zeta)(\hat{r}k + \hat{w}z_2)/y_2]y_2^{1-\zeta}\right).
\end{aligned}
$$

We rewrite equation (A.12) in the following form,

$$\mathbf{X}_c\hat{\mathbf{c}} = \mathbf{D}_c\hat{\mathbf{c}}' + \mathbf{M}_c\hat{\mathbf{c}}'' + \mathbf{Q}_a. \tag{A.13}$$

Thus, we have

$$\hat{\mathbf{c}} = \mathbf{X}_c^{-1}\mathbf{D}_c\hat{\mathbf{c}}' + \mathbf{X}_c^{-1}\mathbf{M}_c\hat{\mathbf{c}}'' + \mathbf{X}_c^{-1}\mathbf{Q}_a, \tag{A.14}$$

We can express equation (A.14) in the following form

$$\hat{\mathbf{c}} = \mathcal{A}_c\hat{\mathbf{c}} + \mathbf{Q}_c, \tag{A.15}$$

where $\mathcal{A}_c = \mathbf{X}_c^{-1}\mathbf{D}_c\frac{d}{dk} + \mathbf{X}_c^{-1}\mathbf{M}_c\frac{d^2}{dk^2}$, and $\mathbf{Q}_c = \mathbf{X}_c^{-1}\mathbf{Q}_a$.

From equation (A.15), we have

$$(\mathbf{I} - \mathcal{A}_c)\hat{\mathbf{c}} = \mathbf{Q}_c,$$

where $\mathbf{I}$ is the identity operator: $\mathbf{I}\cdot\hat{\mathbf{c}} = \hat{\mathbf{c}}$. $\mathcal{A}_c$ is a linear operator. If $\|\mathcal{A}_c\| < 1$, $\mathbf{I} - \mathcal{A}_c$ has a unique bounded linear inverse $(\mathbf{I} - \mathcal{A}_c)^{-1}$ which is a Neumann series,

$$
\begin{aligned}
\hat{\mathbf{c}} &= (\mathbf{I} - \mathcal{A}_c)^{-1}\mathbf{Q}_c \\
&= \sum_{n=0}^{\infty}\mathcal{A}_c^n\mathbf{Q}_c \\
&= \mathbf{Q}_c + \sum_{n=1}^{\infty}\mathcal{A}_c^n\mathbf{Q}_c,
\end{aligned} \tag{A.16}
$$

by Theorem 2 in Chapter II of Yosida (1995). ■

2 Endogenous labor supply

2.1 Economy

The valuation function $v(k)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho v_j(k) = \max_{\tilde{c}_j} u(\tilde{c}_j) + v_j'(k)s_j(k) + \frac{1}{2}v_j''(k)\sigma^2 k^2 + \lambda_j \left(v_{-j}(k) - v_j(k)\right), \; j = 1, 2, \tag{A.17}$$

where $s_j(k) = \varphi(rk + wz_j\ell_j(k))^{1-\zeta} - \delta k - \tilde{c}_j(k) - \gamma(\ell_j(k))$. We adopt the convention that $-j = 2$ when $j = 1$, and $-j = 1$ when $j = 2$.

The stationary distributions $f_j(k)$, $j = 1, 2$, which satisfy

$$0 = \frac{1}{2}\frac{d^2}{dk^2}\left[\sigma^2 k^2 f_j(k)\right] - \frac{d}{dk}\left[s_j(k) f_j(k)\right] - \lambda_j f_j(k) + \lambda_{-j} f_{-j}(k), \; j = 1, 2. \tag{A.18}$$

Same as that of the benchmark model.

We have some results characterizing the policy functions of households with endogenous labor supply.

Proposition 6 *Impose Assumption 2. As $k \to \infty$, the consumption policy function $\tilde{c}_j(k)$ and saving policy function $\tilde{s}_j(k)$, $j = 1, 2$, have the following asymptotic properties.*

1. *If $\zeta > 0$, then*

$$\tilde{c}_j(k) \sim \left[\frac{\rho + (1-\eta)\delta}{\eta} + \frac{\sigma^2(1-\eta)}{2}\right] k, \quad \tilde{s}_j(k) \sim -\left[\frac{\rho + \delta}{\eta} + \frac{\sigma^2(1-\eta)}{2}\right] k.$$

2. *If $\zeta = 0$, then*

$$\tilde{c}_j(k) \sim \left[\frac{\rho + (1-\eta)(\delta - \varphi r)}{\eta} + \frac{\sigma^2(1-\eta)}{2}\right] k, \quad \tilde{s}_j(k) \sim -\left[\frac{\rho + \delta - \varphi r}{\eta} + \frac{\sigma^2(1-\eta)}{2}\right] k,$$

   where $\varphi = 1 - g$.

3. *If $\zeta < 0$, then*

$$\tilde{c}_j(k) \sim \left[\frac{(\frac{1-\eta}{\eta}\varphi r^{1-\zeta})^{-\eta}}{1 - \eta + 2\zeta\eta}\right] k^{1-2\zeta}, \quad \tilde{s}_j(k) \sim -\left[\frac{(\frac{1-\eta}{\eta}\varphi r^{1-\zeta})^{-\eta}}{1 - \eta + 2\zeta\eta}\right] k^{1-2\zeta}.$$

We also characterize the stationary wealth distribution.

Theorem 2 *Under Assumption 2, we have*

1. *If $\zeta > 0$ and $\eta(\eta-1)\sigma^2 < 2(\rho+\delta)$, there exists a unique stationary wealth distribution which follows an asymptotic power law, i.e. $1 - F(k) \sim \kappa_a k^{-\Theta_a}$ as $k \to \infty$, with*
$$\Theta_a = 2 + \frac{2(\rho+\delta)}{\eta\sigma^2} - \eta.$$

2. *If $\zeta = 0$, and $\eta(\eta-1)\sigma^2 < 2\left[(g-1)r + \rho + \delta\right]$. there exists a unique stationary wealth distribution and $1 - F(k) \sim \kappa_b k^{-\Theta_b}$ as $k \to \infty$, with*
$$\Theta_b = -\frac{2(1-g)r}{\eta\sigma^2} + \Theta_a.$$

3. *If $\zeta < 0$, there does not exist a stationary wealth distribution.*

We have the decomposition on social welfare effect as follows,

$$\widehat{WP}_L = \widehat{WP}_{LI} + \widehat{WP}_{LII} + \widehat{WP}_{LIII} + \widehat{WP}_{LIV} - \widehat{WP}_{LV}, \tag{A.19}$$

where

$$\begin{aligned}
\widehat{WP}_{LI} =& (\eta-1)\sum_{j\in\{1,2\}}\int_0^\infty \frac{\varphi_m(r_b k + w_b z_j \ell_{bj}(k))^{1-\zeta_a} - \varphi_b(r_b k + w_b z_j \ell_{bj}(k))^{1-\zeta_b}}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,\\
\widehat{WP}_{LII} =& (1-\eta)\sum_{j\in\{1,2\}}\int_0^\infty \frac{\Delta T_\ell(y_j(k))}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,\\
\widehat{WP}_{LIII} =& (\eta-1)\sum_{j\in\{1,2\}}\int_0^\infty \frac{r_a k + w_a z_j \ell_{aj}(k) - (r_b k + w_b z_j \ell_{bj}(k))}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,\\
\widehat{WP}_{LIV} =& (1-\eta)\sum_{j\in\{1,2\}}\int_0^\infty \frac{s_{aj}(k) - s_{bj}(k)}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,\\
\widehat{WP}_{LV} =& \sum_{j\in\{1,2\}}\int_0^\infty \frac{f_{aj}(k) - f_{bj}(k)}{\tilde{c}_{bj}(k)}\Gamma_{bj}(k)dk,
\end{aligned}$$

with $\Gamma_{bj}(k) = \frac{u(\tilde{c}_{bj}(k))f_{bj}(k)}{\sum_{j\in\{1,2\}}\int_0^\infty u(\tilde{c}_{bj}(k))f_{bj}(k)dk}$ is the weight function, and

$$\begin{aligned}
&\Delta T_\ell(y_j(k))\\
=& r_a k + w_a z_j \ell_{bj}(k) - (r_b k + w_b z_j \ell_{bj}(k)) - \varphi_m\left[(r_a k + w_a z_j \ell_{bj}(k))^{1-\zeta_a} - (r_b k + w_b z_j \ell_{bj}(k))^{1-\zeta_a}\right]\\
&-(\varphi_a - \varphi_m)(r_a k + w_a z_j \ell_{bj}(k))^{1-\zeta_a}.
\end{aligned}$$

2.2 Proof of theoretical results with endogenous labor supply

2.2.1 Proof of Proposition 6

Proof: We have three cases: $\zeta > 0$, $\zeta = 0$, and $\zeta < 0$.
Case 1 $\zeta > 0$.

Lemma 9 *Consider the problem*

$$\rho v_j(k) = \max_{\tilde{c},\ell}\{u(\tilde{c}_j) - v_j'(k)\tilde{c}_j\} + v_j'(k)\left(-\delta k + \varphi(rk)^{1-\zeta}k^{-\zeta}\right) + \frac{1}{2}v_j''(k)\sigma^2 k^2, \tag{A.20}$$

where $u(\cdot)$ satisfies Assumption 2. The optimal policy functions that solve (A.20) are linear in wealth asymptotically and given by

$$\tilde{c}_j(k) \sim \left[\frac{\rho + (1-\eta)\delta}{\eta} + \frac{\sigma^2(1-\eta)}{2}\right]k, \quad \tilde{s}_j(k) \sim -\left[\frac{\rho+\delta}{\eta} + \frac{\sigma^2(1-\eta)}{2}\right]k.$$

Proof: The auxiliary lemma proof is the same as case 1 of Proposition 1. □

Lemma 10 *Consider problem (A.17). For any $\xi > 0$,*

$$v_j(\xi k) = \xi^{1-\eta}v_{\xi,j}(k), \tag{A.21}$$

where $v_{\xi,j}$ solves

$$\begin{aligned}\rho v_{\xi,j}(k) = \max_{\tilde{c}} u(\tilde{c}_j) + v_{\xi,j}'(k)\left[-\delta k + \varphi(rk + wz_j\ell_j/\xi)^{1-\zeta}\xi^{-\zeta} - \tilde{c}\right] \\ + \frac{1}{2}v_{\xi,j}''(k)\sigma^2k^2 + \lambda_j\left(v_{\xi,-j}(k) - v_{\xi,j}(k)\right).\end{aligned} \tag{A.22}$$

Proof: The subsequent proof process also follows the same way as the proof of case 1 in Proposition 1. □
Case 2 $\zeta = 0$.

Lemma 11 *Consider problem*

$$\rho v_j(k) = \max_{\tilde{c}}\{u(\tilde{c}_j) - v_j'(k)\tilde{c}_j\} + v_j'(k)\left[-\delta k + \varphi rk\right] + \frac{1}{2}v_j''(k)\sigma^2k^2, \tag{A.23}$$

where $u(.)$ satisfies Assumption 2. The optimal policy functions that solve (A.23) are linear in wealth and given by

$$\tilde{c}_j(k) = \left[\frac{\rho + (1-\eta)(\delta - \varphi r)}{\eta} + \frac{\sigma^2(1-\eta)}{2}\right]k, \quad \tilde{s}_j(k) = -\left[\frac{\rho + \delta - \varphi r}{\eta} + \frac{\sigma^2(1-\eta)}{2}\right]k.$$

Proof: The auxiliary lemma proof is the same as case 2 of Proposition 1. □

Lemma 12 *Consider problem (A.17). For any $\varrho > 0$,*

$$v_j(\varrho k) = \varrho^{1-\eta} v_{\varrho,j}(k) \tag{A.24}$$

where $v_{\varrho,j}$ solves

$$\begin{aligned}\rho v_{\varrho,j}(k) = \max_{c,0\leq k} u(c,\ell) + v_j'(k)\left[-\delta k + \varphi(rk + wz_j\ell_j/\varrho) - c\right] \\ + \frac{1}{2} v_{\varrho,j}''(k)\sigma^2 k^2 + \lambda_j\left(v_{\varrho,-j}(k) - v_{\varrho,j}(k)\right)\end{aligned} \tag{A.25}$$

Proof: The subsequent proof process also follows the same way as the proof of case 2 in Proposition 1. □

Case 3 $\zeta < 0$.

Lemma 13 *Consider the problem*

$$0 = H(v_j'(k)) + v_j'(k)\varphi(rk)^{1-\zeta}, \tag{A.26}$$

where $H(v_j'(k)) = \max_{\tilde{c}}\{u(\tilde{c}_j) - v_j'(k)\tilde{c}_j(k)\} = \frac{\eta}{1-\eta}(v_j'(k))^{\frac{\eta-1}{\eta}}$. The optimal policy functions that solve (A.26) are given by

$$\tilde{c}_j(k) = \frac{(1-\eta)\varphi r^{1-\zeta}}{\eta} k^{1-\zeta}, \quad s_j(k) = \left[\varphi r^{1-\zeta} - \frac{(1-\eta)\varphi r^{1-\zeta}}{\eta}\right] k^{1-\zeta}.$$

Proof: The auxiliary lemma proof is the same as case 3 of Proposition 1. □

Lemma 14 *Consider Problem (A.17). For any $\omega > 0$,*

$$v_j(\omega k) = \omega^{\vartheta} v_{\omega,j}(k), \tag{A.27}$$

where $v_{\omega,j}$ solves

$$\begin{aligned}\rho\omega^{\zeta} v_{\omega,j}(k) = H\left(v_{\omega,j}'(k)\right) + \omega^{\zeta} v_{\omega,j}'(k)\left[-\delta k + \varphi(rk + wz_j\ell_j/\omega)^{1-\zeta}\omega^{-\zeta}\right] \\ + \frac{1}{2}\omega^{\zeta} v_{\omega,j}''(k)\sigma^2(k)^2 + \lambda_j\omega^{\zeta}\left(v_{\omega,-j}(k) - v_{\omega,j}(k)\right).\end{aligned} \tag{A.28}$$

Proof: The subsequent proof process also follows the same way as the proof of case 3 in Proposition 1. ■

2.2.2 Proof of Theorem 2

Since the KF equation is the same as that in the benchmark model, the process of proving the theorem on the Pareto distribution is exactly the same, we don't repeat it here. ■

2.2.3 Social welfare effect decomposition with endogenous supply

Like the benchmark model, we perform a detailed decomposition on $\hat{\tilde{c}}_j(k)$. The additional consideration here is the effect of the progressivity changes on endogenous labor supply.

