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Comparisons of stationary distributions of linear models☆

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HIGHLIGHTS

- I compare stationary distributions of the linear model $X_{n+1} = a_n X_n + b_n$, where a_n and b_n are non-negative random variables.
- An increase of the variability of a_n and/or b_n causes a less equal stationary distribution in terms of the Lorenz dominance.
- Higher income risk leads to a less equal wealth distribution.
- Redistribution of earnings reduces wealth inequality.

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ABSTRACT

In this note, I compare stationary distributions of the linear model $X_{n+1} = a_n X_n + b_n$, where a_n and b_n are non-negative random variables. I show that an increase of the variability of a_n and/or b_n causes a less equal stationary distribution in terms of the Lorenz dominance. The result is useful in studies of wealth and income distributions.

1. Introduction

In this note, I compare stationary distributions of the linear model $X_{n+1} = a_n X_n + b_n$, where a_n and b_n are non-negative random variables. I show that an increase of the variability of a_n and/or b_n causes a less equal stationary distribution in terms of the Lorenz dominance. Such linear processes are widely used in studying income and wealth distributions; see Becker and Tomes (1979), Davies (1986), and Benhabib et al. (2011).

There are a lot of papers on comparison of distributions; see, for example, Arnold (1987) and Shaked and Shanthikumar (2010). Müller and Stoyan (2002) study comparison methods for stochastic models in general. They also discuss applications of comparison methods in economics and actuarial sciences. Becker and Tomes (1979) and Davies (1986) investigate the impact of redistribution on stationary wealth distributions. My paper presents a new result on comparisons of stationary distributions of linear models. The result may be used to study the influence of income risk and redistribution on wealth distributions.[1]

The rest of this note is organized as follows. I introduce some basic concepts in Section 2. Section 3 contains Theorem 1, the main result of the note. Section 4 contains two applications. Using Theorem 1, I show that higher income risk leads to a less equal wealth distribution and that redistribution of earnings reduces wealth inequality. Section 5 concludes the paper.

2. Basic tools

2.1. The Lorenz ordering

Let $L_X(p)$ be the Lorenz curve of a non-negative random variable X with a finite positive mean.[2] A Lorenz curve satisfies the scale invariance axiom, i.e. for any constant $c > 0$, X and cX have the same Lorenz curve. Thus X and $\frac{X}{E(X)}$ share the same Lorenz curve. By the Lorenz curve, I define the Lorenz ordering as follows.

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[1] Another application of this result to income distributions is in Zhu (2013).

[2] For the definition of the Lorenz curve, $L_X(p)$, see Gastwirth (1971).

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Definition 1. For two non-negative random variables X and Y, X Lorenz dominates Y if, and only if, $L_X(p) \geq L_Y(p)$, for all $p \in [0, 1]$, denoted as $X \succeq_L Y$.

Obviously, the Lorenz ordering is transitive, i.e. $X \succeq_L Y$ and $Y \succeq_L Z$ imply $X \succeq_L Z$. Note that $X \succeq_L Y$ implies that distribution X is more equal than Y and that the Gini coefficient of X is smaller than that of Y.

2.2. Stochastic orders

Following Ok (2013), I define the second order stochastic dominance as follows.

Definition 2. Let $F_X(x)$ and $F_Y(x)$ be the distribution functions of random variables X and Y, respectively. X second order stochastically dominates Y, denoted as $X \succeq_{SSD} Y$, if, and only if,

$$\int_{-\infty}^{x} F_X(u)du \leq \int_{-\infty}^{x} F_Y(u)du$$

for all $x \in \mathbb{R}$, provided that the integrals exist.

Following Müller and Stoyan (2002) I define the convex order of two random variables as follows.

Definition 3. For two random variables X and Y, X is smaller than Y in the convex order, denoted as $X \preceq_{cx} Y$, if, and only if,

$$E[\phi(X)] \leq E[\phi(Y)]$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided that the expectations exist.

Note that the functions ϕ_1 and ϕ_2, defined by $\phi_1(x) = x$ and $\phi_2(x) = -x$, are both convex. Thus $X \preceq_{cx} Y$ implies that $E(X) = E(Y)$, provided that the expectations exist.

