Existence of the Stationary Equilibrium in an Incomplete-market Model with Endogenous Labor Supply

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Abstract

I first study an income fluctuation problem with endogenous labor supply. Let β be the agent's time discount factor and R be the constant gross rate of return of assets. For cases of $\beta R = 1$ and $\beta R < 1$, I find sufficient conditions guaranteeing the existence of an upper bound for the agent's wealth accumulation. For the case of $\beta R < 1$, I prove the existence, uniqueness, and stability of the stationary distribution of state variables. I then show the existence of the stationary general equilibrium in an incomplete-market model with endogenous labor supply. My proof of the existence of the stationary equilibrium also shows that a bisection algorithm can find a general equilibrium.

Keywords: Endogenous labor supply, the incomplete-market model, the bound of wealth accumulation, the stationary distribution, the existence of the stationary equilibrium

JEL classification: D52, D91, E21

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1 Introduction

The aim of this paper is to show the existence of the stationary general equilibrium in an incomplete-market model with endogenous labor supply. There are a continuum of households with measure 1 in the economy. Households have uninsurable idiosyncratic labor efficiency shocks. Each household faces an income fluctuation problem with endogenous labor supply. The labor efficiency shock follows a Markov chain along time and is independent and identically distributed (*i.i.d.*) across households.

Aiyagari and McGrattan (1998) use an incomplete-market heterogeneous agents model with endogenous labor supply to study the optimum quantity of government debt. Marcet et al. (2007) show that incomplete insurance to idiosyncratic employment shocks introduce an ex post wealth effect which reduces labor supply. The ex post wealth effect on labor supply runs counter the precautionary savings motive.[1] These papers did not show the existence of the stationary general equilibrium. This paper fills this gap.

I first study an income fluctuation problem with endogenous labor supply. Let β be the agent's time discount factor and R be the constant gross rate of return of assets. Note that $R = 1 + r$ where r is the net rate of return. For cases of $\beta R = 1$ and $\beta R < 1$, I find sufficient conditions guaranteeing the existence of an upper bound for the agent's wealth accumulation. For cases of $\beta R = 1$, I show that the agent's wealth converges to a finite number almost surely and his labor supply approaches zero almost surely. I show these results in a model with more general utility assumptions and more general labor productivity shocks than Marcet et al. (2007). For the case of $\beta R < 1$, I prove the existence, uniqueness and stability of the stationary distribution of state variables.[2]

To show the existence of the stationary equilibrium, I find the intersection of the "supply" and "demand" curves of the capital-labor ratio in the economy. The aggregate capital supply is the total wealth of households in

[1] In their numerical examples, the wealth effect often dominates the pretionary saving effects. Thus the output and savings in incomplete markets are lower than those in complete markets.

[2] The stability here means that any initial distribution of state variables converges to the unique stationary distribution.

the stationary distribution of state variables. The aggregate labor supply is the total labor supply in the stationary distribution of state variables. The "supply" curve of the capital-labor ratio is the ratio of the aggregate capital supply to the aggregate labor supply. I show that the "supply" curve is a continuous function of the interest rate r and tends to infinity as r approaches $\frac{1}{\beta} - 1$ from below. The "demand" curve of the ratio is from the optimal conditions of the firm's profit-maximization problem. My proof of the existence of the stationary equilibrium also shows that a bisection algorithm can find a general equilibrium. Thus my paper gives guidances to simulation works of incomplete-market models with endogenous labor supply.

1.1 Related literatures

Marcet et al. (2007) show that the agent's wealth converges to a finite number almost surely and his labor supply approaches zero almost surely for the case of $\beta R = 1$. My paper extends these results to a model with more general utility assumptions and more general labor productivity shocks. This paper employs a more general utility function form than Marcet et al. (2007) which uses a separable and homogeneous utility function. For the labor efficiency shocks, Marcet et al. (2007) use a two-state Markov chain which should also satisfy the monotonicity assumption. My results can be applied to multiple-state Markov chains and I do not need the Markov chain to be monotone.

Chamberlain and Wilson (2000) study an income fluctuation problem with stochastic interest rates and labor earnings. They show that if the labor earnings and interest rate processes are sufficiently volatile and the long-run average rate of interest is greater than or equal to the time discount rate, then the household's consumption eventually grows without bound with probability one. Thus the agent's asset also converges to infinity almost surely. Note that $\beta R = 1$ is a special case of their model. However, Chamberlain and Wilson (2000) do not investigate the general equilibrium. They assume that the interest rate process is exogenous.

Schecktman and Escudero (1977) investigate optimal policy functions of an income fluctuation problem with $\beta R < 1$. They also study the long-run property of the wealth accumulation process under $\beta R < 1$.[3] They assume that the labor earnings shock is *i.i.d.* along time and its support is bounded. They show that the wealth accumulation process is bounded if the coefficient of relative risk aversion is bounded as consumption goes to infinity.[4,5]

Aiyagari (1993) proves the existence of the stationary general equilibrium in an incomplete-market model with inelastic labor supply and *i.i.d.* labor efficiency shocks. Aiyagari (1994) uses this model to quantitatively show that precautionary saving is not important for aggregate capital accumulation. Huggett (1993) shows the existence, uniqueness, and stability of the stationary distribution of state variables in an incomplete-market model with serially correlated income shocks. Huggett (1993) assumes that the income shock follows a two-state Markov chain and the Markov chain is monotone.[6]

To show the existence, uniqueness, and stability of the stationary distribution of state variables, Aiyagari (1993), Huggett (1993), and Marcet et al. (2007) employ the monotone-Markov-process method of Hopenhayn and Prescott (1992). Kamihigashi and Stachurski (2014) extend this method to the unbounded state space. I show the existence, uniqueness, and stability of the stationary distribution using a new method and I do not need the monotonicity assumption of the Markov chain.

Miao (2002) shows the existence of the stationary general equilibrium in an incomplete-market model with serially correlated income shocks. Miao (2002) assumes that the transition function of the income shock is monotone and satisfies a smoothness condition.

Kuhn (2013) introduces permanent earnings shocks into the Aiyagari model. In Kuhn (2013) households have inelastic labor supply and their

[3] Rabault (2002) investigates an income fluctuation problem in which the lowest possible level of earnings is zero and the marginal utility of consumption is infinite when consumption is zero.

[4] Schecktman and Escudero (1977) also investigate an example in which the coefficient of relative risk aversion is unbounded as consumption goes to infinity. In this example, they find that the wealth accumulation approaches infinity almost surely even under $\beta R < 1$.

[5] For researches on income fluctuation problems, see also Laitner (1979), Mendelson and Amihud (1982), Sotomayor (1984), Clarida (1987, 1990), and Laitner (1992).

[6] One difference between Aiyagari (1994) and Huggett (1993) is that Aiyagari (1994) has aggregate production while Huggett (1993) is an endowment economy.

labor efficiency shocks have random growth components. They have a utility function of constant relative risk aversion (CRRA). To obtain the stationary general equilibrium Kuhn (2013) assumes that households have a death rate. Kuhn (2013) proves the existence of the stationary general equilibrium in the model. However, Aiyagari (1993), Huggett (1993), Miao (2002), and Kuhn (2013) do not discuss endogenous labor supply.

Acemoglu and Jensen (2015) study comparative statics of a class of heterogeneous agents models, including the Bewley-Aiyagari-Huggett model. They assume that the idiosyncratic shock follows a Markov process with the Feller property. The support of the idiosyncratic shock is compact. They show the existence of the stationary general equilibrium. Acemoglu and Jensen (2015) argue that their proof applies to an Aiyagari model with endogenous labor supply. However, my proof in this paper has its own advantage. Applying my proof one can easily develop a bisection algorithm, similar to that in Aiyagari (1994), to find the stationary general equilibrium. On the contrary, it is difficult to develop an algorithm based on their proof since Acemoglu and Jensen (2015) only prove the existence of the stationary distribution of state variables, but their proof does not give a way to find the stationary distribution.