Using the definition of $\tilde{c}_j(k) = rk + wz_j\ell_j(k) - \left[rk + wz_j\ell_j(k) - \varphi(rk + wz_j\ell_j(k))^{1-\zeta}\right] - s_j(k) - \gamma(\ell_j(k))$ for $j = 1, 2$, we have

$$\begin{aligned}
\hat{\tilde{c}}_j(k) &= \tilde{c}_{aj}(k) - \tilde{c}_{bj}(k) \\
&= r_ak + w_az_j\ell_{aj}(k) - \left[r_ak + w_az_j\ell_{aj}(k) - \varphi_a(r_ak + w_az_j\ell_{aj}(k))^{1-\zeta_a}\right] - s_{aj}(k) - \gamma(\ell_{aj}(k)) \\
&- \left\{r_bk + w_bz_j\ell_{bj}(k) - \left[r_bk + w_bz_j\ell_{bj}(k) - \varphi_b(r_bk + w_bz_j\ell_{bj}(k))^{1-\zeta_b}\right] - s_{bj}(k) - \gamma(\ell_{bj}(k))\right\}.
\end{aligned}$$

Rewriting the above equation, we obtain

$$\begin{aligned}
\hat{\tilde{c}}_j(k) = r_ak + w_az_j\ell_{aj}(k) &- (r_bk + w_bz_j\ell_{bj}(k)) \\
&- \left[r_ak + wz_j\ell_{aj}(k) - \varphi_a(r_ak + w_az_j\ell_{aj}(k))^{1-\zeta_a}\right] \\
&+ \left[r_bk + wz_j\ell_{bj}(k) - \varphi_b(r_bk + w_bz_j\ell_{bj}(k))^{1-\zeta_b}\right] \\
&- (s_{aj}(k) - s_{bj}(k)) \\
&- \left[\gamma(\ell_{aj}(k)) - \gamma(\ell_{bj}(k))\right].
\end{aligned}$$

Introducing φ_m into the above equation, we have

$$\begin{aligned}
\hat{\tilde{c}}_j(k) &= \varphi_m(r_bk + w_bz_j\ell_{bj}(k))^{1-\zeta_a} - \varphi_b(r_bk + w_bz_j\ell_{bj}(k))^{1-\zeta_b} \\
&- \left[r_ak + wz_j\ell_{aj}(k) - \varphi_a(r_ak + w_az_j\ell_{bj}(k))^{1-\zeta_a}\right] \\
&+ \left[r_bk + wz_j\ell_{bj}(k) - \varphi_m(r_bk + w_bz_j\ell_{bj}(k))^{1-\zeta_b}\right] \\
&+ r_ak + w_az_j\ell_{aj}(k) - (r_bk + w_bz_j\ell_{bj}(k)) \\
&- (s_{aj}(k) - s_{bj}(k)) \\
&+ \varphi_a(r_ak + w_az_j\ell_{aj}(k))^{1-\zeta_a} - \gamma(\ell_{aj}(k)) - \varphi_a(r_ak + w_az_j\ell_{bj}(k))^{1-\zeta_a} + \gamma(\ell_{bj}(k)).
\end{aligned} \tag{A.29}$$

Taking Taylor expansion on the last line of the above equation, we obtain

$$\varphi_a(r_ak+w_az_j\ell_{aj}(k))^{1-\zeta_a}-\gamma(\ell_{aj}(k))+\left[\varphi_a(r_ak + w_az_j\ell_{aj}(k))^{1-\zeta_a} - \gamma(\ell_{aj}(k))\right]'(\ell_{bj}-\ell_{aj})+o(\ell_{bj}-\ell_{aj}).$$

Notice that

$$\begin{aligned}
&\left[\varphi_a(r_ak + w_az_j\ell_{aj}(k))^{1-\zeta_a} - \gamma(\ell_{aj}(k))\right]' \\
=&\varphi_a(1 - \zeta_a)(r_ak + w_az_j\ell_{aj})^{-\zeta_a}w_az_j - \gamma'(\ell_{aj}(k)) \\
=&0,
\end{aligned}$$

which comes from the the first-order condition. Hence, the last line in equation (A.29) is approximately zero.

Like the benchmark model, we interpret the first row in equation (A.29) as the mechanical effect, the second and third rows as the efficiency cost $\Delta T_\ell(y_j(k))$, the fourth as the pecuniary externality, and the fifth as the private insurance effect.

We adopt the utilitarian social welfare function:

$$\begin{aligned} W &= \int_0^\infty \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(\tilde{c}_t) dt f(k_0) dk_0 \\ &= \frac{1}{\rho} \int_0^\infty u(\tilde{c}(k)) f(k) dk. \end{aligned} \tag{A.30}$$

We use $\widehat{WP}_L$ to denote $\hat{W}/|W|$. Substituting the decomposition of $\hat{c}_j$ above into

$$\hat{W} = \frac{1}{\rho} \sum_{j\in\{1,2\}} \left[\int_0^\infty u'(\tilde{c}_j(k)) \hat{\tilde{c}}_j(k) f_j(k) dk + \int_0^\infty u(\tilde{c}_j(k)) \hat{f}_j(k) dk \right],$$

we have

$$\widehat{WP}_L \equiv \frac{\hat{W}}{|W|} = \frac{\sum_{j\in\{1,2\}} \left[\int_0^\infty u'(\tilde{c}_j(k)) \hat{\tilde{c}}_j(k) f_j(k) dk + \int_0^\infty u(\tilde{c}_j(k)) \hat{f}_j(k) dk \right]}{\left| \sum_{j\in\{1,2\}} \int_0^\infty u(\tilde{c}_j(k)) f_j(k) dk \right|}.$$

Therefore, we can decompose $\widehat{WP}$ as follows,

$$\begin{aligned} \widehat{WP} =& (\eta - 1) \sum_{j\in\{1,2\}} \int_0^\infty \frac{\varphi_m (r_b k + w_b z_j \ell_{bj}(k))^{1-\zeta_a} - \varphi_b (r_b k + w_b z_j \ell_{bj}(k))^{1-\zeta_b}}{\tilde{c}_{bj}(k)} \Gamma_{bj}(k) dk \\ &+ (1-\eta) \sum_{j\in\{1,2\}} \int_0^\infty \frac{\Delta T_\ell(y_j(k))}{\tilde{c}_{bj}(k)} \Gamma_{bj}(k) dk \\ &+ (\eta - 1) \sum_{j\in\{1,2\}} \int_0^\infty \frac{r_a k + w_a z_j \ell_{aj}(k) - (r_b k + w_b z_j \ell_{bj}(k))}{\tilde{c}_{bj}(k)} \Gamma_{bj}(k) dk \\ &+ (1-\eta) \sum_{j\in\{1,2\}} \int_0^\infty \frac{s_{aj}(k) - s_{bj}(k)}{\tilde{c}_{bj}(k)} \Gamma_{bj}(k) dk \\ &- \sum_{j\in\{1,2\}} \int_0^\infty \frac{f_{aj}(k) - f_{bj}(k)}{f_{bj}(k)} \Gamma_{bj}(k) dk, \end{aligned}$$

and $\Gamma_{bj}(k) = \frac{u(\tilde{c}_{bj}(k) f_{bj}(k)}{\sum_{j\in\{1,2\}} \int_0^\infty u(\tilde{c}_{bj}(k) f_{bj}(k) dk}$ is the weight function. Here we use $u'(\tilde{c}_{bj})\tilde{c}_{bj}/u(\tilde{c}_{bj}) = 1 - \eta$.

3 Inclusion of a safe asset

3.1 Economy

The valuation function $v(k)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho v_j(a) = \max_{c, 0 \leq k \leq a} u(c) + v_j'(a) s_j(k) + \frac{1}{2} v_j''(a) \sigma^2 k^2 + \lambda_j \left(v_{-j}(a) - v_j(a) \right), \; j = 1, 2, \tag{A.31}$$

where $s_j(k) = \varphi \left[w z_j + r_b (a - k) + \phi r k \right]^{1-\zeta} + (1 - \phi) r k - c$. We adopt the convention that $-j = 2$ when $j = 1$, and $-j = 1$ when $j = 2$.

The stationary distributions $g_j(k)$, $j = 1, 2$, which satisfy

$$0 = -\frac{d}{da} \left[s_j(a) g_j(a) \right] + \frac{1}{2} \frac{d^2}{da^2} \left[\sigma^2 k_j(a)^2 g_j(a) \right] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a), \; j = 1, 2. \tag{A.32}$$

We characterize the policy functions of households with a safe asset.

Proposition 7 *Impose Assumption 1. As $k \to \infty$, the consumption policy function $c_j(a)$, saving policy function $s_j(a)$, and portfolio policy function $k_j(a)$, $j = 1, 2$, have the following asymptotic properties.*

1. *If $\zeta > 0$,*

$$c_j(a) \sim \left[\frac{\rho}{\eta} - \frac{1 - \eta}{2\eta^2} \frac{\left[r(1 - \phi) \right]^2}{\sigma^2} \right] a,$$

$$s_j(a) \sim -\left[\frac{\rho}{\eta} - \frac{1 + \eta}{2\eta^2} \frac{\left[r(1 - \phi) \right]^2}{\sigma^2} \right] a,$$

$$k_j(a) \sim \frac{r(1 - \phi)}{\eta \sigma^2} a.$$

2. *$\zeta = 0$,*

$$c_j(a) \sim \left[\frac{\rho - (1 - \eta) \varphi r_b}{\eta} - \frac{1 - \eta}{2\eta^2} \frac{\left[r(1 - \phi) + \varphi(r\phi - r_b) \right]^2}{\sigma^2} \right] a,$$

$$s_j(a) \sim -\left[\frac{\rho - \varphi r_b}{\eta} - \frac{1 + \eta}{2\eta^2} \frac{\left[r(1 - \phi) + \varphi(r\phi - r_b) \right]^2}{\sigma^2} \right] a,$$

$$k_j(a) \sim \frac{r(1 - \phi) + \varphi(r\phi - r_b)}{\eta \sigma^2} a.$$

This result extends Proposition 9 in Achdou et al. (2022) to a general equilibrium setting with the inclusion of the labor market and the imposition of the CRP tax scheme. The key idea of this result is that for large enough wealth a, labor income and the borrowing constraint become irrelevant, and individual behavior will be like in a problem without labor income and without a borrowing constraint. With CRRA utility, this problem is the portfolio allocation problem of Merton (1969) which can be solved analytically with the policy functions.

The extended result corresponding to Theorem 1 is as follows.

Theorem 3 *Impose Assumption 1.*

1. *Let* $\frac{(1+\eta)[r(1-\phi)]^2-\eta^2\sigma^4}{2\eta\sigma^2} < \rho$. *If* $\zeta > 0$, *there exists a unique stationary wealth distribution which follows an asymptotic power law, i.e.* $1 - G(k) \sim \kappa_{ak} k^{-\Theta_{ak}}$ *as* $k \to \infty$, *with*

$$\Theta_{ak} = \eta \left[\frac{2\sigma^2\rho}{[r(1-\phi)]^2} - 1 \right],$$

2. *Let* $\frac{(1+\eta)[r(1-\phi)+(1-g)(r\phi-r_b)]^2-\eta^2\sigma^4}{2\eta\sigma^2} < \rho-(1-g)r_b$. *If* $\zeta = 0$, *there exists a unique stationary wealth distribution and* $1 - G(k) \sim \kappa_{bk} k^{-\Theta_{bk}}$ *as* $k \to \infty$, *with*

$$\Theta_{bk} = \eta \left[\frac{2\sigma^2(\rho - r_b\varphi)}{[r(1-\phi)+\varphi(r\phi-r_b)]^2} - 1 \right],$$

3. *If* $\zeta < 0$, *there does not exist a stationary wealth distribution.*

The household's after-tax income is

$$\varphi \left[r_b b + \phi r k\right]^{1-\zeta} + (1-\phi) r k.$$

When $\zeta = 0$,

$$\begin{aligned} &\varphi r_b b + \varphi\phi r k + (1-\phi) r k \\ =&\varphi r_b b + rk(\varphi\phi + 1 - \phi) \\ =&\varphi r_b a + k\left[(\varphi\phi + 1 - \phi) r - \varphi r_b\right]. \end{aligned}$$

We find that the part of the denominator in the Pareto index under flat tax is $r(1-\phi) + \varphi(r\phi - r_b)$, which exactly equals $(\varphi\phi + 1 - \phi)r - \varphi r_b$. Its economic meaning is the risk premium of risky assets.

This result extends Proposition 10 in Achdou et al. (2022) to a general equilibrium setting with the inclusion of the labor market and the imposition of the CRP tax scheme. Whether $\zeta > 0$ or $\zeta = 0$, the top wealth inequality is decreasing in volatility σ, risk aversion η, and time preference ρ.

3.2 Proof of theoretical results with a safe asset

3.2.1 Derivation of the capital market clearing condition

For the private equity sector, we have

$$w(t) = (1-\alpha)A_p \left(\frac{K(t)}{N(t)}\right)^{\alpha}, \qquad r(t) = \alpha A_p \left(\frac{K(t)}{N(t)}\right)^{\alpha-1}.$$

Thus, we obtain

$$w(t) = (1-\alpha)A_p^{\frac{1}{1-\alpha}} \left(\frac{r(t)}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

And for the public equity sector, we have

$$w(t) = (1-\alpha)A_g \left(\frac{X(t)}{L(t)}\right)^{\alpha}, \qquad r_b(t) = \alpha A_g \left(\frac{X(t)}{L(t)}\right)^{\alpha-1}.$$

Similarly,

$$w(t) = (1-\alpha)A_g^{\frac{1}{1-\alpha}} \left(\frac{r_b(t)}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

Owing they two have the same wage rate, we obtain

$$r(t) = A_p^{1/\alpha} A_g^{-1/\alpha} r_b(t).$$

3.2.2 Proof of Proposition 7

We have three cases: $\zeta > 0, \zeta = 0$ and $\zeta < 0$. The strategy of the proof is similar to that of proving Proposition 1.

Case 1 $\zeta > 0$.

Lemma 15 *Consider the problem*

$$\rho v(a) = \max_{c,k} u(c) + v'(a)\left[(1-\phi)rk - c\right] + \frac{1}{2}v''(a)\sigma^2 k^2 + \lambda_j(v_{-j}(a) - v_j(a)), \tag{A.33}$$

where $u(c) = c^{1-\eta}/(1-\eta), \eta > 0$. The optimal policy functions that solve (A.33) are linear in wealth and given by

$$c(a) = \left[\frac{\rho}{\eta} - \frac{1-\eta}{2\eta^2}\frac{\left[r(1-\phi)\right]^2}{\sigma^2}\right]a, \tag{A.34}$$

$$s(a) = \left[\frac{-\rho}{\eta} + \frac{1+\eta}{2\eta^2}\frac{\left[r(1-\phi)\right]^2}{\sigma^2}\right]a, \tag{A.35}$$

$$k(a) = \frac{r(1-\phi)}{\eta\sigma^2}a. \tag{A.36}$$

Proof: Grouping terms by the relevant maximization problems and solving these, we can write

$$\rho v(a) = H\left(v'(a)\right) + G\left(v'(a), v''(a)\right) + \lambda_j(v_{-j}(a) - v_j(a)), \tag{A.37}$$

$$H(p) = \max_c\{u(c) - pc\} = \frac{\eta}{1-\eta}p^{\frac{\eta-1}{\eta}},$$

$$G(p,q) = \max_k\left\{pr(1-\phi)k + \frac{1}{2}q\sigma^2k^2\right\} = \frac{-1}{2}\frac{p^2}{q}\frac{\left[r(1-\phi)\right]^2}{\sigma^2},$$

and from the first-order conditions

$$u'(c(a)) = v'(a), \quad k(a) = -\frac{v'(a)}{v''(a)}\frac{r(1-\phi)}{\sigma^2}. \tag{A.38}$$

Guess and verify $v(a) = Ba^{1-\eta}$ and hence $v'(a) = (1-\eta)Ba^{-\eta}$,

$$H\left(v'(a)\right) = \frac{\eta}{1-\eta}\left(v'(a)\right)^{\frac{\eta-1}{\eta}} = \frac{\eta}{1-\eta}((1-\eta)B)^{\frac{\eta-1}{\eta}}a^{1-\eta}.$$

Substituting them into equation (A.37) and dividing by $Ba^{1-\eta}$, we have

$$\rho = \eta((1-\eta)B_2)^{-\frac{1}{\eta}} + \frac{1}{2}\frac{\left[r(1-\phi)\right]^2}{\sigma^2}\frac{1-\eta}{\eta}. \tag{A.39}$$

From equation (A.38) $c(a) = ((1-\eta)B_2)^{-\frac{1}{\eta}}a$ and hence using equation (A.39), we obtain equations (A.34) - (A.36). □

Lemma 16 *Consider the problem (A.31) For any* $\xi > 0$*,*

$$v_j(\xi a) = \xi^{1-\eta}v_{\xi,j}(a), \tag{A.40}$$

where $v_{\xi,j}$ *solves*

$$\rho v_{\xi,j}(a) = \max_{c,k} u(c) + v'_{\xi,j}(a)\left(\varphi\left[wz_j/\xi + r(a-k_{\xi,j}) + \phi rk_{\xi,j}\right]^{1-\zeta}\xi^{-\zeta} + r(1-\phi)k_{\xi,j} - c\right)$$
$$+ \frac{1}{2}v''_{\xi,j}(a)\sigma^2k^2_{\xi,j} + \lambda_j\left(v_{\xi,-j}(a) - v_{\xi,j}(a)\right). \tag{A.41}$$