We have the relation between the second order stochastic dominance and the convex order of two random variables with the same mean.

Proposition 4. *Let X and Y be two random variables such that $E(X) = E(Y)$. Then $X \preceq_{cx} Y$ if, and only if, $X \succeq_{SSD} Y$.*

Proof. The proof is carried out using part (c) of Theorem 1.5.13 of Müller and Stoyan (2002) and part (b) of Theorem 3.A.1 of Shaked and Shanthikumar (2010). □

By Theorem 1.5.9 of Müller and Stoyan (2002), we have the following.

Lemma 5. *Let $\{X_j\}$ and $\{Y_j\}$ be two sequences of random variables with $X_j \to_{st} X$ and $Y_j \to_{st} Y$ as $j \to \infty$, where "$\to_{st}$" denotes convergence in distribution. Assume that*

$$E|X_j| \to E|X| \quad \text{and} \quad E|Y_j| \to E|Y| \quad \text{as } j \to \infty.$$

If $X_j \preceq_{cx} Y_j$ for $j = 1, 2, \ldots$, then $X \preceq_{cx} Y$.

For two non-negative random variables the convex order is closely related to the Lorenz ordering. Corollary 1.5.37 of Müller and Stoyan (2002) and Theorem 3.A.10 of Shaked and Shanthikumar (2010) state the following.

Proposition 6. *Let X and Y be two non-negative random variables with equal means. Then $X \preceq_{cx} Y$ if, and only if, $L_X(p) \geq L_Y(p)$ for all $p \in [0, 1]$.*

Note that Proposition 6 can be applied to distributions with unbounded support.

3. The main result

Using Theorem 4.3.9 of Müller and Stoyan (2002) and Lemma 5, we have the main result of this paper.

Theorem 1. *For two linear process $X_{n+1} = a_n X_n + b_n$ and $\tilde{X}_{n+1} = \tilde{a}_n \tilde{X}_n + \tilde{b}_n$, let X and $\tilde{X}$ be the stationary distributions of these two processes, respectively. If*

(i) *a_n, $\tilde{a}_n$, b_n, and $\tilde{b}_n$ are non-negative random variables,*
(ii) *the a_n are i.i.d., the $\tilde{a}_n$ are i.i.d., the b_n are i.i.d., and the $\tilde{b}_n$'s are i.i.d.,*
(iii) *a_n and b_n are independent, $\tilde{a}_n$ and $\tilde{b}_n$ are independent,*
(iv) *$Ea_n = E\tilde{a}_n < 1$, $Eb_n = E\tilde{b}_n < \infty$, $a_n \succeq_{SSD} \tilde{a}_n$, and $b_n \succeq_{SSD} \tilde{b}_n$ for $n = 1, 2, \ldots$,*

then $X \succeq_L \tilde{X}$.

Proof. By Jensen's inequality, we know that $Ea_n = E\tilde{a}_n < 1$ implies that $E \log a_n \leq \log Ea_n < 0$ and $E \log \tilde{a}_n \leq \log E\tilde{a}_n < 0$. Also $E \log^+ b_n \leq Eb_n < \infty$ and $E \log^+ \tilde{b}_n \leq E\tilde{b}_n < \infty$.[3] From Lemma 1.4, Theorem 1.5, and Theorem 1.6 of Vervaat (1979), we know that $E \log a_n < 0$ and $E \log^+ b_n < \infty$ imply that there exists a unique random variable X which satisfies $X =^d a_n X + b_n$.[4] And starting from any distribution X_1, $X_n \to_{st} X$ as $n \to \infty$. A similar result holds for $\{\tilde{X}_n\}$.