Acikgöz (2016) shows the existence of the stationary general equilibrium in an incomplete-market model with production.[7] Acikgöz (2016) assumes that the earnings shock follows a multiple-state Markov chain. And the utility function could be unbounded. Acikgöz (2016) shows that the wealth accumulation process of an income fluctuation problem with $\beta R < 1$, is bounded if the coefficient of absolute risk aversion tends to zero as consumption goes to infinity. Foss et al. (2016) show the boundedness result for an income fluctuation problem with a multiple-state Markov chain and the CRRA utility function. Following my paper, Acikgöz (2016) shows the existence, uniqueness, and stability of the stationary distribution of state variables in his model. Different from Acikgöz (2016) and Foss et al. (2016), my model has endogenous labor supply.

The rest of this paper is organized as follows. In Section 2, I investigate the household's policy functions under three cases: (i) $\beta R > 1$, (ii) $\beta R = 1$, and (iii) $\beta R < 1$. I characterize the stationary general equilibrium and

[7] Acikgöz (2016) also gives an example which has multiple equilibria.

prove the existence of the equilibrium in Section 3. Section 4 concludes the paper. Some long proofs are in the Appendix.

2 An income fluctuation problem with endogenous labor supply

There are a continuum of households with measure 1 in the economy. Each household faces an income fluctuation problem with endogenous labor supply. The household has an instantaneous utility function $u(c, h)$ of consumption c, and leisure h. The utility function $u(c, h)$ satisfies

Assumption 1: $u : R_+ \times [0, 1] \to R$ is twice continuously differentiable.

Assumption 2: $u(c, h)$ is strictly increasing and strictly concave in c and h. And

$$\lim_{c \to 0} u_1(c, h) = +\infty, \quad \forall h \in [0, 1] \text{ and } \lim_{h \to 0} u_2(c, h) = +\infty, \quad \forall c \geq 0.$$

Assumption 3: $u(c, h) \in [0, M]$, $M > 0$.

Each household is endowed with one unit of time. The household is subject to idiosyncratic labor efficiency shocks. The wage rate for the labor efficiency unit is w. We assume that the labor efficiency process $\{e_t\}_{t=0}^{\infty}$ follows a Markov chain with a transition probability $\pi(e'|e)$.

Assumption 4: $e_t \in E \equiv \{e^1, e^2, \cdots, e^n\}$, with $0 < e^1 < e^2 < \cdots < e^n$. $\sum_{e'} \pi(e'|e) = 1$ for all $e \in E$. And $\pi(e'|e) > 0$ for all e, $e' \in E$.

There is only one risk-free asset in the economy. The constant gross rate of return of assets is R. The household's budget constraint is

$$c_t + a_{t+1} = Ra_t + (1 - h_t)e_t w,$$

where a_t is the household's asset. The household cannot borrow assets from others and thus $a_{t+1} \geq 0$. The household's state can be described by $x = (a, e) \in X = A \times E$, where $A = [0, +\infty)$.

Each household has the preference,

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t),$$

where $\beta \in (0, 1)$ is the time discount factor.

I study the household's problem in two steps.[8] Step 1 is an intratemporal problem. The consumer chooses consumption and leisure to maximize the current period utility with respect to the given expenditure. The step is a static maximization exercise and is irrelevant to the dynamic optimization. I derive the indirect utility function $J(y, e)$ in the intratemporal problem. Step 2 is an intertemporal problem which determines the optimal expenditure in every period with respect to the given initial wealth. I use the indirect utility function to transform the original dynamic programming problem with two control variables into a derived dynamic programming problem with only one control variable.[9]

I define the indirect utility function $J(y, e)$ of the intratemporal problem as

$$J(y, e) = \max_{c,h} u(c, h)$$

$$s.t.\ c + hew = y, \ \ 0 \le h \le 1,$$

where y is the expenditure on consumption c and leisure h.

The first-order condition of the intratemporal problem is

$$\frac{u_2(c, h)}{u_1(c, h)} \ge ew \quad \text{with equality when } h < 1.$$

Proposition 1 characterizes the indirect utility function $J(y, e)$.

Proposition 1 *Under Assumptions A1-A4 we have*

1) $J(y, e)$ is bounded.

2) $J(y, e)$ is strictly increasing and strictly concave in y.

[8] I thank Charles Wilson for the suggestions of this two-step procedure.

[9] Foley and Hellwig (1975) employ this two-step procedure to study the income fluctuation problem with endogenous labor supply. The difference between their model and our model is that the asset in their model is money. Therefore, the return to the asset is zero in their model. In my model I study the genereal case of the return to asset. The interest rate in my model could be positive, zero or negative.

3) $J(y,e)$ is continuously differentiable in y. $J_1(y,e) = u_1(c(y,e), h(y,e))$, $\forall y \in (0, +\infty)$.

After we maximize the utility within the period with respect to the budget constraint, the original dynamic utility maximization problem becomes

$$\max E_0 \sum_{t=0}^{\infty} \beta^t J(y_t, e_t)$$

$$s.t. \quad y_t + a_{t+1} = Ra_t + e_t w, \;\; y_t \geq 0.$$

I study the household's problem using standard dynamic programming technique. Let $V(a,e)$ be the optimal value function of the agent's intertemporal optimization problem. The Bellman equation that describes an agent's decision problem is

$$V(a,e) = \max_{a' \in \Gamma(a,e)} \left\{ J(Ra + ew - a', e) + \beta E[V(a', e')|e] \right\},$$

where

$$\Gamma(a,e) = \{a' : 0 \leq a' \leq Ra + ew\}.$$

Let $a(a,e)$, and $y(a,e)$ be the optimal decision rules of the asset for next period and the total expenditure for the current period respectively. $a(a,e)$, and $y(a,e)$ satisfy

$$V(a,e) = J(y(a,e), e) + \beta \sum_{e'} V(a(a,e), e') \pi(e'|e).$$

The Euler equation is

$$V_1(a,e) \geq \beta R E[V_1(a', e')|e] \quad \text{with equality if } a' > 0$$

Proposition 2 characterizes the value function $V(a,e)$ and optimal decision rules, $a(a,e)$ and $y(a,e)$.

Proposition 2 *Under Assumptions A1-A4 we have*

1) $V(a,e)$ is continuous, strictly increasing, strictly concave in a.
2) $V(a,e)$ is continuously differentiable in a. And $V_1(a,e) = RJ_1(y(a,e), e)$

$\forall a \in [0, +\infty)$.[10]

3) $a(a, e)$ is continuous and weakly increasing in a.

4) $y(a, e)$ is strictly increasing in a.

Note that Assumption A2 implies that

$$\lim_{y \to 0} J_1(y, e) = +\infty, \quad \forall e \in E$$

which in turn means that $y > 0$ for the agent's optimal choice. Based on this observation we draw Proposition 2.2.

Assumption 5:

$$u_{12}u_1 - u_{11}u_2 > 0 \text{ and } u_{12}u_2 - u_{22}u_1 > 0.$$

Assumption A5 is to guarantee that both consumption and leisure are normal goods. It also means that marginal rate of substitution $\frac{u_h}{u_c}$ is increasing in c and decreasing in h. This assumption guarantee that both $c(y, e)$ and $h(y, e)$ are increasing functions in y. To show the existence of the stationary equilibrium we need a stronger condition,

Assumption 5′:

$$u_{12} \geq 0.$$

Assumption A5′ means that consumption and leisure are complementary. Assumption A5 is automatically satisfied if Assumption A5′ holds.

We use $c(a, e)$ and $h(a, e)$ to represent $c(y(a, e), e)$ and $h(y(a, e), e)$.

Proposition 3 *Under Assumptions A1-A5′ we have*

1) $c(a, e)$ and $h(a, e)$ are continuous and increasing with respect to a.

2) $h(a, e) = 1$ $\forall e \in E$, when a is sufficiently large.

We use Assumption A5′ to draw Proposition 3.2.

2.1 The case of $\beta R > 1$

We investigate the dynamics of the agent's asset when $\beta R > 1$. We express explicitly the time subscription for each variable. The Euler equation

[10] We use $V_1(0, e)$ to represent $V_1^+(0, e)$.

governs the dynamics of the model,

$$V_1(a_t, e_t) \geq \beta R E[V_1(a_{t+1}, e_{t+1})|e_t] \quad \text{with equality if } a_{t+1} > 0.$$

Assumption A2 guarantees that consumption is strictly positive. Then $V_1(a_t, e_t) = R u_1(c_t, h_t)$.