Proof: From equation (A.40), $v_j(a) = \xi^{1-\eta}v_{\xi,j}(a/\xi), v'_j(a) = \xi^{-\eta}v'_{\xi,j}(a/\xi)$, and $v''_j(a) = \xi^{-\eta-1}v''_{\xi,j}(a/\xi)$. Therefore $H\left(v'_j(a)\right) = H\left(v'_{\xi,j}(a/\xi)\right)\xi^{1-\eta}$. Substituting them into equation (A.31),

$$\begin{aligned}
&\rho\xi^{1-\eta}v_{\xi,j}(a/\xi)\\
=&H\left(v'_{\xi,j}(a/\xi)\right)\xi^{1-\eta} + \xi^{-\eta}v'_{\xi,j}(a/\xi)\varphi\left[wz_j + r_b(a-k) + \phi rk\right]^{1-\zeta} + \xi^{-\eta}v'_{\xi,j}(a/\xi)r(1-\phi)k\\
+&\frac{1}{2}\xi^{-\eta-1}v''_{\xi,j}(a/\xi)\sigma^2k^2 + \xi^{1-\eta}\lambda_j(v_{\xi,-j}(a/\xi), v_{\xi,j}(a/\xi)),
\end{aligned}$$

dividing by $\xi^{1-\eta}$,

$$\begin{aligned}&\rho v_{\xi,j}(a/\xi)\\ =&H\left(v'_{\xi,j}(a/\xi)\right)+v'_{\xi,j}(a/\xi)\varphi\left[wz_j+r_b(a-k)+\phi rk\right]^{1-\zeta}/\xi+v'_{\xi,j}(a/\xi)r(1-\phi)k/\xi\\ +&\frac{1}{2}v''_{\xi,j}(a/\xi)\sigma^2(k/\xi)^2+\lambda_j(v_{\xi,-j}(a/\xi),v_{\xi,j}(a/\xi)),\end{aligned}$$

and utilizing

$$v'_{\xi,j}(a/\xi)\left[wz_j+r_b(a-k)+\phi rk\right]^{1-\zeta}/\xi=v'_{\xi,j}(a/\xi)\left[wz_j/\xi+r_b(a-k)/\xi+\phi rk/\xi\right]^{1-\zeta}\xi^{-\zeta},$$

yields

$$\begin{aligned}\rho v_{\xi,j}(a/\xi)=\max_{c,k}u(c)+v'_{\xi,j}(a/\xi)&\left(\varphi\left[wz_j/\xi+r_b(a-k)/\xi+\phi rk/\xi\right]^{1-\zeta}\xi^{-\zeta}+r(1-\phi)k/\xi-c\right)\\ &+\frac{1}{2}v''_{\xi,j}(a/\xi)\sigma^2k^2+\lambda_j\left(v_{\xi,-j}(a/\xi)-v_{\xi,j}(a/\xi)\right).\end{aligned}\tag{A.42}$$

Hence, we have equation (A.41).

Consider the consumption policy function $c_j(a)$ and the portfolio policy function $k_j(a)$. From equation (A.40), $v_j(a)=\xi^{1-\eta}v_{\xi,j}(a/\xi), v'_j(a)=\xi^{-\eta}v'_{\xi,j}(a/\xi)$, $v''_j(a)=\xi^{-\eta-1}v''_{\xi,j}(a/\xi)$, and therefore, we have

$$c_j(a)=\left(v'_j(a)\right)^{-1/\eta}=\xi\left(v'_{\xi,j}(a/\xi)\right)^{-1/\eta}=\xi c_{\xi,j}(a/\xi).$$

F.O.C of k for equation (A.31) is

$$\frac{(1-\zeta)\varphi(wz_j+r_b(a-k)+\phi rk)^{-\zeta}(\phi r-r_b)+(1-\phi)r}{k_j}=-\frac{v''_j(a)\sigma^2}{v'_j(a)}.\tag{A.43}$$

Rewriting equation (A.43), we obtain

$$k_j=-\frac{v'_j(a)\left[(1-\zeta)\varphi(wz_j+r_b(a-k)+\phi rk)^{-\zeta}(\phi r-r_b)+(1-\phi)r\right]}{v''_j(a)\sigma^2}.\tag{A.44}$$

Since $v'_j(a)=\xi^{-\eta}v'_{\xi,j}(a/\xi)$, $v''_j(a)=\xi^{-\eta-1}v''_{\xi,j}(a/\xi)$, we have

$$k_j=-\frac{\xi v'_{\xi,j}(a/\xi)\left[(1-\zeta)\varphi(wz_j+r_b(a-k)+\phi rk)^{-\zeta}(\phi r-r_b)+(1-\phi)r\right]}{v''_{\xi,j}(a/\xi)\sigma^2}.\tag{A.45}$$

F.O.C of k for equation (A.42) is

$$\begin{aligned}&-v''_{\xi,j}(a/\xi)\sigma^2k_{\xi,j}\\ =&v'_{\xi,j}(a/\xi)\left\{\varphi\xi^{-\zeta}(1-\zeta)\left[wz_j/\xi+r_b(a-k)/\xi+\phi rk/\xi\right]^{-\zeta}(\phi r/\xi-r_b/\xi)+r(1-\phi)/\xi\right\}.\end{aligned}\tag{A.46}$$

Rearranging equation (A.46), we obtain

$$k_{\xi,j} = -\frac{v'_{\xi,j}(a/\xi)\left\{\varphi\xi^{-\zeta}(1-\zeta)\left[wz_j/\xi + r_b(a-k)/\xi + \phi rk/\xi\right]^{-\zeta}(\phi r/\xi - r_b/\xi) + r(1-\phi)/\xi\right\}}{v''_{\xi,j}(a/\xi)\sigma^2}. \tag{A.47}$$

Thus, we have

$$k_{\xi,j} = -\frac{v'_{\xi,j}(a/\xi)\left\{\varphi(1-\zeta)\left[wz_j + r_b(a-k) + \phi rk\right]^{-\zeta}(\phi r/\xi - r_b/\xi) + r(1-\phi)/\xi\right\}}{v''_{\xi,j}(a/\xi)\sigma^2}. \tag{A.48}$$

In particular with $\xi = a$, we have

$$c_j(a) = ac_{a,j}(1).$$

Equation (A.45) becomes

$$\frac{k_j}{a} = -\frac{v'_{a,j}(1)\left[\varphi(1-\zeta)(wz_j + r_b(a-k) + \phi rk)^{-\zeta}(\phi r - r_b) + (1-\phi)r\right]}{v''_{a,j}(1)\sigma^2}. \tag{A.49}$$

And equation (A.48) becomes

$$k_{\xi,j} = -\frac{v'_{\xi,j}(1)\left\{\varphi(1-\zeta)\left[wz_j + r_b(a-k) + \phi rk\right]^{-\zeta}(\phi r/a - r_b/a) + r(1-\phi)/a\right\}}{v''_{\xi,j}(1)\sigma^2}. \tag{A.50}$$

Hence, we obtain

$$\lim_{a\to\infty}\frac{c_j(a)}{a} = \lim_{\xi\to\infty} c_{\xi,j}(1) = c(1) = \frac{\rho}{\eta} - \frac{1-\eta}{2\eta^2}\frac{\left[r(1-\phi)\right]^2}{\sigma^2}, \tag{A.51}$$

$$\lim_{a\to\infty}\frac{k_j(a)}{a} = \lim_{\xi\to\infty} k_{\xi,j}(1) = k(1) = \frac{r(1-\phi)}{\eta\sigma^2}, \tag{A.52}$$

where the second equality of equations (A.51) and (A.52) uses that problem (A.41) converges to that with no labor income as $\xi \to \infty$ and therefore also $c_{\xi,j}(a) \to c(a)$ and $k_{\xi,j}(a) \to k(a)$ for all a as $\xi \to \infty$. □

Case 2 $\zeta = 0$.

Lemma 17 *Consider the problem*

$$\rho v(a) = \max_{c,k} u(c) + v'(a)\left\{\varphi r_b a + \left[r(1-\phi) + \varphi(r\phi - r_b)\right]k - c\right\} + \frac{1}{2}v''(a)\sigma^2 k^2, \tag{A.53}$$

where $u(.)$ satisfies Assumption 1. The optimal policy functions that solve (A.53) are linear in wealth and given by

$$
\begin{aligned}
c(a) &= \left[\frac{\rho-(1-\eta)\varphi r_b}{\eta} - \frac{1-\eta}{2\eta^2}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{\sigma^2}\right]a, \\
s(a) &= \left[\frac{\varphi r_b-\rho}{\eta} + \frac{1+\eta}{2\eta^2}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{\sigma^2}\right]a, \\
k(a) &= \frac{r(1-\phi)+\varphi(r\phi-r_b)}{\eta\sigma^2}a. \qquad \text{(A.54)}
\end{aligned}
$$

Proof: Write (A.53) as

$$
\begin{aligned}
\rho v(a) &= H\left(v'(a)\right) + G\left(v'(a), v''(a)\right) + v'(a)\varphi r_b a, \qquad \text{(A.55)}\\
H(p) &= \max_c\{u(c)-pc\} = \frac{\eta}{1-\eta}p^{\frac{\eta-1}{\eta}}, \\
G(p,q) &= \max_k\left\{p\left[r(1-\phi)+\varphi(r\phi-r_b)\right]k + \frac{1}{2}q\sigma^2k^2\right\} = \frac{-1}{2}\frac{p^2}{q}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{\sigma^2}.
\end{aligned}
$$

The first-order conditions give

$$
u'(c(a)) = v'(a), \quad k(a) = -\frac{v'(a)}{v''(a)}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]}{\sigma^2}. \qquad \text{(A.56)}
$$

Guess and verify $v(a) = B_2a^{1-\eta}$ and hence $v'(a) = (1-\eta)B_2a^{-\eta}$, $v''(a) = -\eta(1-\eta)B_2a^{-\eta-1}$,

$$
\begin{aligned}
H(v'(a)) &= \frac{\eta}{1-\eta}(v'(a))^{\frac{\eta-1}{\eta}} = \frac{\eta}{1-\eta}((1-\eta)B_2)^{\frac{\eta-1}{\eta}}a^{1-\eta}, \\
-\frac{(v'(a))^2}{v''(a)} &= \frac{(1-\eta)B_2}{\eta}a^{1-\eta}, \\
G(v'(a),v''(a)) &= -\frac{(v'(a))^2}{2v''(a)}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{\sigma^2} = \frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{2\sigma^2}\frac{(1-\eta)B_2}{\eta}a^{1-\eta}.
\end{aligned}
$$

Substituting the above into equation (A.55) and dividing by $B_2a^{1-\eta}$, we have

$$
\rho = \eta((1-\eta)B_2)^{-\frac{1}{\eta}} + \frac{1}{2}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{\sigma^2}\frac{1-\eta}{\eta} + (1-\eta)\varphi r_b. \qquad \text{(A.57)}
$$

From equation (A.56), $c(a) = ((1-\eta)B_2)^{-\frac{1}{\eta}}a$ and hence using (A.57) gives (A.54). □

Lemma 18 *Consider the problem (A.31). For any $\psi > 0$,*

$$
v(\varrho a) = \varrho^{1-\eta}v_{\varrho,j}(a), \qquad \text{(A.58)}
$$

where $v_{\varrho,j}$ solves

$$\rho v_{\varrho,j}(a) = \max_{c,k} u(c) + v'_{\varrho,j}(a)\left\{(1-g)r_b a + \left[r(1-\phi)+\varphi(r\phi - r_b)\right]k + \varphi w z_j/\varrho - c\right\} \\ + \frac{1}{2}v''_{\varrho,j}(a)\sigma^2 k^2 + \lambda_j\left(v_{\varrho,-j}(a) - v_{\varrho,j}(a)\right). \tag{A.59}$$

Proof: Write (A.31) as

$$\rho v(a) = H\left(v'(a)\right) + G\left(v'(a), v''(a)\right) + v'(a)\varphi(wz_j + r_b a), \tag{A.60}$$
$$H(p) = \max_c\{u(c) - pc\} = \frac{\eta}{1-\eta}p^{\frac{\eta-1}{\eta}},$$
$$G(p,q) = \max_k\left\{p\left[r(1-\phi)+\varphi(r\phi-r_b)\right]k + \frac{1}{2}q\sigma^2k^2\right\} = \frac{-1}{2}\frac{p^2}{q}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{\sigma^2},$$

From equation (A.58), $v_j(a) = \varrho^{1-\eta}v_{\varrho,j}(a/\varrho), v'_j(a) = \varrho^{-\eta}v'_{\varrho,j}(a/\varrho)$, and $v''_j(a) = \varrho^{-\eta-1}v''_{\varrho,j}(a/\varrho)$. Therefore $H\left(v'_j(a)\right) = H\left(v'_{\varrho,j}(a/\varrho)\right)\varrho^{1-\eta}$, and $G(v'(a), v''(a)) = \varrho^{1-\eta}G(v'_{\varrho,j}(a/\varrho), v''_{\varrho,j}(a/\varrho))$. Substituting them into equation (A.60), dividing by $\varrho^{1-\eta}$, yields equation (A.59).

With Lemmas 17 and 18 in hand, consider first the asymptotic behavior of the consumption policy function $c_j(a)$ and $k_j(a)$. From equation (A.58), $v_j(a) = \varrho^{1-\eta}v_{\varrho,j}(a/\varrho), v'_j(a) = \varrho^{-\eta}v'_{\varrho,j}(a/\varrho)$, $v''_j(a) = \varrho^{-\eta-1}v''_{\varrho,j}(a/\varrho)$, and therefore, we have

$$c_j(a) = \left(v'_j(a)\right)^{-1/\eta} = \varrho\left(v'_{\varrho,j}(a/\varrho)\right)^{-1/\eta} = \varrho c_{\varrho,j}(a/\varrho),$$
$$k_j(a) = -\frac{v'(a)}{v''(a)}\frac{(1-g)(r-r_b)}{\sigma^2} = -\frac{\varrho^{-\eta}v'_{\varrho,j}(a/\varrho)}{\varrho^{-\eta-1}v''_{\varrho,j}(a/\varrho)}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]}{\sigma^2} = \varrho k_{\varrho,j}(a/\varrho).$$

In particular, let $\varrho = a$. We then have

$$c_j(a) = ac_{\varrho,j}(1),$$
$$k_j(a) = ak_{\varrho,j}(1).$$

Hence, we obtain

$$\lim_{a\to\infty}\frac{c_j(a)}{a} = \lim_{\varrho\to\infty}c_{\varrho,j}(1) = c_j(1) = \frac{\rho-(1-\eta)(1-g)r}{\eta} - \frac{1-\eta}{2\eta^2}\frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{\sigma^2},$$
$$\lim_{a\to\infty}\frac{k_j(a)}{a} = \lim_{\varrho\to\infty}k_{\varrho,j}(1) = k_j(1) = \frac{r(1-\phi)+\varphi(r\phi-r_b)}{\eta\sigma^2}, \text{ for } j = 1,2,$$

where the second equality uses that problem (A.59) converges to problem (A.53) as $\varrho\to\infty$ and therefore $c_{\varrho,j}(a)\to c_j(a)$ and $k_{\varrho,j}(a)\to k_j(a)$ for all a as $\varrho\to\infty$. The asymptotic behavior of $s_j(a)$ can be proved in an analogous fashion. ■

3.2.3 Proof of Theorem 3

As in the proof of Theorem 1 , Now we define $m_j(a) = \sigma^2 k_j(a)^2 g_j(a)/2$, and

$$m_1'(a) + m_2'(a) = s_1(a)g_1(a) + s_2(a)g_2(a) = \frac{2s_1(a)}{\sigma^2 k_1(a)^2} m_1(a) + \frac{2s_2(a)}{\sigma^2 k_2(a)^2} m_2(a). \tag{A.61}$$

Define $m(a) = m_1(a) + m_2(a)$. After collecting the leading term, equation (A.61) is written as

$$m'(a) = \frac{\theta_p}{a} m(a) + h_1(a)m_1(a) + h_2(a)m_2(a), \tag{A.62}$$

$$\theta_p = \frac{2\bar{s}_p}{\sigma^2 \bar{k}_p^2}, \quad h_j(a) = \frac{2}{\sigma^2}\left(\frac{\tilde{s}_j + \bar{s}_p a}{\left(\tilde{k}_j + \bar{k}_p a\right)^2} - \frac{\bar{s}_p}{\bar{k}_p^2 a}\right),$$

for $j = 1, 2$ and $p = a, b$, where

$$\begin{aligned}
\bar{s}_a &= \frac{-\rho}{\eta} + \frac{1+\eta}{2\eta^2}\frac{[r(1-\phi)]^2}{\sigma^2}, \\
\bar{s}_b &= \frac{\varphi r_b - \rho}{\eta} + \frac{1+\eta}{2\eta^2}\frac{[r(1-\phi) + \varphi(r\phi - r_b)]^2}{\sigma^2}, \\
\bar{k}_a &= \frac{r(1-\phi)}{\eta\sigma^2}, \\
\bar{k}_b &= \frac{r(1-\phi) + \varphi(r\phi - r_b)}{\eta\sigma^2}.
\end{aligned}$$