Note that condition (iv) implies that $a_n \preceq_{cx} \tilde{a}_n$ and $b_n \preceq_{cx} \tilde{b}_n$, by Proposition 4. By Theorem 7.A.4 of Shaked and Shanthikumar (2010), $\begin{pmatrix} a_n \\ b_n \end{pmatrix} \preceq_{cx} \begin{pmatrix} \tilde{a}_n \\ \tilde{b}_n \end{pmatrix}$, since a_n and b_n are independent. $\tilde{a}_n$ and $\tilde{b}_n$ are independent.[5]

For any convex function $\phi(x)$ of $x \in \mathbb{R}$, $\phi(ax + b)$ is a convex function of $x \in \mathbb{R}$. For any convex function $\phi(x)$ of $x \in \mathbb{R}$, $\phi(ax + b)$ is a convex function of $(a, b) \in \mathbb{R}^2$. Pick $X_1 = \tilde{X}_1$. Then $X_n \preceq_{cx} \tilde{X}_n$ by Theorem 4.3.9 of Müller and Stoyan (2002). By Lemma 5, we have $X \preceq_{cx} \tilde{X}$. Thus $X \succeq_L \tilde{X}$ by Proposition 6. □

4. Income risk, redistribution, and wealth distributions

Assume that there is a continuum of measure-1 agents in an economy. Agents live for one period. At the end of the period, an agent gives birth to one child so that the population keeps constant. An agent in period t receives bequests w_t from his/her parents at the beginning of the period and invests his/her wealth in an individual-specific project. The rate of return on investment is R_t. R_t is i.i.d. across generations. The agent's labor earnings is y_t. y_t is also i.i.d. across generations. R_t is independent of y_t. Both R_t and y_t are idiosyncratic shocks. After R_t and y_t realize, the agent allocates his/her total resources optimally to his/her own consumption c_t and bequests to his/her child w_{t+1}. The agent's problem is

$$\max_{c_t, w_{t+1}} \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \chi \frac{w_{t+1}^{1-\gamma} - 1}{1-\gamma}$$
$$St. \quad c_t + w_{t+1} = R_t w_t + y_t,$$

[3] $\log^+ b_n = \max(0, \log b_n)$.

[4] Here, $=^d$ denotes equality in distribution.

[5] For the definition and properties of the convex order of random vectors, see chapter 3 of Müller and Stoyan (2002) and chapter 7 of Shaked and Shanthikumar (2010).

where $\gamma > 0$ is the coefficient of relative risk aversion and χ is the bequest motive intensity. Solving the agent's problem, we have

$$w_{t+1} = \frac{1}{1+\chi^{-\frac{1}{\gamma}}} R_t w_t + \frac{1}{1+\chi^{-\frac{1}{\gamma}}} y_t.$$

4.1. Income risk and wealth distributions

Suppose that there are two economies, A and B. Economy A has R_t and y_t. Economy B has $\tilde{R}_t$ and $\tilde{y}_t$. Assume that $R_t \succeq_{SSD} \tilde{R}_t$, $y_t \succeq_{SSD} \tilde{y}_t$, $ER_t = E\tilde{R}_t < 1 + \chi^{-\frac{1}{\gamma}}$, and $Ey_t = E\tilde{y}_t < \infty$. Let $a_t = \frac{1}{1+\chi^{-\frac{1}{\gamma}}} R_t$, $\tilde{a}_t = \frac{1}{1+\chi^{-\frac{1}{\gamma}}} \tilde{R}_t$, $b_t = \frac{1}{1+\chi^{-\frac{1}{\gamma}}} y_t$, and $\tilde{b}_t = \frac{1}{1+\chi^{-\frac{1}{\gamma}}} \tilde{y}_t$. Thus we know that $Ea_t = E\tilde{a}_t < 1$ and $Eb_t = E\tilde{b}_t < \infty$. Let W be the stationary wealth distribution of economy A and $\tilde{W}$ be that of economy B. Applying Theorem 1, we have the following.

Proposition 7. $W \succeq_L \tilde{W}$.

Proof. By Proposition 4, we know that $R_t \preceq_{cx} \tilde{R}_t$ and $y_t \preceq_{cx} \tilde{y}_t$. Thus $a_t \preceq_{cx} \tilde{a}_t$ and $b_t \preceq_{cx} \tilde{b}_t$.[6] Also, $Ea_t = E\tilde{a}_t$ and $Eb_t = E\tilde{b}_t$. Thus $a_t \succeq_{SSD} \tilde{a}_t$, and $b_t \succeq_{SSD} \tilde{b}_t$ by Proposition 4. Then Proposition 7 follows from Theorem 1. □

For two random variables, X and Y, with the same mean, $X \succeq_{SSD} Y$ means that Y has more risk than X. Proposition 7 implies that the higher the income risk, the less equal the stationary wealth distribution.