Theorem 1 *If $\beta R > 1$ and Assumptions A1-A4 hold, we have*

$$\lim_{t\to\infty} a_t = \infty \ a.s. \tag{1}$$

The agent's asset grows without bound as long as the interest rate exceeds the time discount rate. Theorem 1 does not depend on Assumption 5. The proof of Theorem 1 uses the Supermartingale Convergence Theorem, which is widely used in literatures of the income fluctuation problem, such as Schechtman (1976), Mendelson and Amihud (1982), Sotomayor (1984), and Chamberlain and Wilson (2000).[11] The asset accumulation approaches infinity since the agent is too patient and/or the interest rate is too high.

2.2 The case of $\beta R = 1$

There are two purposes of investigating the asymptotic properties of agent's behavior when $\beta R = 1$. On one hand, the asymptotic properties of optimal policies help us understand the agent's long-run behavior. On the other hand, they give us the limiting values of the per capita variables as interest rate approaches the time preference rate from below. Especially, this clue help us obtain the limit of per capita asset-labor ratio when $R \to \frac{1}{\beta}$ from below.

The objective function is concave and the feasible set is convex in this problem. Then the first-order conditions and the transversality condition are necessary and sufficient for an optimum. Plugging the following policies in the first-order conditions and the transversality condition, we can verify that they are optimal as $a \geq \bar{k}$,

$$h(a, e) = 1, \ c(a, e) = ra, \ \text{and } a(a, e) = a.$$

[11] For the Supermartingale Convergence Theorem see Billingsley (1986).

These policy functions imply that the asset accumulation has an upper bound. This follows from Proposition 3.2 which is due to Assumption 5′. We will investigate more about this point later.

The following lemma describes the asymptotic property of the marginal return of assets.

Lemma 1 $\lim_{t\to+\infty} V_1(a_t, e_t)$ *exists and is finite almost surely when* $\beta R = 1$.

Proof. Applying supermartingale convergence theorem and the Euler equation. ■

Lemma 1 implies that the process of the marginal return of assets, $\{V_1(a_t, e_t)\}_{t=0}^{\infty}$, has a finite limit almost surely. We then investigate the asset accumulation and labor supply when $V_1(a_t, e_t)$ is fixed. Thus $u_1(c_t, h_t)$ is fixed since $V_1(a_t, e_t) = Ru_1(c_t, h_t)$. Combining this relationship and the intratemporal labor-leisure decision, we have

$$u_1(c, h) = \lambda, \tag{2}$$

where λ is a constant that may depend on the path $\{(a_t, e_t)\}_{t=0}^{\infty}$, and

$$u_2(c, h) \geq \lambda ew, \text{ with equality when } h < 1. \tag{3}$$

By Assumption A2**,** $u(c, h)$ is strictly concave. The Hessian of $u(c, h)$ is negative definite. From Equations (2) and (3) we can derive

$$c = \xi(\lambda, e),$$

and

$$h = g(\lambda, e).$$

Thus $\xi(\lambda, e)$ and $g(\lambda, e)$ are the consumption and leisure when the marginal utility of consumption is fixed as λ.

If $u_1(c(a, e), h(a, e)) = \lambda$, then $V_1(a, e) = R\lambda$. $V_1(a, e)$ is a strictly decreasing function of a. From Proposition 3 we know that $g(\lambda, e)$ is a decreasing function of λ and $g(\lambda, e) = 1$ when λ is low enough. Since the realizations of e in E are finite, there exists $\bar{\lambda} > 0$ such that $g(\lambda, e) = 1$, $\forall e \in E$, for $\lambda \leq \bar{\lambda}$.

We define a dissaving function $\chi(\phi, e)$ as consumption minus labor income,

$$\chi(\phi, e) = \xi(\phi, e) - (1 - g(\phi, e))ew.$$

We know that E is a finite set. Thus $\exists \hat{e}$ and $\check{e}$ such that $\chi(\phi, \hat{e}) = \max_{e \in E}\{\chi(\phi, e)\}$ and $\chi(\phi, \check{e}) = \min_{e \in E}\{\chi(\phi, e)\}$.

Lemma 2 *For $\phi > \bar{\lambda}$, $\chi(\phi, \hat{e}) > \chi(\phi, \check{e})$.*

Lemma 3 *There is an $\varepsilon > 0$ such that for any $\alpha \in R$ and any $\phi > \bar{\lambda}$ we have*

$$P\left(\alpha \leq \sum_{j=1}^{+\infty} \chi(\phi, e_{t+j-1})\beta^j \leq \alpha + \varepsilon | e_t\right) < 1 - \varepsilon,$$

for all e_t, $t \geq 0$.

From the policy functions for $a \geq \bar{k}$, we know that the asset stays in the compact set $[0, \bar{k}]$, if $a_0 \in [0, \bar{k}]$. Iterating the budget constraint, $a_{t+1} = Ra_t + (1 - h_t)e_t w - c_t$, we have

$$\frac{a_t}{R^{t-\tau}} = a_\tau + \sum_{j=1}^{t-\tau} \frac{(1 - h_{\tau+j-1})e_{\tau+j-1}w - c_{\tau+j-1}}{R^j} \text{ for } \tau \leq t.$$

Letting $t \to +\infty$, we have

$$a_\tau = \sum_{j=1}^{+\infty} [c_{\tau+j-1} - (1 - h_{\tau+j-1})e_{\tau+j-1}w]R^{-j}.$$

Theorem 2 characterizes the long-run labor supply in the economy when $\beta R = 1$.

Theorem 2 *$h_t \to 1$ a.s., when $\beta R = 1$.*

Corollary 2.1: If $a_0 \leq \bar{k}$, $a_t \to \bar{k}$ *a.s.*, when $\beta R = 1$.

When $\beta R = 1$, the agent accumulates his asset until it reaches a fixed finite level if his starting asset is lower than this fixed level. If his starting asset is higher than this fixed level, the agent holds the starting asset forever. The agents whose assets are higher or equal to this critical level, do not supply labor. Then the agent's labor supply converges to zero and the asset-labor ratio approaches infinity.

This theorem says that the agent's labor supply asymptotically approaches zero. For the agents whose starting assets are higher than $\bar{k}$, they do not want supply labor since they are rich enough. They just consume the interest of their wealth. Their labor supply is zero. Then they do not have to face the labor productivity shock. For the agents whose starting asset levels are lower than $\bar{k}$, they first accumulate asset and reach the target level, $\bar{k}$. Then they stop working and reach a perfect self-insurance state. The endogenous labor supply opens a door for the agents to enter a perfect insurance situation. If the agents do not supply labor, the productivity shocks have no impact on the agents. Only when agents have accumulated enough assets, can their optimal choice of labor supply be zero. This gives the agents strong incentive for accumulating asset if their wealth is less than the target level.

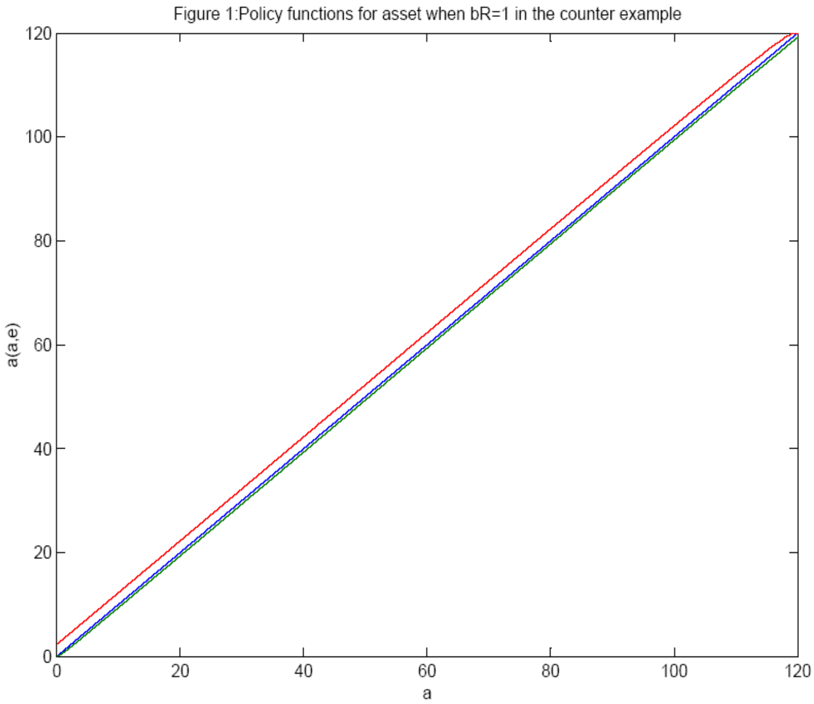
The upper bound of the asset accumulation is due to Assumption A5$'$. If Assumption A5$'$ does not hold, it seems that there exist counterexamples. We run a simulation for a specific utility function,