The subscript a represents the case of $\zeta > 0$, and the subscript b denotes the case of $\zeta = 0$. Dividing equation (A.62) by $m(a)$ and integrating both sides from a_1 to a_2 where $a_1 < a_2$ are large enough, we have

$$\ln\left(\frac{m(a_2)}{a_2^{\theta_p}}\right) - \ln\left(\frac{m(a_1)}{a_1^{\theta_p}}\right) = \int_{a_1}^{a_2} \frac{h_1(x)m_1(x)}{m(x)}dx + \int_{a_1}^{a_2} \frac{h_2(x)m_2(x)}{m(x)}dx. \tag{A.63}$$

There exists a positive constant $\bar{C}_k$ such that $|h_j(a)| \le \bar{C}_k/a^2, j = 1, 2$ and $m_j > 0$. Therefore, we have

$$\left|\ln\left(\frac{m(a_2)}{a_2^{\theta_p}}\right) - \ln\left(\frac{m(a_1)}{a_1^{\theta_p}}\right)\right| \le \int_{a_1}^{a_2} \frac{\bar{C}_k}{x^2}\left(\frac{m_1(x)}{m(x)} + \frac{m_2(x)}{m(x)}\right)dx \le \bar{C}_k\left(\frac{1}{a_1} - \frac{1}{a_2}\right).$$

Hence there exists $\bar{\xi}_k$ such that

$$\lim_{a\to\infty} \ln\left(\frac{m(a)}{a^{\theta_p}}\right) = \bar{\xi}_k.$$

Recall the definition of $m(a) = \sigma^2 f(a) \left(k_1(a)^2 + k_2(a)^2\right)/2$.
Consider $\zeta > 0$ and $\zeta = 0$, respectively.
Case 1 $\zeta > 0$.

$$g_j(a) \sim \xi a^{-\Theta_{ak}-1}, \quad \Theta_{ak} = 1 - \theta_a = 1 - \frac{2\bar{s}_a}{\sigma^2 \bar{k}_a^2} = \eta \left[\frac{2\sigma^2 \rho}{\left[r(1-\phi)\right]^2} - 1 \right].$$

Thus, we have

$$\lim_{a\to\infty} \frac{1-F(a)}{a^{-\Theta_{ak}}} = \lim_{a\to\infty} \frac{\int_a^\infty f(z)dz}{a^{-\Theta_{ak}}} = \lim_{a\to\infty} \frac{f(a)}{\Theta_{ak} a^{-\Theta_{ak}-1}} = \kappa_{ak}.$$

Case 2 $\zeta = 0$.

$$g_j(a) \sim \xi a^{-\Theta_{bk}-1}, \quad \Theta_{bk} = 1 - \theta_b = 1 - \frac{2\bar{s}_b}{\sigma^2 \bar{k}_b^2} = \eta \left[\frac{2\sigma^2 (\rho - r_b \varphi)}{\left[r(1-\phi) + \varphi(r\phi - r_b)\right]^2} - 1 \right].$$

Thus, we have

$$\lim_{a\to\infty} \frac{1-F(a)}{a^{-\Theta_{bk}}} = \lim_{a\to\infty} \frac{\int_a^\infty f(z)dz}{a^{-\Theta_{bk}}} = \lim_{a\to\infty} \frac{f(a)}{\Theta_{bk} a^{-\Theta_{bk}-1}} = \kappa_{bk}.$$

Numerical results

4 Benchmark model

In this section, we show the taxation for those whose wealth rank is below 95%. We report the algorithm of tax incidence and the algorithm about welfare effect decomposition. We show the numerical results about the derivative of wealth distribution (high-type households). We further elaborate on how does the progressivity of income taxation affect consumption. We report the Gini coefficient of wealth at different progressivities. We also perform comparative static analysis for different idiosyncratic investment risks and relative risk aversion coefficients.

4.1 Taxation for low- and high-type households

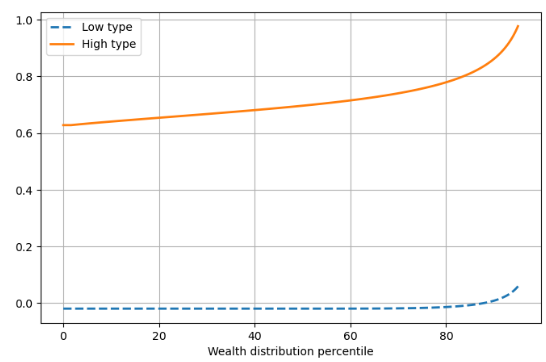


Figure A.1: Taxation for low- and high-type households

Figure A.1 shows that for low-type groups, more than 80% of people receive tax subsidies. All the high-type households are taxed.

4.2 Tax incidence

4.2.1 Algorithm of tax incidence

Calculate $\hat{c}_j$, $\hat{s}_j$, $\hat{v}_j$, and $\hat{f}_j$ for $j = 1, 2$. When $\zeta = \zeta_x$, we can obtain $c_j(\zeta_x), s_j(\zeta_x)$, $v_j(\zeta_x)$, and $f_j(\zeta_x)$ when reaching stationary wealth distribution under the current progressivity of taxation by solving the HJB and KF equations. Next we will give a 0.01 taxation perturbation under the current tax system, which is definited by ζ_x^p. Under the new taxation ζ_x^p, we obtain $c_j(\zeta_x^p), s_j(\zeta_x^p)$, $v_j(\zeta_x^p)$, and $f_j(\zeta_x^p)$ similarly. Then, we can calculate $\hat{c}_j$, $\hat{s}_j$, $\hat{v}_j$, and $\hat{f}_j$ from $c_j(\zeta_x^p) - c_j(\zeta_x)$, $s_j(\zeta_x^p) - s_j(\zeta_x)$, $v_j(\zeta_x^p) - v_j(\zeta_x)$,and $f_j(\zeta_x^p) - f_j(\zeta_x)$ respectively. Therefore, we can draw Figures 3 and 4 in the text.

4.2.2 The derivative of wealth distribution with high-type households

The figure below shows the derivative of the wealth distribution for high-type households when the progressivity of income taxation is equal to 0.181.

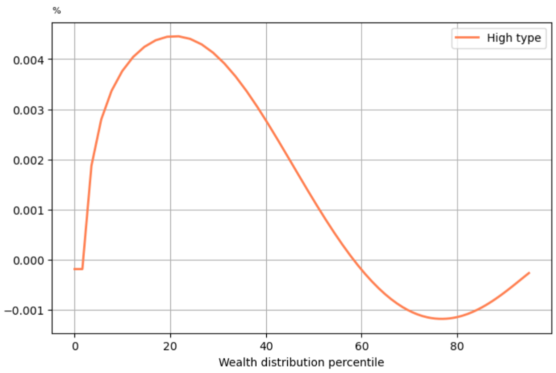


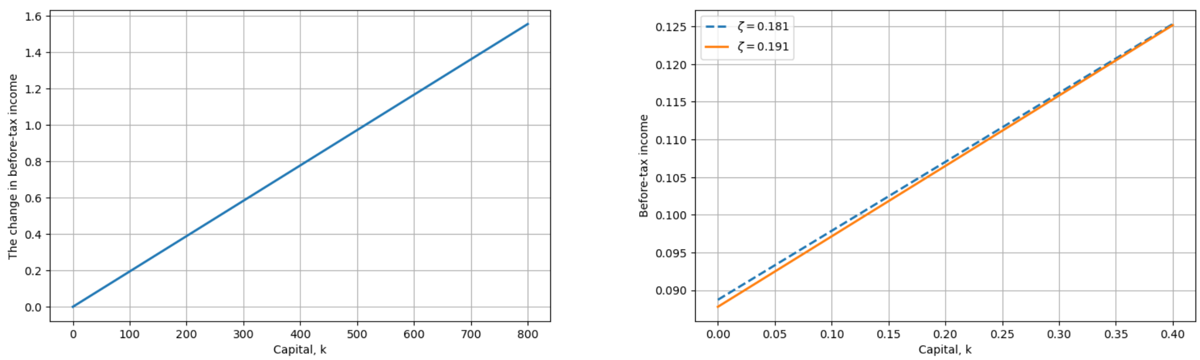
Figure A.2: The derivative of wealth distribution with high-type households, $\hat{f}_2$

We can find that the trend in the change of wealth distribution for high-type households due to the change in the progressivity is the same as the trend of low-type households. When the progressivity of income taxation increases, the population share of the low wealth with low- and high-type households decreases and the middle-wealth population share increases, indicating that for the all households, the increase in the progressivity improves the equality of wealth distribution.

4.2.3 How does the progressivity of income taxation affect consumption?

The impact of the progressivity of income taxation ζ on consumption c_j is analyzed in terms of this transmission mechanism as follows. Firstly, the progressivity affects the price vectors in equilibrium, the interest rate r and the wage rate w, thus affecting the before-tax income. With a given before-tax income, the progressivity affects φ. Under the combined influence of φ and ζ , the after-tax income changes and eventually affects consumption. Let's first analyze the effect of progressivity of income taxation ζ on the before-tax income of low-type households.

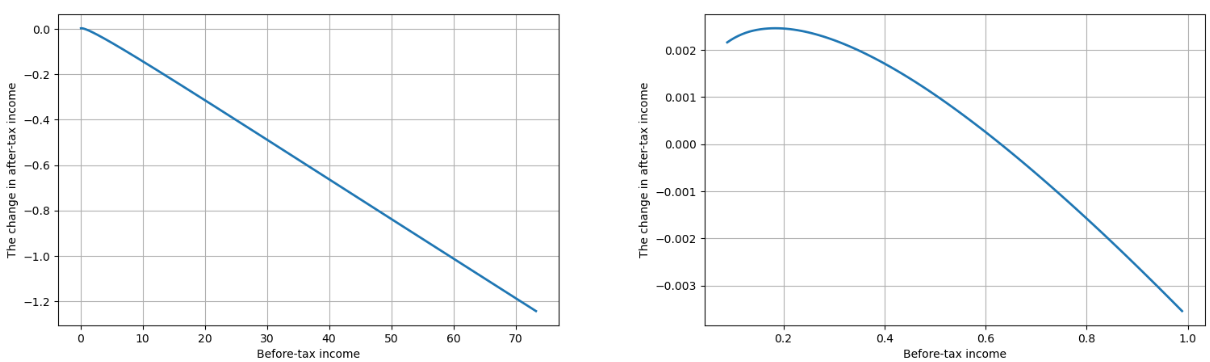
Panel (a)in Figure (A.3) draws the difference between before-tax income after the tax reform minus the before-tax income before the tax reform for all low-type households. We find that the change in the progressivity affects the price vectors the interest rate r and the wage rate w, then affects before-tax income through $y_j = rk + wz_j$. This is also the pecuniary externalities that we discuss in detail in the decomposition of the social welfare effect. And from the simulation results show that the rise in ζ makes r rise and w decrease, because the rise in ζ inhibits efficiency, the steady capital K decreases, thus, r rises, and at the same time w decreases. For the low wealth group, since their main income is derived from labor income, the rising capital income is not enough to make up for the falling wage income, and eventually the before-tax income of people with wealth less than 0.4 falls due to the rise in the progressivity.[2] This message can be seen in panel (b) in Figure A.3.



(a) All low-type households (b) Low wealth group

Figure A.3: Before-tax income with different progressivities for low-type households

Figure A.4 shows that changes in the progressivity lead to increases or decreases in after-tax income depending on the value of before-tax income.[3] For this segment of the population with before-tax income less than 0.29, the tax is actually a subsidy.[4] The progressivity increase instead makes their after-tax income increase, which can be seen in the right panel in Figure A.4.[5]



(a) All low-type households (b) Low wealth group

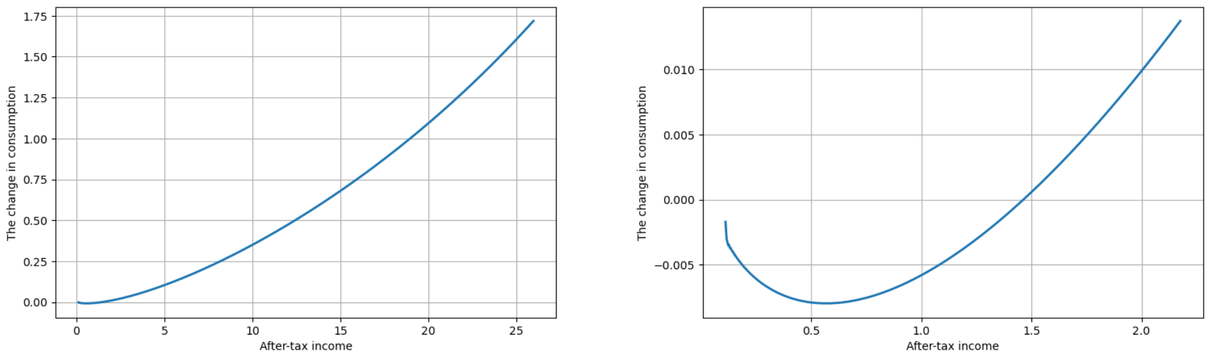
Figure A.4: The change in after-tax income for low-type households

[2]The wealth rank of the population at this point is 69.33%.

[3]Since $\varphi y_j^{1-\zeta}$, it is worth noting that after-tax income is affected by two parameters, φ, and a progressivity, ζ, where φ varies with ζ, but the variation is not monotonic and ultimately this after-tax income is affected by the combined effect of these two parameters.

[4]That is to say, in the low-type households, the population whose wealth ranking is lower than 88.42% receives tax subsidies.

[5]The right panel is a zoomed-in display of the low-wealth group on the left panel.



(a) All low-type households

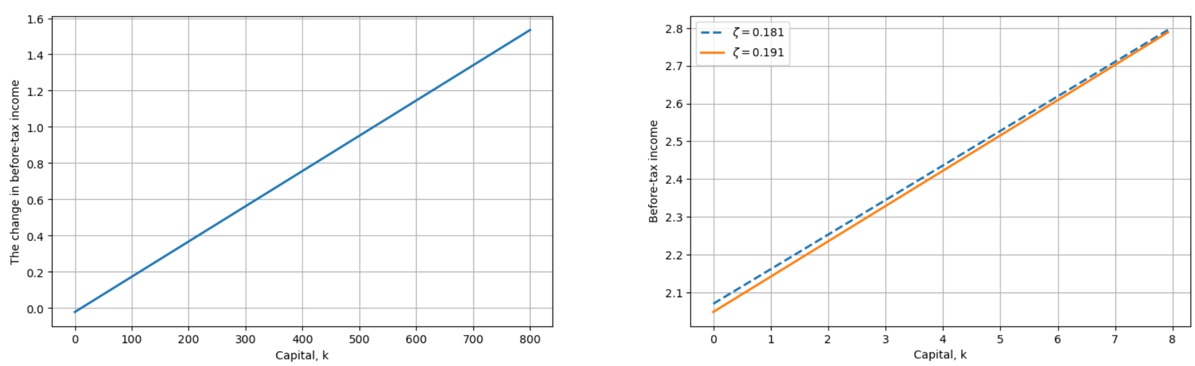
(b) Low wealth group

Figure A.5: The change in consumption for low-type households

As well as from the right panel in Figure A.5 we find that at a given after-tax income, for those with wealth ranking less than 99.73% of low-type households, higher progressivity bring lower consumption, and higher progressivity bring higher consumption for the rest.

Hence combining Figure A.3 to Figure A.5, we can find that the increase in consumption of the population at the mass point in the low-type households depends mainly on the redistribution. The decline in consumption among the middle-wealth group, on the other hand, depends mainly on the insurance effect. An increase in after-tax income makes the middle wealth population consume less and a decrease in after-tax income makes the high wealth population consume more.

Next we analyze the high-type households in the same way.