4.2. Redistribution and wealth distributions

Suppose that there is a government in the economy. It levies a labor income tax with a flat rate τ and gives a lump-sum transfer $T(\tau)$ to all agents. Thus the agent receives

$$\hat{y}_t^\tau = (1-\tau)y_t + T(\tau).$$

The government has a balanced budget, and $T(\tau) = \tau E y_t$. We have $E\hat{y}_t^\tau = Ey_t$. It is easy to show that $\hat{y}_t^{\tau_1} \succeq_{SSD} \hat{y}_t^{\tau_2}$ for $\tau_1 \geq \tau_2$.[7] Let W^{τ_1} be the stationary wealth distribution of an economy with τ_1, and W^{τ_2} be that of an economy with τ_2. Applying Theorem 1, we have the following.

Proposition 8. *$W^{\tau_1} \succeq_L W^{\tau_2}$ for $\tau_1 \geq \tau_2$.*

Davies (1986), using the coefficient of variation, shows that redistribution of earnings reduces wealth inequality. Here, I obtain a stronger result. I show that redistribution of earnings decreases wealth inequality, even when we use the Lorenz ordering.

5. Conclusion

For the linear model $X_{n+1} = a_n X_n + b_n$, where a_n and b_n are non-negative random variables, I use Theorem 4.3.9 of Müller and Stoyan (2002) to show that an increase of the variability of a_n and/or b_n causes a less equal distribution of X_n in terms of the Lorenz dominance. Then I use a simple continuity argument of Lemma 5 to show that it also leads to a less equal stationary distribution of $X_{n+1} = a_n X_n + b_n$. The result is useful in studies of wealth and income distributions.

References

Arnold, B., 1987. Majorization and the Lorenz Order: A Brief Introduction. Springer-Verlag, Berlin.
Benhabib, J., Bisin, A., Zhu, S., 2011. The distribution of wealth and fiscal policy in economies with finitely lived agents. Econometrica 79, 123–157.
Becker, G., Tomes, N., 1979. An equilibrium theory of the distribution of income and intergenerational mobility. Journal of Political Economy 87, 1153–1189.
Davies, J., 1986. Does redistribution reduce inequality? Journal of Labor Economics 4, 538–559.
Gastwirth, J., 1971. A general definition of the Lorenz curve. Econometrica 39, 1037–1039.
Müller, A., Stoyan, D., 2002. Comparison Methods for Stochastic Models and Risks. John Wiley & Sons, West Sussex, UK.
Ok, E.A., 2013. Probability Theory with Economic Applications. NYU, Mimeo.
Shaked, M., Shanthikumar, J.G., 2010. Stochastic Orders. Springer, New York, NY.
Vervaat, W., 1979. On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. Advances in Applied Probability 11, 750–783.
Wan, J., Zhu, S., 2012. Bequests, Estate Taxes, and Wealth Distributions. NUS, Mimeo.
Zhu, S., 2013. Pure Luck: A Human Capital Theory of Labor Earnings Distributions. NUS, Mimeo.

[6] $X \preceq_{cx} Y$ implies that $cX \preceq_{cx} cY$ for any $c \in \mathbb{R}$. Note that $\phi(cx)$ is a convex function of $x \in \mathbb{R}$ if $\phi(x)$ is a convex function of $x \in \mathbb{R}$.

[7] By Lemma 6 of Wan and Zhu (2012), we have $\hat{y}_t^{\tau_1} \preceq_{cx} \hat{y}_t^{\tau_2}$. By Proposition 4, we have $\hat{y}_t^{\tau_1} \succeq_{SSD} \hat{y}_t^{\tau_2}$, since $E\hat{y}_t^{\tau_1} = E\hat{y}_t^{\tau_2}$.