$$u(c,h) = -e^{-(c+\sqrt{h})}.$$

under a two-state Markov chain shock. This utility function does not satisfy our complementarity assumption, Assumption A5$'$, since $u_{12}(c,h) < 0$ for $\forall c \geq 0$ and $h \in [0,1]$. We set the parameters as: $\beta = 0.96$, $e \in E = \{0.585, 4.248\}$, $R = \frac{1}{\beta} = 1.0417$ and wage $w = 1.0815$. The transition probability of the shocks is

$$\pi = \begin{pmatrix} 0.9181 & 0.0819 \\ 0.0819 & 0.9181 \end{pmatrix}.$$

I solved this dynamic problem numerically. Plotting the policy functions of the asset accumulation in Figure 1, I find that the policy function for the high shock has no intersection with 45^0 line. Thus there is no upper bound for the asset accumulation in this economy.



The utility function in this example represents a quasilinear preference. There is no income effect for consumption on the leisure. All of the income effect is reflected on the consumption level. The leisure level can be determined only by the labor-leisure decision equation itself. In this example the leisure demand in both high shock sate and lower shock state is strictly less than one. Then the agent always supply labor no matter what state of shock realizes and no matter how rich he is. The agent always faces the income fluctuation shocks. This gives the agent a strong precautionary savings incentive. Then the agent wants to accumulate assets until he has infinite assets then he will not care about the income risk. Actually any utility function representing the quasilinear preference can work as the counterexample by this reason. Then the GHH utility functions, first introduced into the real-business-cycle literature by Greenwood, Hercowitz and Huffman (1988), can serve as the counterexamples.

2.3 The case of $\beta R < 1$

When $\beta R < 1$, agent's optimal decision rules have interesting asymptotic properties, as in Aiyagari (1994) and Huggett (1993). We present the characteristics of the agent's policy functions, which determine the long-run behavior of the economy. We then investigate the stochastic stability of the dynamic system for $\beta R < 1$.

The following lemma says that for any state of asset, there is a state of labor efficiency in which the agent dissaves. This implies that there is positive probability for the agent's asset to be lower than a given level in finite steps.

Lemma 4 *For $a > 0$, $a(a,e) < a$ for some $e \in E \equiv \{e^1, e^2, \cdots, e^n\}$, when $\beta R < 1$.*

Lemma 5 *For sufficiently large a, $a(a,e) < a$, $\forall e \in E$, when $\beta R < 1$.*

When $\beta R < 1$, I find a sufficient condition to guarantee that there is an upper bound for the asset accumulation. This condition is different from the conditions for the income fluctuation problem with exogenous labor supply. In the literature, the usual condition is the boundedness of the coefficient of risk aversion. My sufficient condition is $u_{12}(c,h) \geq 0$. I do not need the restriction on the coefficient of the risk aversion. Under my condition, when the agent has a sufficient large asset, he does not supply labor and then he acts as in the deterministic situation. In the deterministic situation, $\beta R < 1$ implies that the agent dissaves when he has a sufficiently large asset.

Let $\hat{a}(a) = \max_{e \in E}\{a(a,e)\}$. Then $\hat{a}(a)$ is continuous in a, since $a(a,e)$ is continuous in a by Proposition 2.2. By Lemma 5 we have $\hat{a}(a) < a$ for any sufficiently large a. This guarantees that there is an upper bond for asset accumulation. Assumption A5$'$ is a sufficient condition to guarantee the existence of this upper bound. Our condition is different from that of the existing literature. In the literature, such as Schechtman and Escudero (1977), Clarida (1987) and Aiyagari (1994), bounded coefficient of relative risk aversion is used to guarantee that there is an upper bound of the asset accumulation. We find another sufficient condition, the complementarity between consumption and leisure, to guarantee the existence of the upper bound of the asset accumulation. The difference between our situation and theirs is that labor supply is endogenous in our model. If consumption and leisure are complementary, when the agent accumulate enough asset, her labor supply will be zero and she will stop accumulating asset because of impatience. Lemma 5 implies that $\hat{a}(a)$ has a fixed point as a function of a. Let $\bar{a}$ be the smallest fixed point of $\hat{a}(a)$.

Lemma 6 *When $\beta R < 1$, $\exists \tilde{e} \in E$, and $\tilde{a} > 0$, $a(a, \tilde{e}) = 0$, for $a \in [0, \tilde{a}]$.*

State $(0, \tilde{e})$ plays a crucial role in the following results of the process $\{(a_t, e_t)\}_{t=0}^{\infty}$, which prove the existence and uniqueness of the stationary distribution and characterize the stationary distribution.

Lemmas 4-6 show that the decision rules of the asset accumulation have the shape in Figure 2.

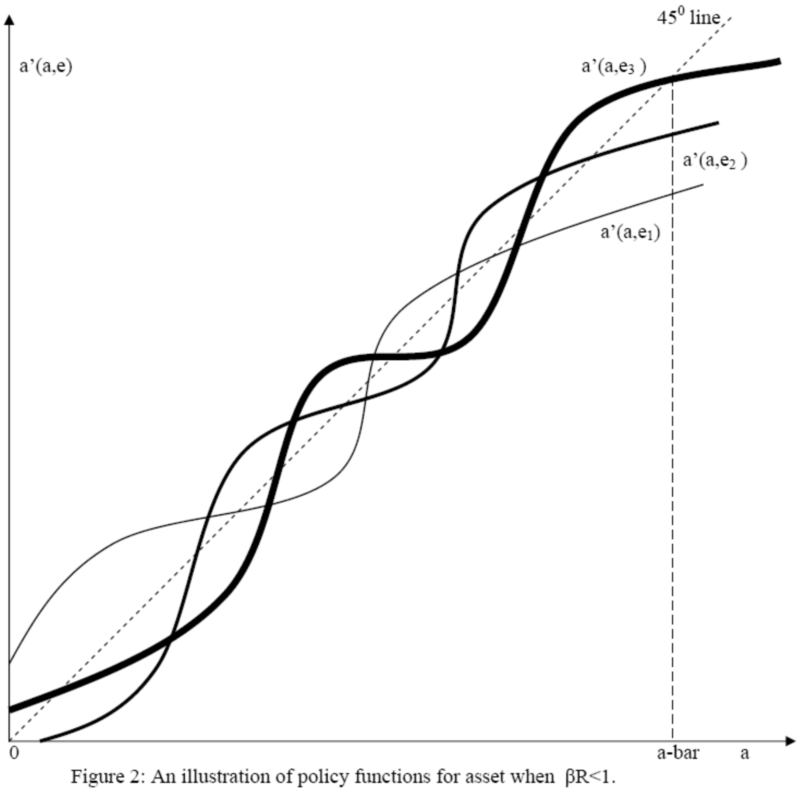


Figure 2: An illustration of policy functions for asset when βR<1.

These lemmas imply that there is $S = [0, \bar{a}] \times E$ such that if an agent starts with state s in S, then the agent stays in S. If an agent starts outside

S, the agent will finally arrive in S almost surely. Lemma 6 shows that if an agent starts with $s = (a, e)$ in S, next period's state must also be in S because $a(a, e) \leq \hat{a}(a) \leq \hat{a}(\bar{a}) = \bar{a}$. Lemma 5 shows that if an agent starts somewhere outside S, there is positive probability for her to reach S in finite steps. Then S is the unique ergodic set. I define the transition function P as

$$P((a,e), A \times \{e'\}) = \begin{cases} \pi(e'|e) & \text{if } a(a,e) \in A \\ 0 & \text{otherwise} \end{cases},$$

for all $(a, e) \in S$, $A \times \{e'\} \in \mathbf{B}(S)$.