(a) All high-type households

(b) Low wealth group

Figure A.6: Before-tax income with different progressivities for high type

Similar to the low-type households, the increase in the progressivity reduces the income of the lower and middle wealth groups, which are dominated by labor income and account for 96.89% of the high-type households.

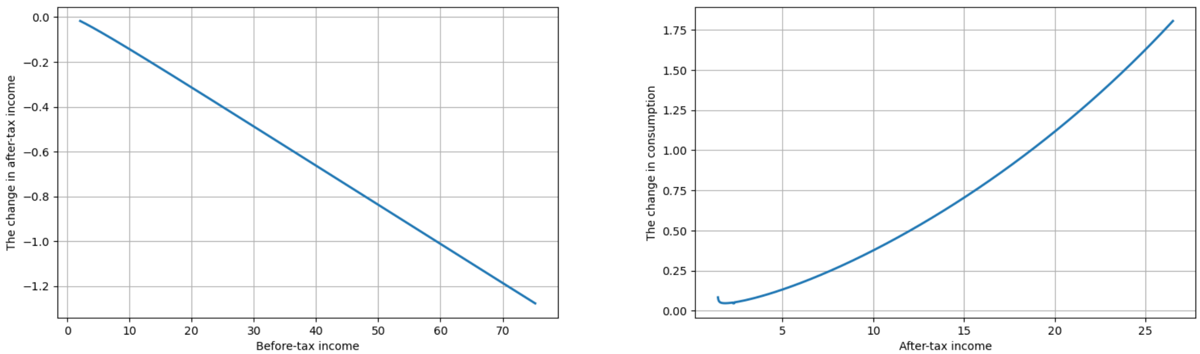
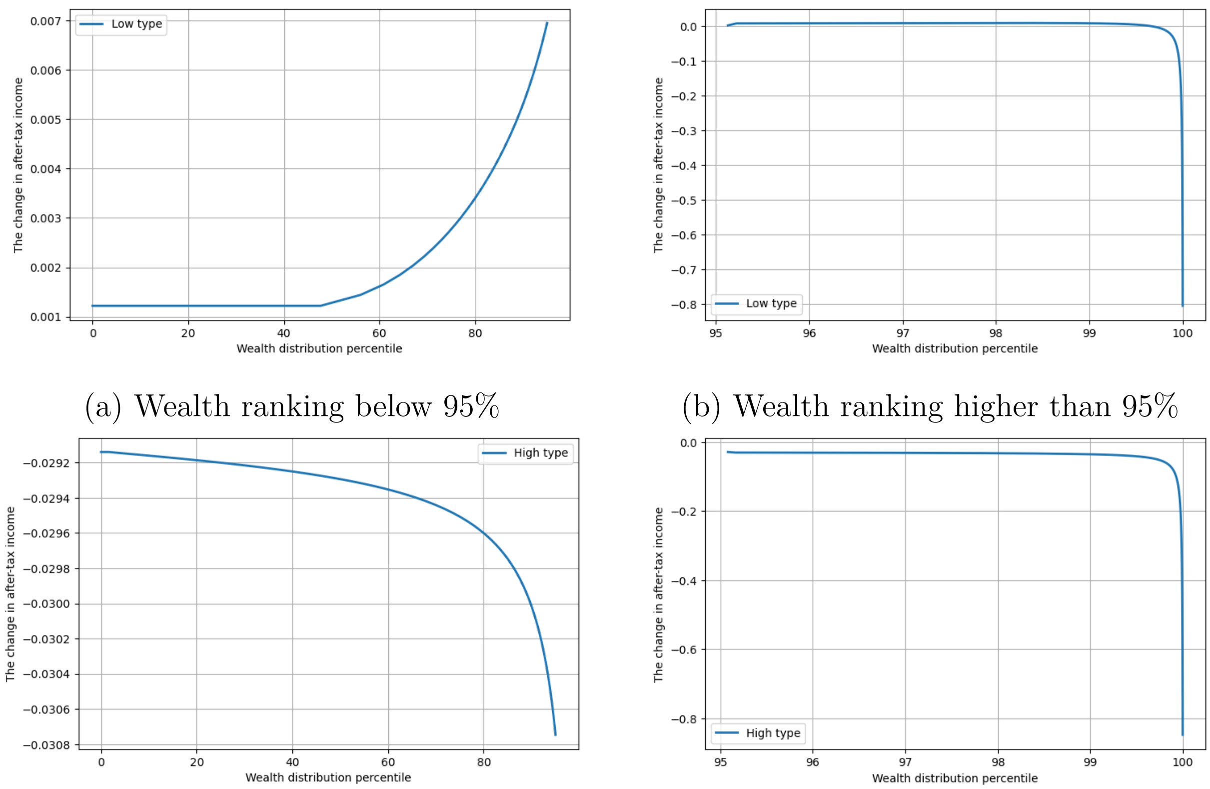


Figure A.7: Changes due to the progressivity for high type

The left panel of Figure A.7 shows that when the progressivity increase, after-tax income decreases for everyone in the high-type households, which is distinct from the increase in after-tax income for the population at the mass point of the wealth spectrum in the low-type households. The right panel shows that consumption increases for a given after-tax income for all. The main reason that makes consumption decrease in the high-type households is the decrease in after-tax income due to the increase in the progressivity.



(a) Wealth ranking below 95%

(b) Wealth ranking higher than 95%

(c) Wealth ranking below 95%

(d) Wealth ranking higher than 95%

Figure A.8: The change in after-tax income with different types

4.3 Welfare effect decomposition

4.3.1 Algorithm of welfare effect decomposition

We give a 0.01 taxation perturbation under the current tax system $\zeta = \zeta_x$, which is definited by ζ_x^p. In the text we decompose $\widehat{WP}(\zeta_x)$ into five parts, $\widehat{WP}_I(\zeta_x)$, $\widehat{WP}_{II}(\zeta_x)$, $\widehat{WP}_{III}(\zeta_x)$, $\widehat{WP}_{IV}(\zeta_x)$ and $\widehat{WP}_V(\zeta_x)$. They are helpful in finding the optimal progressivity of income taxation.

1. We obtain φ_m by $g\int_0^\infty yf(y;\zeta_x)dy = \int_0^\infty \left(y - \varphi_m y^{1-\zeta_x^p}\right) f(y;\zeta_x)dy$.

2. Based on step 1, we can calculate $\widehat{WP}_I(\zeta), \widehat{WP}_{II}(\zeta), \widehat{WP}_{III}(\zeta), \widehat{WP}_{IV}(\zeta)$ and $\widehat{WP}_V(\zeta)$ by equation (16). We perform channel decomposition and analyze the impact of each effect on the social welfare,

$$\widehat{WP} = \widehat{WP}_I + \widehat{WP}_{II} + \widehat{WP}_{III} + \widehat{WP}_{IV} - \widehat{WP}_V.$$

4.3.2 Welfare effect decomposition at the optimal progressivity

Figure A.9 plots the contribution of different $k's$ to $\widehat{WP}_I(\zeta)$, $\widehat{WP}_{II}(\zeta)$, $\widehat{WP}_{III}(\zeta)$, $\widehat{WP}_{IV}(\zeta)$, and $\widehat{WP}_V(\zeta)$ at the optimal tax progressivity $\zeta^* = 0.38$.

With an increase in progressivity from 0.38 to 0.39, panel (a) of Figure A.9 shows that the mechanical effect channel leads to a improvement in welfare for low-type households with wealth ranks below 93.21%, owing to the increased redistribution resulting from the tax reform. However, this mechanical effect weakens for high-type households. The low-wealth group of low-type households benefit the most from the tax reform, while for high-type households, the low-wealth group experiences the largest reduction in social welfare due to this channel.

Turning to panel (b) in Figure A.9, we observe a decrease in social welfare for low-type households as a result of the efficiency cost channel. In this case, the tax collected from these households increases, leading to a decrease in their social welfare. On the other hand, the efficiency cost channel has a positive effect on the social welfare of high-type households with wealth rankings below 87.67%, as the tax collected from this group is reduced.

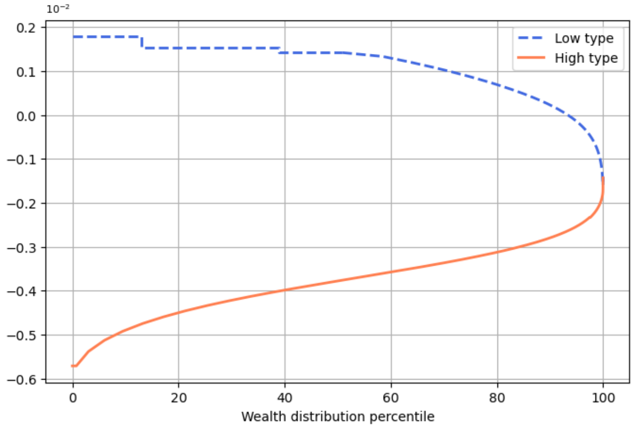
In Panel (c) of Figure A.9, the pecuniary externality channel shows a decline in social welfare for low-type households below the 58.49% wealth ranking and for high-type households below the 97.54% ranking. This channel is associated with a higher interest rate and lower wage rate resulting from an increase in progressivity. Since labor income is the main source of income for these two groups, their pre-tax income decreases. As a result, the welfare of these households decreases due to the pecuniary externality effect.

Panel (d) in Figure A.9 indicates that the all high-type households have the increase in this channel. Among the low-type households, the savings of households whose wealth ranks between 38.91% to 58.49% remain unchanged and stay at 0, meaning there is no private insurance effect for this group, and welfare remains unchanged through this channel. People whose wealth rank is lower than 38.91% and those whose wealth rank is between 58.49% and 85.95% have increased savings and decreased consumption, leading to reduced welfare through this channel.[6]

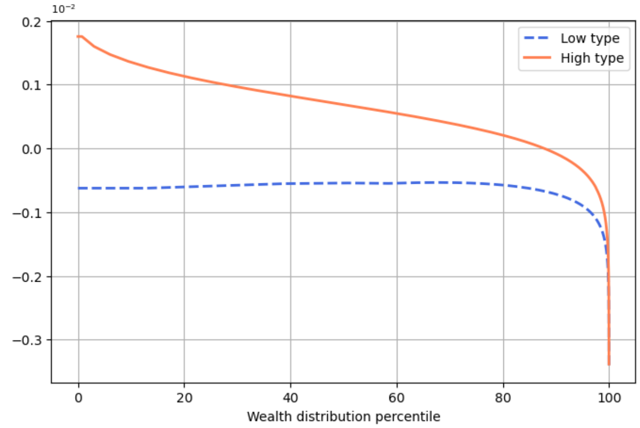
In panel (e) of Figure A.9 shows the percentage change of the population size due to the tax reform. The distribution effect in panel (e) is larger than the other components in panels (a)-(d). Progressivity of changes cause larger changes of population for middle-wealth groups than for low-wealth groups, in terms of percentage.

Panel (f) of Figure A.9 shows the curve of the weight function $\Gamma_{bj}(k)$. The welfare weights of high-type households are relatively small compared to those of low-type households, and the population of the middle-wealth group has a large proportion, making it a dominant group. This means that the tax progressivity $\zeta^* = 0.38$ results in an olive-shaped society, where a large middle class coexists with small percentages of the wealthy and the poor.

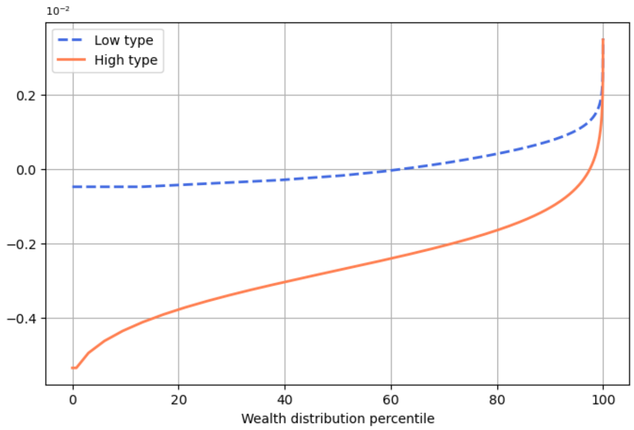
[6]The line of the high-type households whose color is coral has a kink near the wealth ranking of 100, which is caused by the rapid movement of k at this point.



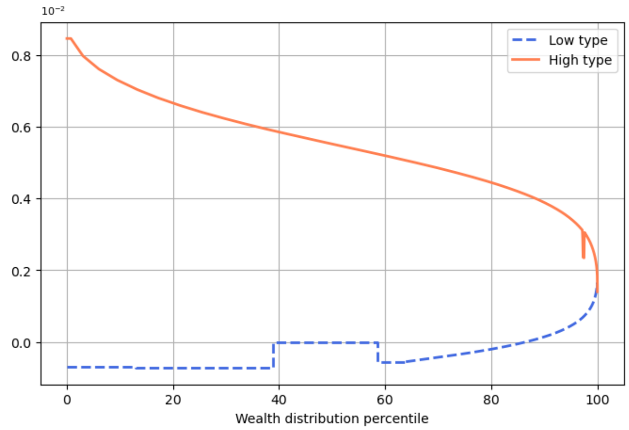
(a) Mechanical effect



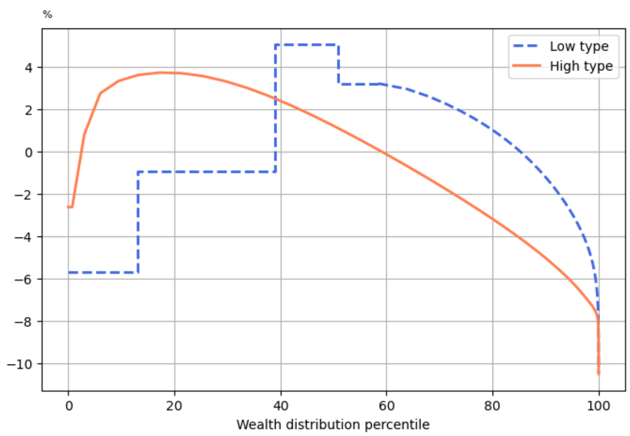
(b) Efficiency cost



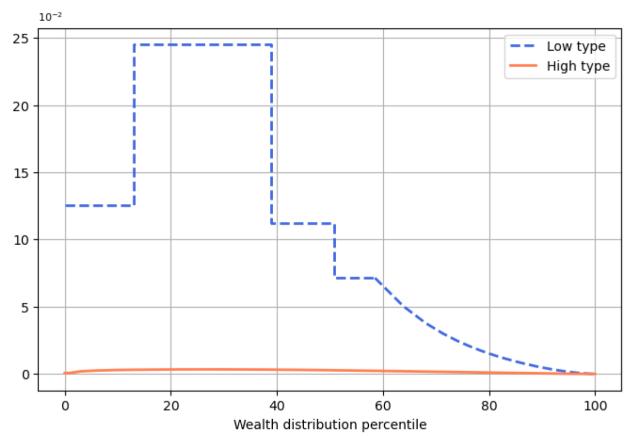
(c) Pecuniary externalities



(d) Private insurance



(e) Distribution effect



(f) Weight function

Figure A.9: Welfare effect decomposition at $\zeta^* = 0.38$

4.3.3 The derivative of the value function

Differentiating the HJB equation (A.1) with respect to ζ gives

$$(\rho+\lambda_j)\,\hat{v}_j(k)-\lambda_j\hat{v}_{-j}=u'\left(c_j(k)\right)\hat{c}_j(k)+\left(\hat{v}_j(k)\right)'s_j(k)+v_j'(k)\hat{s}_j(k)+\frac{1}{2}\left(\hat{v}_j(k)\right)''\sigma^2k^2,$$

for $j=1,2$. Using $v_j'(k)=u_j'\left(c_j(k)\right), v_j''(k)=u_j''\left(c_j(k)\right)c_j'(k)$, under Assumption 1 we have

$$\begin{aligned}(\rho+\lambda_j)\,\hat{v}_j(k)-\lambda_j\hat{v}_{-j}(k)&=c_j(k)^{-\eta-1}\left[c_j(k)\hat{c}_j(k)-\eta\hat{c}_j(k)s_j(k)+c_j(k)\hat{s}_j(k)\right]\\&\quad+\frac{1}{2}\eta\sigma^2k^2c_j'(k)c_j(k)^{-\eta-1}\left[(\eta+1)\frac{\hat{c}_j(k)}{c_j(k)}-\frac{\hat{c}_j'(k)}{c_j'(k)}\right],j=1,2.\end{aligned} \tag{A.64}$$

This then leads to

$$\hat{v}_j(k)=\frac{\Psi_j(k)}{\rho}+\frac{\lambda_j}{\rho+\lambda_j+\lambda_{-j}}\left(\frac{\Psi_{-j}(k)-\Psi_j(k)}{\rho}\right),j=1,2, \tag{A.65}$$

where $\Psi_j(k)$ denotes the right-hand side of (A.64).

Consider a variable m. We let $\varepsilon(m)\equiv\frac{\hat{m}}{m}$ in our analysis. Note that $\varepsilon(m)$ represents the semi-elasticity of variable m with respect to tax progressivity ζ. Rewrite $\Psi_j(k)$ as

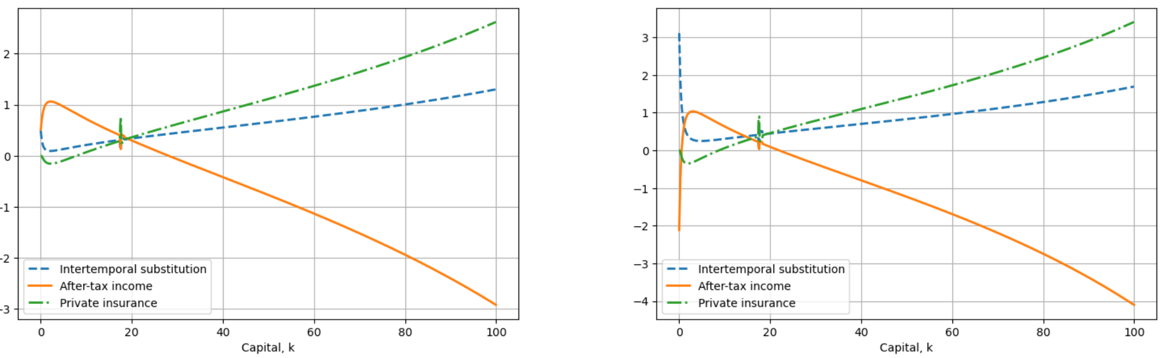
$$\Psi_j(k)=c_j(k)^{-\eta}\left\{\begin{array}{c}-\eta s_j(k)\varepsilon\left(c_j\right)+\varphi y_j^{1-\zeta}\left[(1-\zeta)\varepsilon\left(y_j\right)+\varepsilon(\varphi)-\ln y_j\right]\\+\frac{1}{2}\eta\sigma^2k^2\frac{c_j'(k)}{c_j(k)}\left[(\eta+1)\varepsilon\left(c_j\right)-\varepsilon\left(c_j'\right)\right]\end{array}\right\}. \tag{A.66}$$

From equation (A.65), we then have

$$\begin{aligned}\hat{v}_j(k)=&\tfrac{\rho+\lambda_{-j}}{\rho(\rho+\lambda_j+\lambda_{-j})}c_j(k)^{-\eta}\left\{\begin{array}{c}-\eta s_j(k)\varepsilon\left(c_j\right)+\varphi y_j^{1-\zeta}\left[(1-\zeta)\varepsilon\left(y_j\right)+\varepsilon(\varphi)-\ln y_j\right]\\+\frac{1}{2}\eta\sigma^2k^2\frac{c_j'(k)}{c_j(k)}\left[(\eta+1)\varepsilon\left(c_j\right)-\varepsilon\left(c_j'\right)\right]\end{array}\right\}\\&+\tfrac{\lambda_j}{\rho(\rho+\lambda_j+\lambda_{-j})}c_{-j}(k)^{-\eta}\left\{\begin{array}{c}-\eta s_{-j}(k)\varepsilon\left(c_{-j}\right)+\varphi y_{-j}^{1-\zeta}\left[(1-\zeta)\varepsilon\left(y_{-j}\right)+\varepsilon(\varphi)-\ln y_{-j}\right]\\+\frac{1}{2}\eta\sigma^2k^2\frac{c_{-j}'(k)}{c_{-j}(k)}\left[(\eta+1)\varepsilon\left(c_{-j}\right)-\varepsilon\left(c_{-j}'\right)\right]\end{array}\right\}.\end{aligned} \tag{A.67}$$

From the equations (A.65) and (A.66), we find that the household's utility consists of two components. As exemplified by the utility expression for the low-type households, the term $\Psi_j(k)/\rho$ or $\Psi_{-j}(k)/\rho$ comes from the lifetime utility discount of the low-type households in the present, and the other part comes from the fact that the proportion of the population switching from low type to high type in the future will lose the lifetime utility discount of low type, as well as gain the lifetime discount of high-type utility. In equation (A.67), the first term in the curly braces $\eta s_j(k)\varepsilon\left(c_j\right)$ denotes the intertemporal substitution, the second term $\varphi y_j^{1-\zeta}\left[(1-\zeta)\varepsilon\left(y_j\right)+\varepsilon(\varphi)-\ln y_j\right]$ represents the change in after-tax income due to the progressivity changes, and the third term $\frac{1}{2}\eta\sigma^2k^2\frac{c_j'(k)}{c_j(k)}\left[(\eta+1)\varepsilon\left(c_j\right)-\varepsilon\left(c_j'\right)\right]$ indicates private insurance.

Here we report the numerical results about the decomposition of $\hat{v}_j$ for low- and high-type households.



(a) Value function change of low type (b) Value function change of high type

Figure A.10: Value function Changes due to the variation of ζ

For low-type households, the lifetime utility of this group not only considers the current low-type utility but also considers the possibility of jumping to a high-type utility in the future. Private insurance is not needed when $k = 0$, therefore, private insurance is zero. However, when k is large, we can see that the private insurance is strong. When the income is small, that is, when k is small, this group receives a tax subsidy. Hence, the after-tax income at the lower end of k first increases and then decreases. Intertemporal substitution is mainly affected by savings. Since the possibility of jumping to a high-type utility in the future should be considered, savings is positive when $k = 0$ and the intertemporal substitution effect is positive. Be similar for high-type households.

4.4 The Gini coefficient of wealth

Figure A.11 reports the Gini coefficient of wealth, which measures the fairness of the distribution of social wealth.[7] As the progressivity of income taxation increases, the Gini coefficient decreases, which means that the social wealth is more equitable.

[7]From the data, we find that the gini coefficient of capital with U.S. data is 0.816, and that in our model is 0.846.

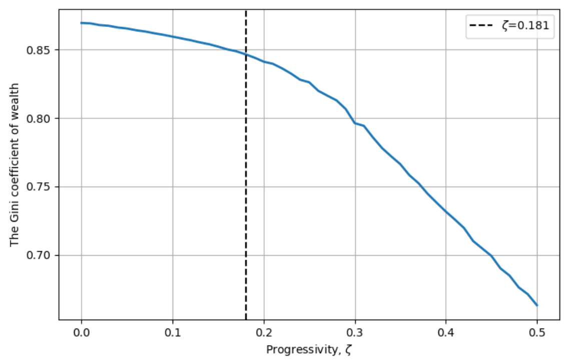


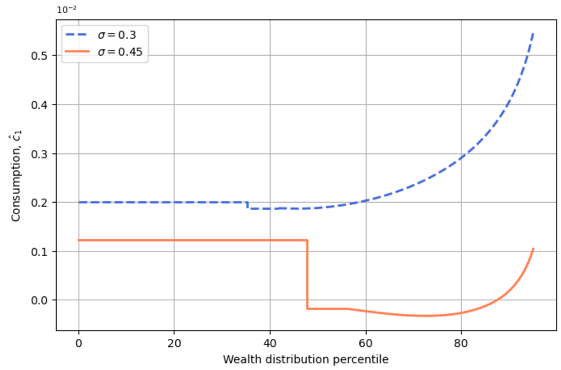
Figure A.11: The Gini coefficient of wealth

4.5 Comparative static analysis of benchmark model

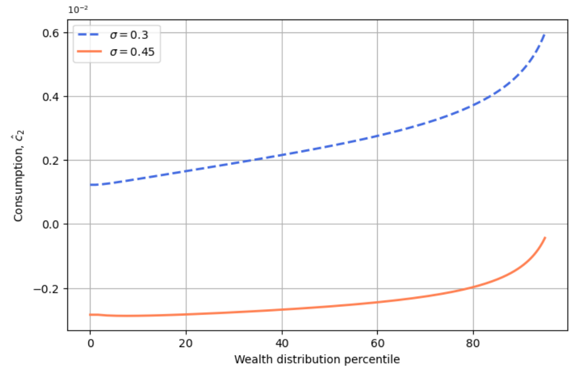
We next explore the variation of $\hat{c}_j$ and $\hat{s}_j$ under different idiosyncratic investment risks σ.[8]

Panel (a) in Figure A.12 shows that the increase in consumption is greater for lower uncertainty, especially in the benchmark model, where the middle-wealth group in the low-type households tends to save more when the progressivity of income taxation increases, resulting in a situation where consumption decreases despite the increase in after-tax income, which stems from the insurance effect. For high-type households, the increase in the progressivity provides a greater degree of cushion against higher capital income uncertainty, high-type households tend to consume less. The change in savings in panel (c) corresponds to the change in consumption in panel (a). Lower uncertainty about capital incomes leads to higher consumption, and that is accompanied by lower savings. Similarly, the change in savings in panel (d) corresponds to the change in consumption in panel (b).

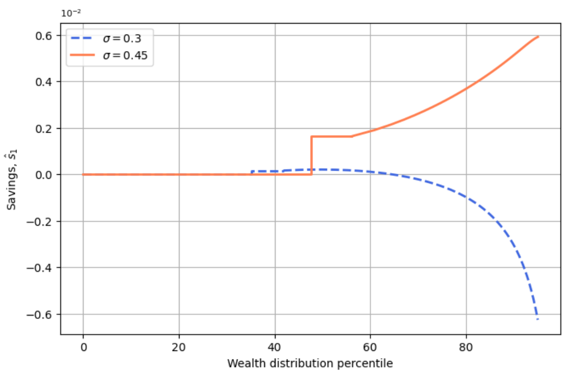
[8]We have σ in the benchmark model is 0.45.



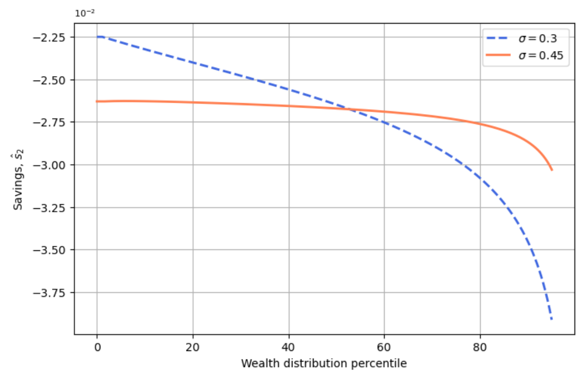
(a) Consumption change of low type



(b) Consumption change of high type



(c) Savings change of low type



(d) Savings change of high type

Figure A.12: Changes due to the variation of σ

We also compare social welfare under different idiosyncratic investment risks and find that there is no significant positive or negative relationship between idiosyncratic investment risks and the optimal progressivity.

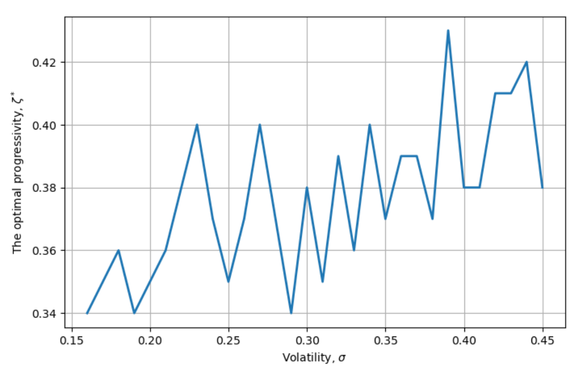


Figure A.13: Comparative static analysis on σ

We similarly explore the impact of different relative risk aversion coefficients η on households' policy functions and wealth distribution.[9]

when $\zeta = 0.181$, the savings function and wealth distribution of low-type households are plotted as follows.

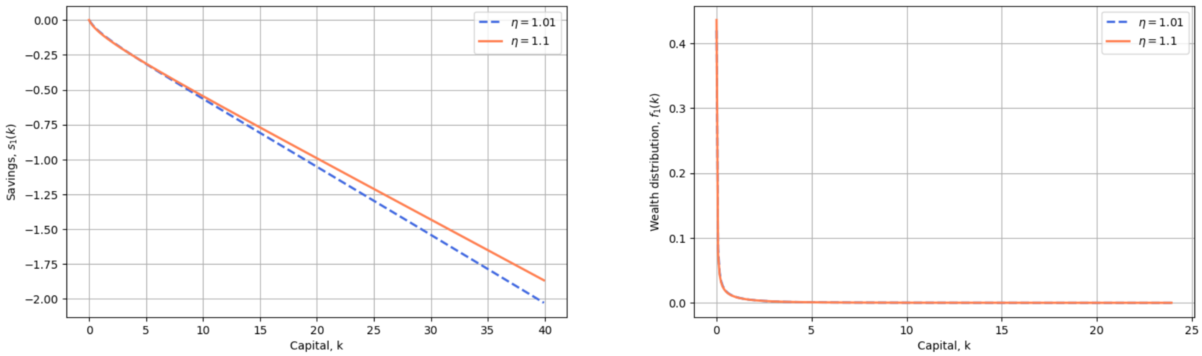


Figure A.14: Savings and distribution of low-type households with different η

The savings function and wealth distribution of high-type households are as follows.

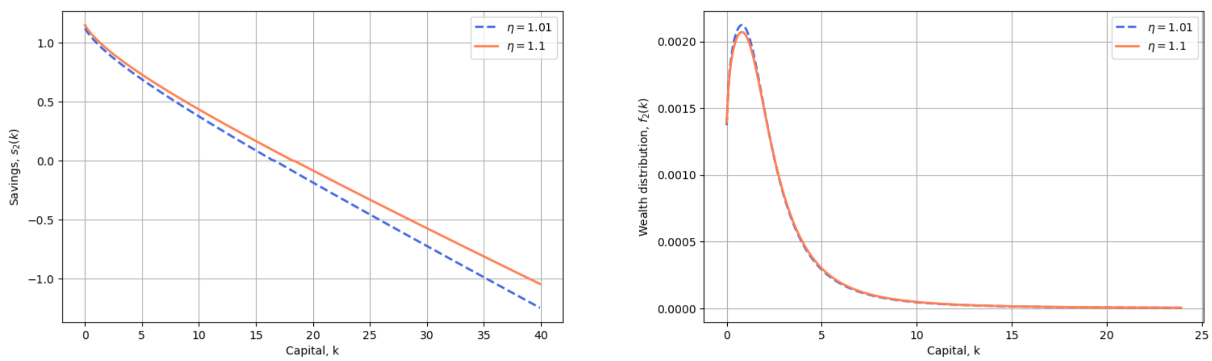


Figure A.15: Savings and distribution of high-type households with different η

We can see from Figures A.14 and A.15 that higher relative risk aversion coefficient makes households tend to save more, both for low-type and high-type groups. A higher relative risk aversion coefficient also leads to a more inequitable distribution of wealth.

In the figure below we also plot the optimal progressivity for different relative risk aversion coefficients, we find that there is no significant positive or negative relationship between the relative risk aversion coefficient and the optimal progressivity.

[9]We have η in the benchmark model is 1.1.

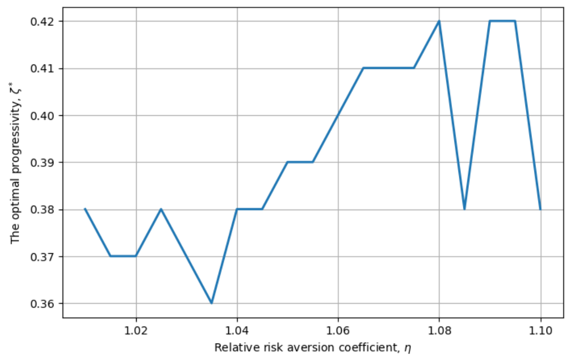


Figure A.16: Comparative static analysis on η

5 Endogenous labor supply

We calibrate the parameters of the model with endogenous labor supply. $\gamma(\ell)$ is set as

$$\gamma(\ell) = \frac{\ell^{1+1/\Phi}}{1+1/\Phi}.$$

We conduct quantitative analysis by choosing the same values for the parameters in the benchmark model along with $\chi = 0.53$, and $\Phi = 0.5$ from Chang and Park (2021).