Lemmas 4-6 imply that an agent, who starts with any positive asset, has a positive probability to hit the lower bound of the asset space in finite steps. This is the crucial point for us to prove Theorem 3.

Theorem 3 *When $\beta R < 1$, the process $\{(a_t, e_t)\}_{t=0}^{\infty}$ is uniformly ergodic. Precisely, $\{(a_t, e_t)\}_{t=0}^{\infty}$ has a unique stationary distribution μ on S, and moreover, for some $\rho \in (0, 1)$*

$$||P^n(s, \cdot) - \mu|| \leq \rho^n \quad \forall s \in S$$

where $|| \cdot ||$ is the total variation norm.

In the literature researchers usually use the Monotone Markov Processes method to prove the existence and uniqueness of the stationary distribution. Aiyagari (1993) prove the result for *i.i.d.* shocks. Huggett (1993) prove the result for two-state Markov chain shocks. For multiple-state Markov chain shocks, the monotonicity condition is very restrictive. And also it is impossible to verify the monotonicity condition for the joint Markov chain of the state variables including exogenous shocks and endogenous asset variable. The method we use here does not require the monotonicity of the Markov chain of the state variables. The crucial observation is that the lower bound of the state space is an accessible atom. Starting from any asset level, the state variables have positive probability to hit the lower bound in finite periods. This follows from the boundedness of the marginal utility of asset, $V_1(a, e)$, and $\beta R < 1$. The role of

Assumption $5'$ is to guarantee the existence of the upper bound of the asset accumulation. Assumption $5'$ is only a sufficient condition. There may be other conditions guaranteeing that the state space is compact. For any problem with a compact state space and bounded $V_1(a,e)$, Lemma 8 should go through for finite-state Markov shocks. The proof of Theorem 3 provides a new method to show the existence and uniqueness of the stationary distribution for the Markov dynamic systems. This method can apply to more general economic situations.

After I weaken the monotonicity of the Markov chain shocks, the results become more applicable in the simulation exercises. In the simulation, researchers use the Tauchen method, first introduced by Tauchen (1986), to approximate the autoregressive processes estimated from the real economic data. The Tauchen method chooses values for the state variables and the transition probabilities so that the resulting finite-state Markov chain mimics closely an underlying continuous-valued autoregression. A discrete-valued multiple-state Markov chain surely fits the autoregressive processes better than a two-state Markov chain.

Proposition 4 *Let μ be the unique stationary distribution for $\{(a_t, e_t)\}_{t=0}^{\infty}$. Then the Law of Large Numbers (LLN) holds for any $\mathbf{B}(S)-$measurable function f satisfying $\int_S |f| d\mu < +\infty$, i.e.*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(a_k, e_k) = \int_S f d\mu \qquad \mathbf{P}_\mu - a.s.$$

Since S is compact, any continuous function is integrable with respect to the probability measure. Then all the moments of the asset satisfy the Law of Large Numbers. This result implies that in order to compute the mean of wealth in the stationary economy we do not have to simulate the asset accumulation processes for a lot of household to find the approximate stationary distribution and then compute the mean value of the asset. We can simulate an asset accumulation process for one household for a long enough period and then compute the sample path mean of the asset to approximate the cross section mean in the stationary distribution.[12] Proposition

[12] A question may be raised: How long is long enough? There are two practical answers

4 shows that the path mean converges to the cross section mean almost surely.

Proposition 5 *Let μ be the unique stationary distribution for $\{(a_t, e_t)\}_{t=0}^{\infty}$. $\mu(\{s = (a, e) : a = 0\}) > 0$, i.e. the lower bound of the asset space is a mass point in the stationary distribution.*

Schechtman and Escudero (1977) show that the bounded coefficient of relative risk aversion can guarantee that the asset accumulation has an upper bound if the agent faces income fluctuation shocks. They also gave a counterexample to show that asset accumulation can approaches infinity almost surely even though $\beta R < 1$. Assumption A5$'$ is a sufficient condition guaranteeing the existence of the upper bound for the asset accumulation when labor supply is endogenous. We also find that if Assumption A5$'$ does not hold, it seems that there exist counterexamples. We run a simulation for a specific utility function,

$$u(c, h) = -e^{-(c+\sqrt{h})}.$$

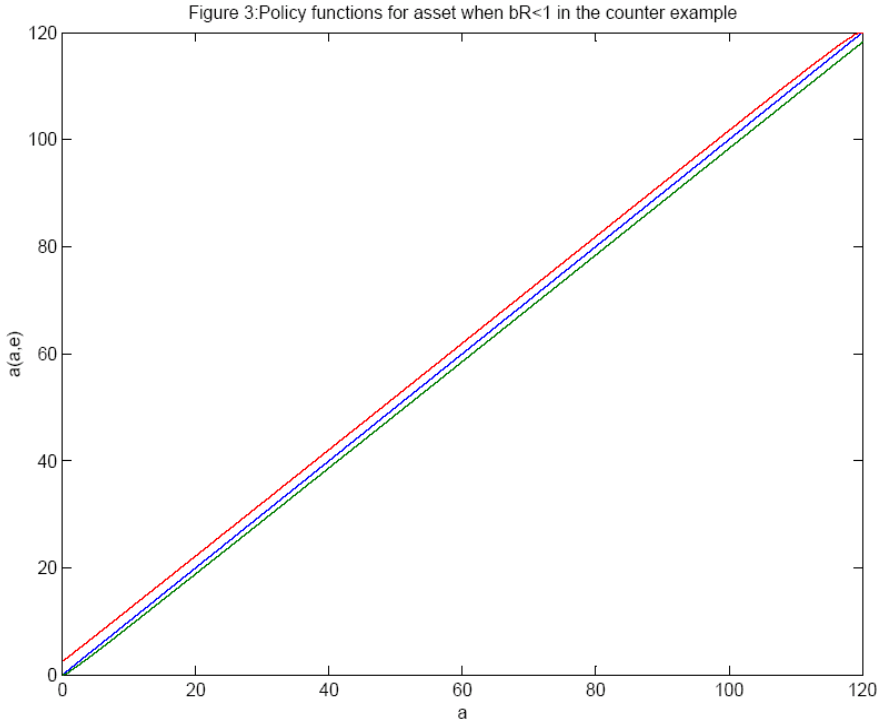
under a two-state Markov chain shock. This utility function does not satisfy our complementarity assumption A5$'$ since $u_{12}(c, h) < 0$ for $\forall c \geq 0$ and $h \in [0, 1]$. We set the parameters as: $\beta = 0.96$, $e \in E = \{0.585, 4.248\}$, $R = 1.0217$ and wage $w = 1.1781$. The parameters satisfy $\beta R < 1$. The transition probability of the shocks is

$$\pi = \begin{pmatrix} 0.9181 & 0.0819 \\ 0.0819 & 0.9181 \end{pmatrix}$$

Figure 3 shows that the asset accumulation function of the high shock has no intersection with 45^0 line. Thus there is no upper bound for the asset

to this question: 1) Set an arbitrary large number, say 3000, for simulation period, and after simulating for those periods, then 2) Set an arbitrary small convergence criterion to test whether path average of the simulated data for two consecutive periods is smaller than that convergence creterion. If the test result is true, the simulation can stop.

accumulation in this economy.



As we have discussed in section 2.2., this utility function implies that the agent's labor supply is fixed in a given shock state. The agent always faces the income fluctuation shocks. And the exponential utility function has an unbounded coefficient of relative risk aversion. This gives the agent such a strong precautionary savings incentive that the asset accumulation approaches infinity.

3 The general equilibrium

There are a continuum of households with measure 1 in the economy. Each household faces an income fluctuation problem with endogenous labor supply. There is uncertainty at the individual level but there is no aggregate uncertainty.[13] There is a single firm in the economy.

[13]To facilate the Law of Large Numbers in the economy with a continuum of households with measure 1, I use the construction in Sun (2006).