Table A.1: Calibration from literature

Preferences of the labor supply	$\chi = 0.53$
Frisch elasticity of the labor supply	$\Phi = 0.5$

Compared with the setup of the benchmark model, there is an additional step of calculating the endogenous labor supply. The policy functions of households are

$$\ell_j^{1/\Phi} = \varphi(1-\zeta)(rk + wz_j\ell_j)^{-\zeta} wz_j/\chi, \tag{A.68}$$

and

$$c_j = (v_j')^{-1/\eta} + \chi\ell^{1+1/\Phi}/(1+1/\Phi). \tag{A.69}$$

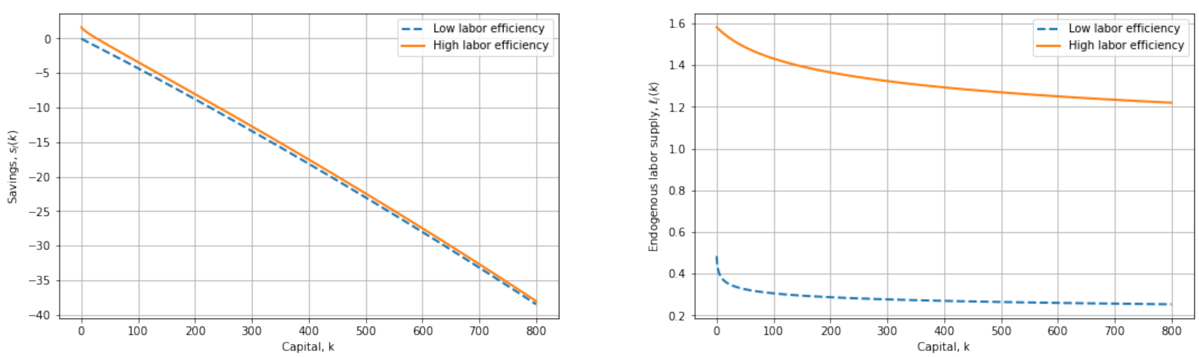
Equations (A.68) and (A.69) are used in our algorithm for calculating the endogenous labor supply and consumption of the household. Since equation (A.68) is a nonlinear equation, we can use the bisection method. Bringing ℓ_j into equation (A.69) finally calculates the household's consumption.

Table A.2 reports the wealth and income inequality resulting from the extended model and the comparison with the data. Note that the top 95-100% groups share in the wealth distribution fits the data well.

Table A.2: Wealth and income distribution

	Partition							
Percentile	0-20	20-40	40-60	60-80	80-90	90-95	95-99	99-100
Wealth share (data)	-0.002	0.011	0.045	0.112	0.120	0.111	0.267	0.336
Wealth share (model)	0	0.001	0.023	0.114	0.150	0.142	0.251	0.319
Income share (data)	0.028	0.067	0.113	0.183	0.138	0.102	0.159	0.210
Income share (model)	0.037	0.038	0.048	0.086	0.127	0.223	0.206	0.235

Figure A.17 reports the savings function and endogenous labor supply at $\zeta = 0.181$, showing that both household's savings and endogenous labor supply are decreasing as capital increases, and it is particularly noteworthy that endogenous labor supply is higher for households with high labor efficiency than that with low labor efficiency. This result is consistent with the prediction of Proposition 4, $\ell_1 < \ell_2$, and when $\zeta > 0$, $\partial \ell_j(k)/\partial k < 0$, for $j = 1, 2$.



(a) Savings

(b) Endogenous labor supply

Figure A.17: Savings and endogenous labor supply

As shown in Figure A.18 the optimal progressivity of income taxation is 0.31, which is lower than the optimal progressivity of income taxation of 0.38 in the benchmark model. The reasons for this are (i) there is an additional tax distortion margin with endogenous labor supply and hence an additional cost with raising ζ, and (ii) adjusting endogenous labor supply provides households a margin to insure against idiosyncratic shocks and hence there is no need for a higher ζ to insure against idiosyncratic shocks.

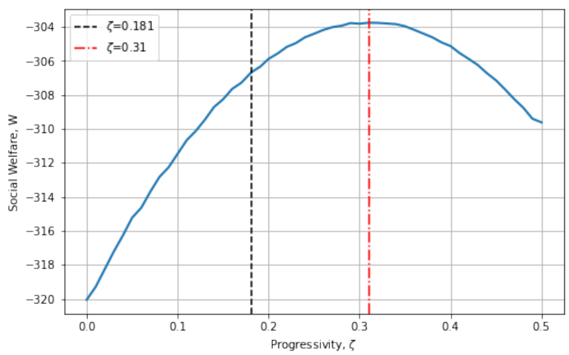


Figure A.18: Social welfare

We can see from Figure A.19 that as the progressivity increases, the Gini coefficient of wealth is decreasing, indicating that the distribution of wealth is more equitable.

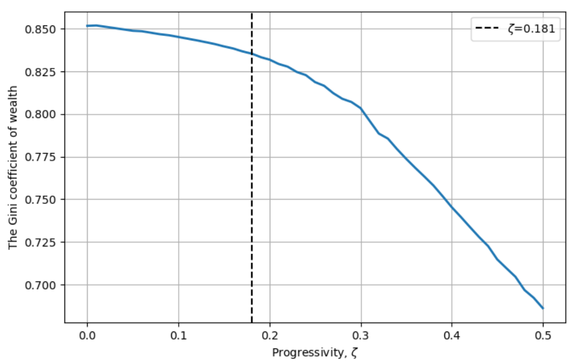


Figure A.19: The Gini coefficient of capital

Figure A.20 plots the contribution of different $k's$ to $\widehat{WP}_{LI}(\zeta)$, $\widehat{WP}_{LII}(\zeta)$, $\widehat{WP}_{LIII}(\zeta)$, $\widehat{WP}_{LIV}(\zeta)$, and $\widehat{WP}_{LV}(\zeta)$ when $\zeta = 0.181$. It is seen from equation (A.19) that each decomposition component of $\widehat{WP}_L$ are all weighted by individual utility like the benchmark model. The horizontal coordinate of the figure represents the wealth distribution percentile in the order of wealth k in each type population.

When the progressivity of income taxation increases from the progressivity of 0.181 to 0.191, panel (a) in Figure A.20 displays the curve of $(\eta-1)\frac{\varphi_m(r_bk+w_bz_j\ell_{bj}(k))^{1-\zeta_a}-\varphi_b(r_bk+w_bz_j\ell_{bj}(k))^{1-\zeta_b}}{\tilde{c}_{bj}(k)}$, indicating that the mechanical effect channel corresponds to an increase of welfare in the wealth ranks below 98.17% in the low-type households, since these households receive more redistribution from the tax reform. The mechanical effect decreases in the group of the high-type. For the low-type households, the benefits are greatest for the low-wealth group. And for the high-type households, the low-wealth group has the greatest reduction in social welfare affected through this channel.

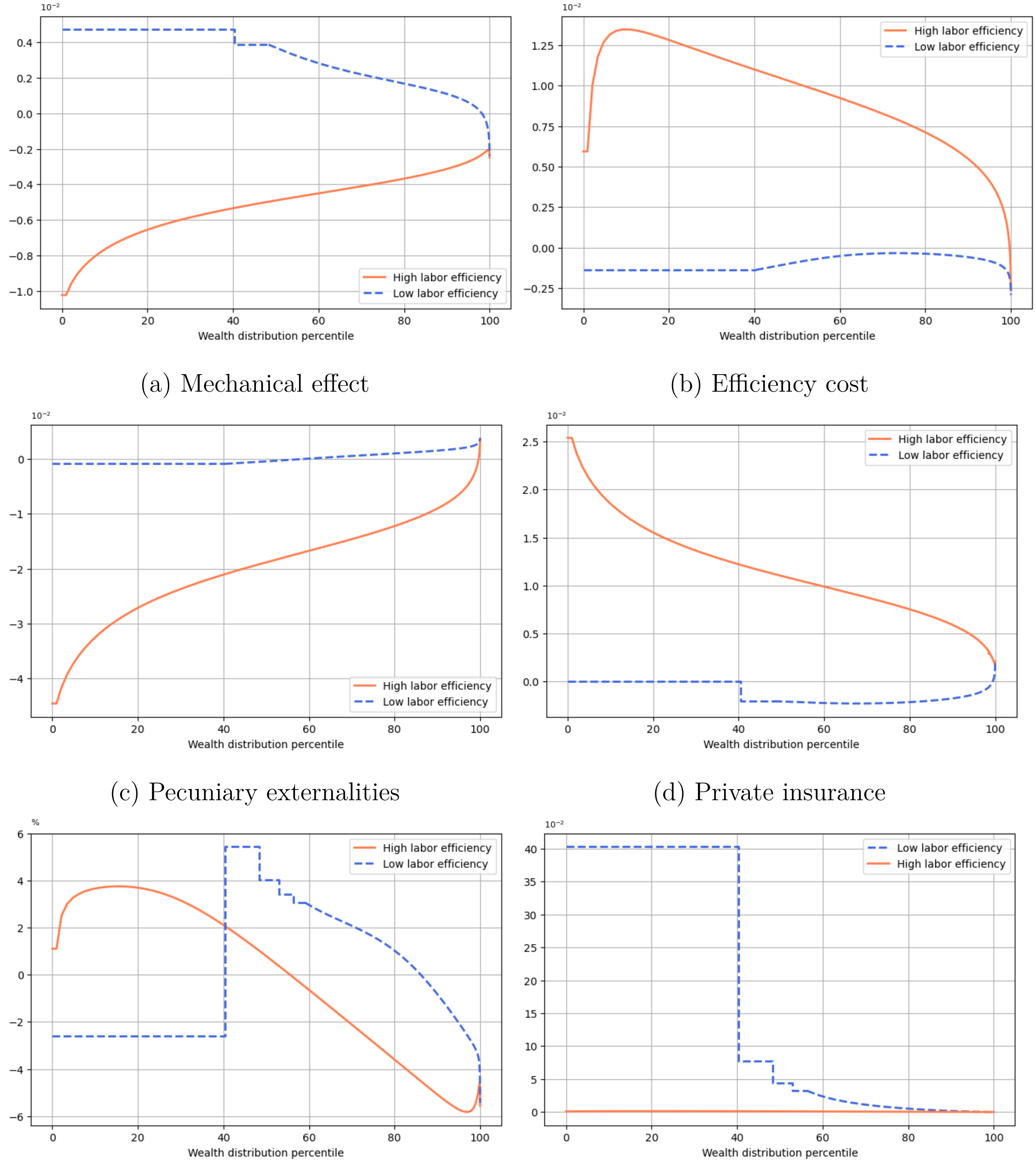
In panel (b) of Figure A.20, we present the curve of $(1-\eta)\frac{\Delta T_\ell(y_j(k))}{\tilde{c}_{bj}(k)}$. The social welfare of the low-type households decreases due to the efficiency cost channel. The tax that could be collected from the low-type households increased. The social welfare of the population whose wealth ranking is lower than 99.7% in the high-type households increases due to this channel, the tax collection from this group is reduced.

Panel (c) in Figure A.20 displays the curve of $(\eta-1)\frac{r_a k+w_a z_j \ell_{aj}(k)-(r_b k+w_b z_j \ell_{bj}(k))}{\tilde{c}_{bj}(k)}$, indicating that the pecuniary externality channel has a negative impact on the social welfare of those whose wealth ranking is lower than 56.4% in low-type households, as well as those whose ranking is lower than 99.64% in high-type households. Higher progressivity results in a higher interest rate and a lower wage rate, which has a direct impact on labor income, the main source of income for these two groups. As a result, the pre-tax income of these groups is reduced, leading to a decrease in social welfare.

Panel (d) in Figure A.20 draws the curve of $(1-\eta)\frac{s_{aj}(k)-s_{bj}(k)}{\tilde{c}_{bj}(k)}$, manifesting that the all high-type households has the increase in this channel. Since the savings of the low-wealth population in the low-type households has not changed and is still zero, the private insurance of this group is zero. Among the low-type households, those whose wealth ranks from 40.37% to 99.42% reduce social welfare through this channel.

Panel (e) in Figure A.20 depicts the impact of the tax reform on population size through the curve of $\frac{f_{aj}(k)-f_{bj}(k)}{f_{bj}(k)}$. Progressivity of changes cause larger changes of population for middle-wealth groups than for low-wealth groups, in terms of percentage.

Panel (f) of Figure A.20 shows the curve of the weight function $\Gamma_{bj}(k)$. The welfare weights of the high-type household are rather small compared to those of the low-type households. The population of the low-wealth group has a large proportion, which leads to the dominant role of that group.



(a) Mechanical effect

(b) Efficiency cost

(c) Pecuniary externalities

(d) Private insurance

(e) Distribution effect

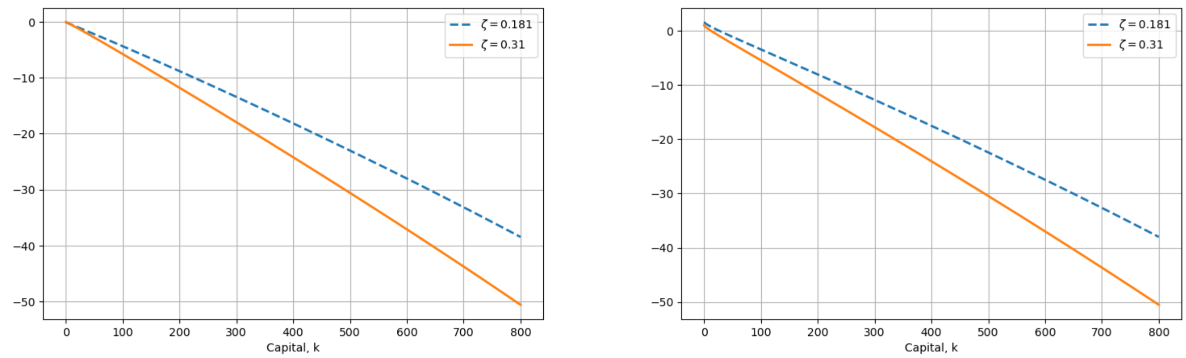
(f) Weight function

Figure A.20: Welfare effect decomposition with endogenous labor supply at $\zeta = 0.181$

5.1 Comparative static analysis with endogenous labor supply

In the following four figures we compare the differences in the consumption, saving, and value functions of individual policy functions under different progressivity of income taxation, as well as the differences in the endogenous wealth distribution.

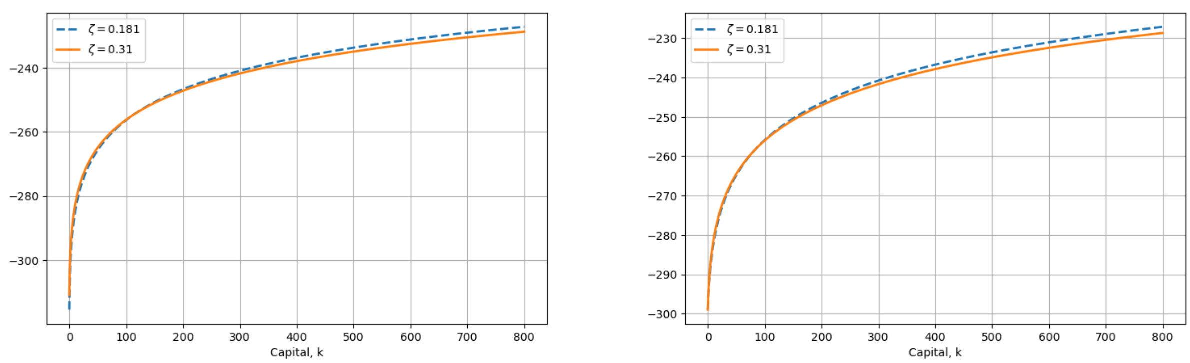
We show the effect of changes in the progressivity of income taxation on households' savings.