3.1 The firm's problem

The single firm has an aggregate production function $F(K, L)$ satisfying

Assumption 6: F displays constant returns to scale, with $F_1, F_2 > 0$, and $F_{11}, F_{22} < 0$. And F satisfies Inada conditions $\lim_{K\to+\infty} F_1(K, 1) = 0$ and $\lim_{K\to 0} F_1(K, 1) = +\infty$.

The firm maximizes its profits in each period. It rents capital and hires labor from competitive markets. Thus its profit maximization problem is

$$\max_{K,L} \; \{F(K, L) - (r + \delta)K - wL\}$$

where $\delta \in (0, 1)$ is the depreciation rate of capital. The first-order conditions are

$$F_1(K, L) = r + \delta \tag{4}$$

and

$$F_2(K, L) = w. \tag{5}$$

By the property of constant returns to scale, Equation (4) implies that

$$F_1\left(\frac{K}{L}, 1\right) = r + \delta \tag{6}$$

and Equation (5) implies that

$$F_2\left(\frac{K}{L}, 1\right) = w. \tag{7}$$

Equation (6) determines a negative relationship between the capital-labor ratio $\frac{K}{L}$ and the interest rate r by Assumption A6. From Equations (6) and (7) we see that the wage rate w has a one-to-one relationship with the interest rate r.

3.2 The stationary equilibrium

Let μ be a probability measure (time-invariant in a stationary equilibrium) defined on $(S, \mathbf{B}(S))$ where $S = [0, \bar{a}] \times E$ and $\mathbf{B}(S)$ is the Borel σ-algebra on S. Thus, for $B \in \mathbf{B}(S)$, $\mu(B)$ indicates the mass of agents whose individual state is lies in B. $P(s, B)$ is the transition function giving the probability that an agent in individual state s at time t will have an individual state that lies in the set $B \in \mathbf{B}(S)$ next period. Let $K(\mu)$ and $L(\mu)$ denote the aggregate stock of capital and labor as a function of the distribution μ. I then introduce the definition of stationary competitive equilibrium for the economy.

Definition 1 *A stationary competitive recursive competitive equilibrium with incomplete markets is a list of functions* $(c(s), h(s), a(s), \mu, K, L)$ *and a pair of prices* (r, w) *such that:*

(1) $c(s), h(s), a(s)$ *are optimal decision rules given* (r, w)*.*

(2) (r, w) *satisfy firm's profit-maximization conditions.*

(3) Market clearing conditions are satisfied.

(i) $\int_S a(s)d\mu = K(\mu)$,

(ii) $\int_S e(1 - h(s))d\mu = L(\mu)$.

(4) μ *is a stationary distribution under the transition function* P *implied by the household's decision rules. Formally,* μ *satisfies*

$$\mu(B) = \int_S P(s, B)d\mu \quad \forall\ B \in \mathbf{B}(S).$$

When $\beta R > 1$, agents' assets converge to infinity by Theorem 1. Therefore there is no general equilibrium. When $\beta R = 1$, agents' labor supplies converge to zero by Theorem 2. Therefore there is no general equilibrium. Thus we must have $\beta R < 1$ in a general equilibrium. We obtain the following result similar to Proposition 3 of Marcet et al. (2007).

Theorem 4 *In a stationary equilibrium under incomplete markets,* $\beta R < 1$*.*

The interest rate has to be smaller than the time discount rate. While in the equilibrium of complete markets, $\beta R = 1$. Thus the capital-labor ratio is higher under incomplete markets than complete markets.

3.2.1 The existence of the stationary equilibrium

From the firm's profit-maximization conditions, we can view the wage rate w as a function of the interest rate r. For a general equilibrium, we only need to find the equilibrium interest rate r.

In household's problem, for a given r, there is a unique stationary distribution for the state variables. The stationary distribution can be expressed as $\mu(r)$ to emphasize the its dependence on the interest rate r. We compute the aggregate asset with respect to $\mu(r)$ and obtain the aggregate capital supply. Similarly we can compute the aggregate labor supply. Thus we can define the "supply" of the capital-labor ratio as

$$\zeta(r) \equiv \frac{\int_S a d\mu(r)}{\int_S e(1-h(s;r))d\mu(r)}.$$

Note that $\zeta(r)$ is a function of only r since w is a function of r.

Lemma 7 is a general result for real functions. The proof of this lemma is provided by Professor Jushan Bai.

Lemma 7 *$\{f_n\}_{n=1}^{+\infty}$ is a sequence of weakly increasing functions on $[b,d] \subseteq R^1$. And $f_n(x) \to f(x)$ pointwisely. $f(x)$ is continuous. Then f_n uniformly converges to f.*

Lemma 8 *If $\mu(r_n)$ weakly converges to $\mu(r)$ as $r_n \to r$. Then $L(r_n) = \int_S e(1-h(s;r_n))d\mu(r_n) \to L(r) = \int_S e(1-h(s;r))d\mu(r)$ as $r_n \to r$.*

If $\mu(r_n)$ weakly converges to $\mu(r)$ as $r_n \to r$, Lemma 8 implies that the aggregate labor supply moves continuously with respect to r. At the same time, the aggregate capital supply is a continuous function of r by the definition of the weak convergence.

Lemma 9 *When $\beta R < 1$, $\zeta(r)$ moves continuously with respect to r and $\limsup \zeta(r_n) = +\infty$ as $r_n \uparrow (\frac{1}{\beta} - 1)$.*

We derive the "demand" for the capital-labor ratio, $D(r)$, from Equation (6) of Section 3.1. The "demand" curve $D(r)$ approaches the horizontal line of $r = -\delta$, as the capital-labor ratio tends to infinity. We combine the "supply" and "demand" curves of the capital-labor ration to determine the equilibrium interest rate r and the equilibrium capital-labor ratio $\frac{K}{L}$.

Theorem 5 *There exists a stationary equilibrium.*

Proof. Lemma 9 implies that $\zeta(r)$ moves continuously with respect to r. And there exists a sequence of $\{r_n\}_{n=1}^{+\infty}$ such that $\zeta(r_n)$ tends to infinity as $r_n \uparrow (\frac{1}{\beta}-1)$. The firm's profit-maximization problem gives us a downward continuous curve of $\frac{K}{L}$ and r. There exists at least an intersection of these two curves. ∎

The "supply" and "demand" curves of the capital-labor ratio are illustrated in Figure 4.

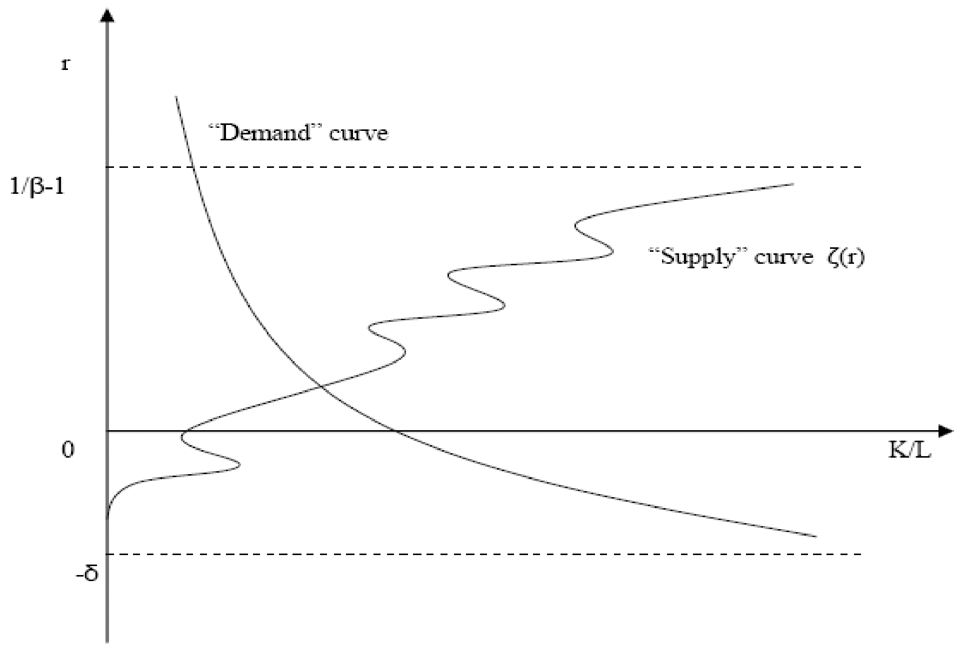


Figure 4: "Supply" and "Demand" curves of capital-labor ratio.

My proof of the existence of the stationary equilibrium also shows that a bisection algorithm can find a general equilibrium. We can follow the following bisection algorithm to find the equilibrium r^*.

Step (1): Guess initial $r_1 > -\delta$ and close to $-\delta$, and $r_2 < \frac{1}{\beta} - 1$ and close to $\frac{1}{\beta} - 1$.

Step (2): Set

$$r_3 = \frac{r_1 + r_2}{2}.$$

Step (3): If $D(r_3) - \zeta(r_3) > 0$, let $r_1 = r_3$. Otherwise let $r_2 = r_3$.

Step (4): If $r_2 - r_1 < \varepsilon$, stop the algorithm and let $r^* = r_3$. Otherwise go back to step (2).

Thus my paper gives guidances to simulation works of incomplete-market models with endogenous labor supply.

4 Conclusion

This paper shows that under the condition of complementarity of consumption and leisure, the asset accumulation is bounded. I first investigate an income fluctuation problem with endogenous labor supply. To show the existence of the stationary equilibrium, I find the intersection of the "supply" and "demand" curves of the capital-labor ratio in the economy. The aggregate capital supply is the total wealth of households in the stationary distribution of state variables. The aggregate labor supply is the total labor supply in the stationary distribution of state variables. The "supply" curve of the capital-labor ratio is the ratio of the aggregate capital supply to the aggregate labor supply. I show that the "supply" curve is a continuous function of the interest rate r and tends to infinity as r approaches $\frac{1}{\beta} - 1$ from below. The "demand" curve of the ratio is from the optimal conditions of the firm's profit-maximization problem. My proof of the existence of the stationary equilibrium also shows that a bisection algorithm can find a general equilibrium. Thus my paper gives guidances to simulation works of incomplete-market models with endogenous labor supply.

This paper concludes that the "supply" curve of the capital-labor ratio is continuous. It is still not clear whether this curve is monotone. Thus the uniqueness of the equilibrium is ambiguous. The uniqueness is an important starting point to study the movement of the equilibrium variables, such as aggregate production and consumption, when the fundamentals of the economy, such as preferences and technology, change. In the future we could find conditions which determines the uniqueness of the equilibrium.

In this paper we assume that the interest rate is deterministic. We show that the asset accumulation is bounded in the general equilibrium. Setting a Bewley model with stochastic investment returns Benhabib, Bisin, and Zhu (2015) show that the stationary wealth distribution has unbounded support. In the future we could also investigate an income fluctuation problem with endogenous labor supply and stochastic investment returns.

5 Appendix

5.1 Proof of Proposition 1

Proof:

1) $J(y,e)$ is bounded since $u(c,h)$ is bounded.

2) Suppose that $y_1 < y_2$. $c(y_1,e)$ and $h(y_1,e)$ are the optimal choices for the intratemporal problem. Then

$$\begin{aligned} J(y_1,e) &= u(c(y_1,e),h(y_1,e)) \\ &< u(c(y_1,e)+y_2-y_1,h(y_1,e)) \\ &\leq J(y_2,e). \end{aligned}$$

For any y_1 and y_2, and $\lambda \in (0,1)$, we have

$$\begin{aligned} & J(\lambda y_1+(1-\lambda)y_2,e) \\ \geq\ & u(\lambda c(y_1,e)+(1-\lambda)c(y_2,e),\lambda h(y_1,e)+(1-\lambda)h(y_2,e)) \\ >\ & \lambda u(c(y_1,e),h(y_1,e))+(1-\lambda)u(c(y_2,e),h(y_2,e)) \\ =\ & \lambda J(y_1,e)+(1-\lambda)J(y_2,e). \end{aligned}$$

3) To prove that $J(y,e)$ is differentiable at $y_0 \in (0,+\infty)$, note that A2 implies that $c_0 > 0$ which in turn means that $y_0 - h(y_0,e)ew > 0$. Then for any y belonging to a neighborhood D of y_0, $h(y_0,e)$ is still feasible. Define $H(y,e)$ on D as

$$H(y,e) = u(y-h(y_0,e)ew,h(y_0,e)).$$

Thus $H(y,e)$ is concave and differentiable in y. Since $h(y_0,e)$ is still feasible for all $y \in D$, it follows that

$$H(y,e) \leq \max_{h\in[0,1]} u(y-hew,h) = J(y,e), \text{ all } y \in D,$$

with equality at y_0.

Now any subgradient p of J at y_0 must satisfy

$$p(y-y_0) \geq J(y,e)-J(y_0,e) \geq H(y,e)-H(y_0,e), \text{ all } y \in D,$$

where the first inequality uses the definition of a subgradient and the second uses the fact that $H(y,e) \leq J(y,e)$, with equality at y_0. Since $H(y,e)$ is differentiable at y_0, p is unique. By Theorem 25.1 of Rockafellar (1970), any concave function with a unique subgradient at an interior point y_0 is differentiable at y_0. Then $J(y,e)$ is differentiable at y_0.

Furthermore, we know that $J_1(y_0,e) = H_1(y_0,e) = u_1(c(y_0,e), h(y_0,e))$ for $y_0 \in (0,+\infty)$. By the Theorem of the Maximum, $c(y_0,e)$ and $h(y_0,e)$ are continuous in $y_0 \in [0,+\infty)$. Thus $J_1(y_0,e)$ is continuous in $y_0 \in (0,+\infty)$. ■

5.2 Proof of Proposition 2

Proof:

1) This is a direct result of the Theorems 9.6, 9.7 and 9.8 of Stokey and Lucas (1989).

2) To prove that $V(a,e)$ is differentiable at $a_0 \in (0,+\infty)$, note that Assumption A2 implies that $y_0 > 0$ which in turn means that $Ra_0 + ew - a(a_0,e) > 0$. Then for any a belonging to a neighborhood D of a_0, $a(a_0,e)$ is still feasible. Define $W(a,e)$ on D as

$$W(a,e) = J(Ra + ew - a(a_0,e), e) + \beta E[V(a(a_0,e), e')|e].$$

Then $W(a,e)$ is concave and differentiable in a. Since $(h(a_0,e), a(a_0,e))$ is still feasible for all $a \in D$, it follows that

$$W(a,e) \leq \max_{a' \in \Gamma(a,e)} \{J(Ra + ew - a', e) + \beta E[V(a',e')|e]\} = V(a,e), \text{ all } a \in D,$$

with equality at a_0.

Now any subgradient p of V at a_0 must satisfy

$$p(a - a_0) \geq V(a,e) - V(a_0,e) \geq W(a,e) - W(a_0,e), \text{ all } a \in D,$$

where the first inequality uses the definition of a subgradient and the second uses the fact that $W(a,e) \leq V(a,e)$, with equality at a_0. Since $W(a,e)$ is differentiable at a_0, p is unique. By Theorem 25.1 of Rockafellar (1970), any concave function with a unique subgradient at an interior point a_0 is differentiable at a_0. Then $V(a,e)$ is differentiable at a_0. Furthermore, we know that $V_1(a_0,e) = W_1(a_0,e) = RJ_1(y(a_0,e),e)$ for $a_0 \in (0,+\infty)$.

By Proposition 6.7.4 of Florenzano and Van (2001), $V(a,e)$ is differentiable in $(0,+\infty)$ implies that $V(a,e)$ is continuously differentiable in $(0,+\infty)$. We know that $V(a,e)$ is continuous in a from 1). By Proposition 6.7.4 of Florenzano and Van (2001), we know that $\lim_{a\to 0} V_1(a,e) = V_1^+(0,e)$. By the Theorem of the Maximum, $y(a,e)$ is continuous in $a \in [0,+\infty)$. Let $a_0 \to 0$ on the both sides of $V_1(a_0,e) = RJ_1(y(a_0,e),e)$. We have $V_1^+(0,e) = RJ_1(y(0,e),e)$. Then $V(a,e)$ is continuously differentiable and $V_1(a,e) = RJ_1(y(a,e),e)$, $\forall a \in [0,+\infty)$[14].