(a) Low labor efficiency (b) High labor efficiency

Figure A.21: Savings of low and high labor efficiency households with different ζ

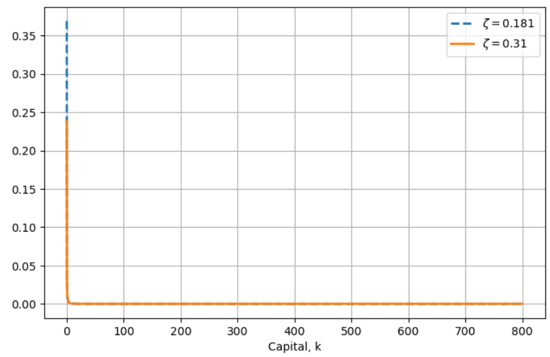
In Figure A.21, we find that the higher the progressivity, the lower the savings for both low and high labor efficiency households.



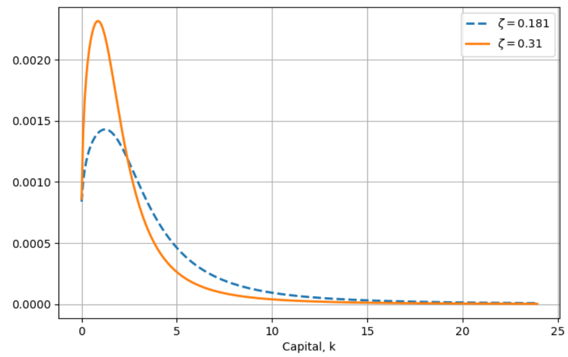
(a) Low labor efficiency (b) High labor efficiency

Figure A.22: Value function of low and high-type households with different ζ

Figure A.22 shows that for low and high labor efficiency households, when the progressivity of income taxation is increased, welfare increases for those low- and middle-wealth households, while welfare decreases for wealthy households.



(a) Low labor efficiency



(b) High labor efficiency

Figure A.23: Distribution of low and high labor efficiency households with different ζ

Finally, we can see from the above figure that an increase in the progressivity of income taxation helps to reduce social wealth inequality. Figure (A.23, a) shows that for the low labor efficiency households, the percentage of the population at the mass point, which is the bottom wealth, decreases, and for the high labor efficiency households, the percentage of the population with medium wealth increases in Figure (A.23, b).

6 Inclusion of a safe asset

As before, we have two groups of parameters. The parameters in Table A.3 are consistent with the benchmark model, except for an additional extra parameter, the proportion of private equity returns levied ϕ. The parameters in Table A.4 are intended to match the data.

Table A.3: Calibration from literature

Coeffecient of relative risk aversion	$\eta = 1.1$
Time discount factor	$\rho = 0.04$
Capital income Share	$\alpha = 1/3$
Progressivity of income taxation	$\zeta = 0.181$
Government Purchase to GDP Ratio	$g = 0.189$
The proportion of private equity returns levied	$\phi = 0.3$

Table A.4: Calibration from matching the targets

Parameter values	Targets
Volatility of Brownian motion $\sigma = 0.27$	Top 1% capital shares.
Probability of transition for income {0.05,0.5}	Top 1% income shares.
Labor productivities {0.1,3.5}	The 90-95 and 95-99 capital shares.
Private total factor productivity $A_p = 0.8$	Mean rate return to private equity $r = 0.052$.
Public total factor productivity $A_g = 0.68$	Mean rate return to public equity $r_b = 0.032$.

Optimal consumption and choice of risky assets of households are

$$c_j(a) = v_j'(a)^{-1/\eta},$$

and

$$k_j(a) = \min\left\{0 = v_j'(a)\varphi(1-\zeta)(r-r_b)\left[wz_j + ra + (r-r_b)k_j(a)\right]^{-\zeta} + v_j''(a)\sigma^2 k_j(a), a\right\}, \tag{A.70}$$

where $k_j(a)$ is the implicit solution.

Boundary Conditions The HJB equation (A.31) is defined on $(0,\infty)$ but in practice it has to be solved on a bounded interval $(0, a_{\max})$. A non-trivial issue concerns the question what boundary condition to impose at $a_{\max}$. We use the asymptotic behavior of the value function to motivate boundary conditions as follows. For large a, we have

$$v_j(a) = \tilde{v}_{0,j} + \tilde{v}_{1,j} a^{1-\eta}$$

for unknown constants $\tilde{v}_{0,j}$ and $\tilde{v}_{1,j}$. Hence, we impose the following boundary condition

$$v_j''(a_{\max}) = -\eta v_j'(a_{\max})/a_{\max}. \tag{A.71}$$

To solve equation (A.31), what we really need is a boundary condition for the term $\frac{\sigma^2}{2} v_j''(a)k(a)^2$. From equations (A.70) and (A.71), when $\zeta > 0$, $k_{\max}$ is solved by equation (A.70). Therefore, we obtain

$$k_{\max} = k_j(a_{\max}), \tag{A.72}$$

and

$$\frac{\sigma^2}{2} k_j(a_{\max})^2 v_j''(a_{\max}) = v_j'(a_{\max})\xi, \quad \xi = -\frac{\sigma^2 k_j(a_{\max})^2 \eta}{2a_{\max}}. \tag{A.73}$$

We use equation (A.73) when solving equation (A.31). Finally, it helps numerical stability to impose a state constraint $a \le a_{\max}$. This is equivalent to

$$c_j(a_{\max}) \ge \varphi\left[wz_j + r_b(a_{\max} - k_{\max}) + \phi r k_{\max}\right]^{1-\zeta} + (1-\phi) r k_{\max},$$

or using equation (A.72)

$$v_j'(a_{\max}) \le \left(\varphi\left[wz_j + r_b(a_{\max} - k_{\max}) + \phi r k_{\max}\right]^{1-\zeta} + (1-\phi) r k_{\max}\right)^{-\eta}.$$

When $\zeta > 0, k_j(a) \sim \frac{r(1-\phi)}{\eta\sigma^2}a$, then, we have

$$\xi_0 = \frac{-\left[r(1-\phi)\right]^2}{2\eta\sigma^2}a_{max},$$

and

$$v'_{j0}\left(a_{\max}\right) \leq \left[\varphi\left[wz_j + r_b a_{\max}\left(1 - \frac{r(1-\phi)}{\eta\sigma^2}\right) + \phi\frac{r^2(1-\phi)}{\eta\sigma^2}a_{\max}\right]^{1-\zeta} + \frac{\left[r(1-\phi)\right]^2}{\eta\sigma^2}a_{\max}\right]^{-\eta}.$$

When $\zeta = 0, k_j(a) \sim \frac{r(1-\phi)+\varphi(r\phi-r_b)}{\eta\sigma^2}a$, then, we obtain

$$\xi_0 = \frac{-\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{2\eta\sigma^2}a_{max},$$

and

$$v'_{j0}\left(a_{\max}\right) \leq \left\{\varphi(wz_j + r_b a_{\max}) + \frac{\left[r(1-\phi)+\varphi(r\phi-r_b)\right]^2}{\eta\sigma^2}a_{\max}\right\}^{-\eta}.$$

We numerically compute the equilibrium of our model including a safe asset. Our computing algorithm runs as follows,

1. Guess r_b^n.
2. Calculate r^n and w^n from $r^n = r_b^n A_p^{\frac{1}{\alpha}} A_g^{-\frac{1}{\alpha}}$, and $w^n = (1-\alpha)A_g^{\frac{1}{1-\alpha}}\left(\frac{r_b^n}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$.
3. Obtain the household's policy functions $k(a;r^n)$ and $x(a;r_b^n)$ based on $-v''_j(a)\sigma^2 k = v'_j(a)\left[(1-\zeta)\varphi(wz_j + r_b(a-k) + \phi rk)^{-\zeta}(\phi r - r_b) + (1-\phi)r\right], x = a - k$.
4. Compute $K = \int_0^\infty k(a;r^n)f(a;r^n)da$ and $X = \int_0^\infty x(a;r_b^n)f(a;r_b^n)da$.
5. Calculate $r_b^{n+1} = \left(\frac{X + KA_g^{-\frac{1}{\alpha}}A_p^{\frac{1}{\alpha}}}{(z_1\lambda_2+z_2\lambda_1)/(\lambda_1+\lambda_2)}\right)^{\alpha-1}\alpha A_g$. Since $N(r_b^{n+1}) + L(r_b^{n+1}) = \frac{z_1\lambda_2+z_2\lambda_1}{\lambda_1+\lambda_2}$, where $N(r_b^{n+1}) = \frac{K}{\left(\frac{r_b^{n+1}}{\alpha}\right)^{\frac{1}{\alpha-1}}A_g^{\frac{1}{\alpha(1-\alpha)}}A_p^{-1/\alpha}}$, and $L(r_b^{n+1}) = \frac{X}{\left(\frac{r_b^{n+1}}{\alpha}\right)^{\frac{1}{\alpha-1}}A_g^{\frac{1}{1-\alpha}}}$.
6. If $\mid r_b^{n+1} - r_b^n \mid < 0.005$, then stop. Otherwise use r_b^{n+1} repeat step 2-5.

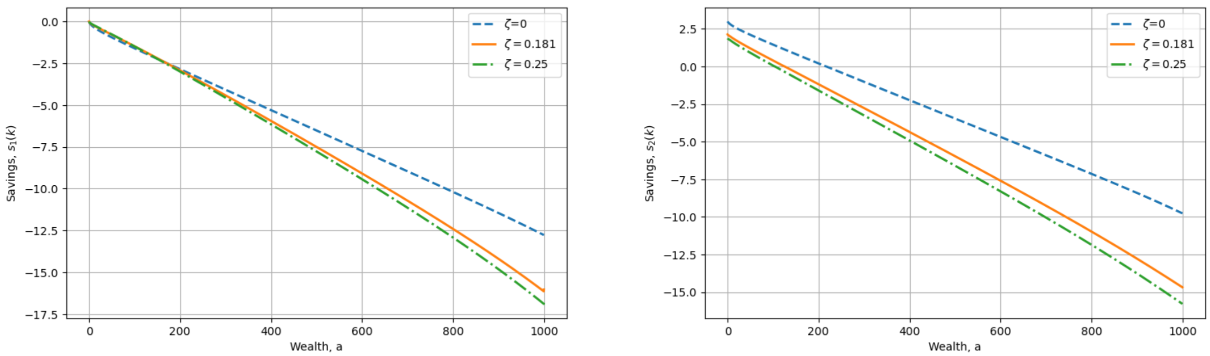
Based on the two parameter tables A.3 and A.4, we obtain the stationary distribution of wealth and income at the progressivity of 0.181.

Table A.5: Wealth and income distribution

	Partition							
Percentile	0-20	20-40	40-60	60-80	80-90	90-95	95-99	99-100
Wealth share (data)	-0.002	0.011	0.045	0.112	0.120	0.111	0.267	0.336
Wealth share (model)	0.005	0.051	0.122	0.233	0.193	0.139	0.169	0.088
Income share (data)	0.028	0.067	0.113	0.183	0.138	0.102	0.159	0.210
Income share (model)	0.024	0.036	0.057	0.095	0.092	0.188	0.175	0.333

6.1 Comparative static analysis with the portfolio problem

We compare the policy functions and the endogenous wealth distribution under different progressivities of income taxation .

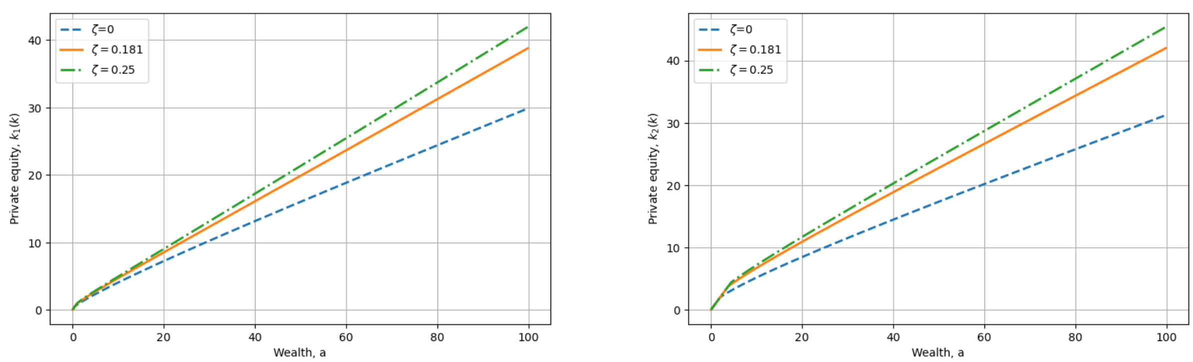


(a) Low-type households

(b) High-type households

Figure A.24: Savings of low and high-type households with different ζ

Higher progressivity is accompanied by lower savings, for both low- and high-type households.

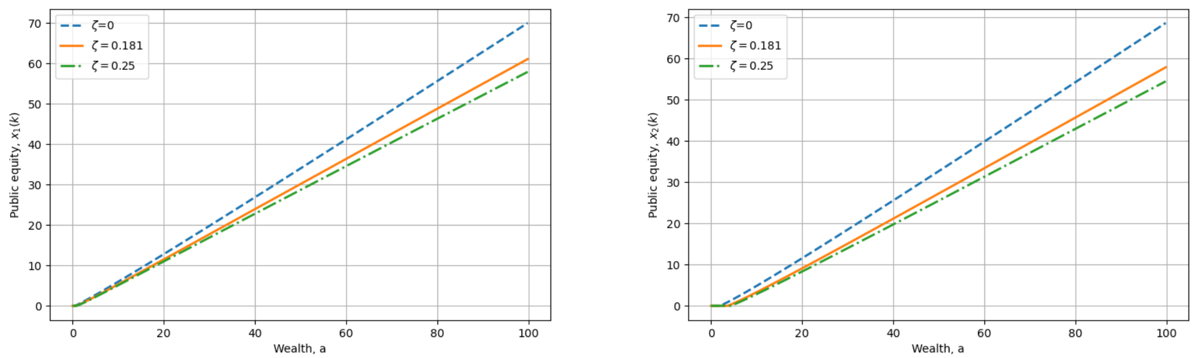


(a) Low-type households

(b) High-type households

Figure A.25: Private equity of low and high-type households with different ζ

Higher progressivity leads to both low- and high-type households tending to invest more in private equity. The increase in the progressivity provides a greater degree of cushion against higher capital income uncertainty.

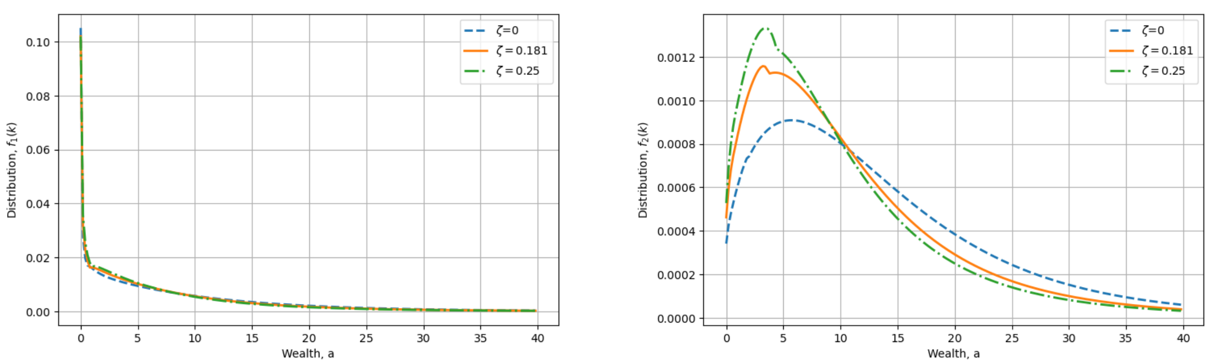


(a) Low-type households

(b) High-type households

Figure A.26: Public equity of low and high-type households with different ζ

Correspondingly, higher progressivity makes both low- and high-type households tend to invest in less public equity.



(a) Low-type households

(b) High-type households

Figure A.27: Distribution of low and high-type households with different ζ

Higher progressivity leads to a more equal distribution of wealth. Figure (A.27, b) shows that as the progressivity increases, for the high labor-type households, the percentage of the population with medium wealth increases.

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