3) By the Theorem of the Maximum, $a(a,e)$ is continuous in a. $\forall 0 \leq a_1 < a_2$, either $a(a_1,e) = 0$ or $a(a_1,e) > 0$. If the first occurs, $a(a_1,e) \leq a(a_2,e)$.

If $a(a_1,e) > 0$, we have $V_1(a_1,e) = \beta RE[V_1(a(a_1,e),e')|e]$. $a(a_2,e) > 0$. Suppose that $a(a_1,e) > a(a_2,e)$. $V_1(a_1,e) = \beta RE[V_1(a(a_1,e),e')|e] < \beta RE[V_1(a(a_2,e),e')|e] \leq V_1(a_2,e)$, a contradiction with the fact that $V(a,e)$ is strictly concave in a. Then we have $a(a_1,e) \leq a(a_2,e)$.

4) This result follows directly from 2). ■

5.3 Proof of Proposition 3

Proof:

1) $V(a,e)$ is bounded. We know that $V_1(a,e) \to 0$, when $a \to +\infty$. Then $u_1(c(a,e),h(a,e)) \to 0$, when $a \to +\infty$, since $V_1(a,e) = Ru_1(c(a,e),h(a,e))$. Suppose that $h(a,e) < 1\, \forall a > 0$. Then $u_2(c(a,e),h(a,e)) = u_1(c(a,e),h(a,e))ew$. We have $\lim_{a\to+\infty} u_2(c(a,e),h(a,e)) = 0$.

On the other hand, for some $\hat{a} > 0$, by lemma (10), $c(a,e) \geq c(\hat{a},e)$ when $a > \hat{a}$. Then $u_2(c(a,e),h(a,e)) \geq u_2(c(\hat{a},e),h(a,e))$ for $a > \hat{a}$, since $u_{12} \geq 0$. $h(a,e)$ is increasing in a, and is bounded in $[0,1]$. Then $\exists \bar{h} \in [0,1]$, such that $h(a,e) \to \bar{h}$, when $a \to +\infty$. We have $\lim_{a\to+\infty} u_2(c(a,e),h(a,e)) \geq \lim_{a\to+\infty} u_2(c(\hat{a},e),h(a,e)) = u_2(c(\hat{a},e),\bar{h}) > 0$. This contradicts with $\lim_{a\to+\infty} u_2(c(a,e),h(a,e)) = 0$. Then we have $h(a,e) = 1$ when a is sufficiently large.

2) Applying Proposition 2.4 and Assumption A5. ■

[14] Again, we use $V_1(0,e)$ to represent $V_1^+(0,e)$.

5.4 Proof of Theorem 1

Proof. We know that $V_1(a_t, e_t)$ is finite since $V_1(a_t, e_t) = Ru_1(c_t, h_t)$. Let $d_t = (\beta R)^t V_1(a_t, e_t)$. The Euler equation implies that $\{d_t\}_{t=0}^{+\infty}$ is a nonnegative supermartingale. But since $d_0 = V_1(a_0, e_0)$, it follows that there exists a random variable d_∞ with $E[d_\infty] < V_1(a_0, e_0)$ such that $\lim_{t\to+\infty} d_t = d_\infty$ $a.s.$. Then $\lim_{t\to+\infty}(\beta R)^t V_1(a_t, e_t) = d_\infty$ $a.s.$. Since $\beta R > 1$ we have $\lim_{t\to+\infty} V_1(a_t, e_t) = 0$ $a.s.$.

Now suppose that $a_t \nrightarrow +\infty$ with positive probability. Let $D = \{\omega : a_t(\omega) \nrightarrow +\infty\ \}$. $P(D) > 0$. For each $\omega \in D$, there exists a bounded subsequence $\{a_{t_k}(\omega)\}_{k=1}^{+\infty}$. $\exists\ B(\omega) > 0$, $a_{t_k}(\omega) < B(\omega)$. We know that $V_1(a_{t_k}(\omega), e_{t_k}(\omega)) \to 0$ except a zero-measure subset of D. Pick ω in D but not in the previous zero-measure subset of D. For convenience I omit ω in the following derivation. Thus $V_1(a_{t_k}, e_{t_k}) \geq V_1(B, e_{t_k}) \geq \min_{e\in E}\{V_1(B, e)\} > 0$, a contradiction with $\lim_{k\to+\infty} V_1(a_{t_k}, e_{t_k}) = 0$. ■

5.5 Proof of Lemma 2

Proof: Pick $\bar{e}$, such that $g(\phi, \bar{e}) < 1$. We have

$$\frac{\partial \chi(\phi, \bar{e})}{\partial e} = -\frac{u_{12}u_1 - u_{11}u_2}{u_{11}u_{22} - u_{12}^2}\frac{\lambda w}{u_1} - (1 - g(\phi, \bar{e}))w < 0,$$

by Assumptions A2 and A5.[15] Thus $\chi(\phi, \hat{e}) > \chi(\phi, \check{e})$. ■

5.6 Proof of Lemma 3

Proof: Denote $\bar{P} = \min_{(e,e')\in E\times E} \pi(e'|e)$. Choose T such that $\beta^T < \frac{1}{4}$.Let $\varepsilon = \min\{\bar{P}^T, \frac{\beta^2}{1-\beta}\frac{1}{4}(\chi(\phi, \hat{e}) - \chi(\phi, \check{e}))\}$.

Denote $\bar{\alpha} = \beta\chi(\phi, e_t) + \frac{\beta^2}{1-\beta}\frac{1}{2}(\chi(\phi, \hat{e}) + \chi(\phi, \check{e}))$. We then prove this lemma in two cases.

Case (i) $\alpha \leq \bar{\alpha}$.

[15] E has finite $e's$. We can extend E to a closed interval $[e_1, e_n]$ so that we can use derivative to do the comparative statics analysis.

Pick event $D_1 = \{e_t,\, e_{t+j-1} = \hat{e} \text{ for } j = 2, \cdots, T+1\}$. On D_1 we have

$$
\begin{aligned}
& \sum_{j=1}^{+\infty} \chi(\phi, e_{t+j-1})\beta^j \\
= \;& \beta\chi(\phi, e_t) + \sum_{j=2}^{+\infty} \chi(\phi, \hat{e})\beta^j + \sum_{j=T+2}^{+\infty} (\chi(\phi, e_{t+j-1}) - \chi(\phi, \hat{e}))\beta^j \\
\geq \;& \beta\chi(\phi, e_t) + \sum_{j=2}^{+\infty} \chi(\phi, \hat{e})\beta^j + \sum_{j=T+2}^{+\infty} (\chi(\phi, \check{e}) - \chi(\phi, \hat{e}))\beta^j \\
= \;& \beta\chi(\phi, e_t) + \sum_{j=2}^{+\infty}[\frac{1}{2}(\chi(\phi, \hat{e}) + \chi(\phi, \check{e}))]\beta^j + \sum_{j=2}^{+\infty}[\frac{1}{2}(\chi(\phi, \hat{e}) - \chi(\phi, \check{e}))]\beta^j \\
& - \sum_{j=T+2}^{+\infty} (\chi(\phi, \hat{e}) - \chi(\phi, \check{e}))\beta^j \\
\geq \;& \beta\chi(\phi, e_t) + \frac{\beta^2}{1-\beta}\frac{1}{2}(\chi(\phi, \hat{e}) + \chi(\phi, \check{e})) + \sum_{j=2}^{T+1}[\frac{1}{2}(\chi(\phi, \hat{e}) - \chi(\phi, \check{e}))]\beta^j \\
& - \sum_{j=T+2}^{+\infty} [\frac{1}{2}(\chi(\phi, \hat{e}) - \chi(\phi, \check{e}))]\beta^j \\
\geq \;& \bar{\alpha} + (1 - 2\beta^T)2\varepsilon \\
> \;& \bar{\alpha} + \varepsilon \\
\geq \;& \alpha + \varepsilon.
\end{aligned}
$$

We know $P(D_1|e_t) = P\left(e_{t+j-1} = \hat{e} \text{ for } j = 2, \cdots, T+1 | e_t\right) \geq \bar{P}^T \geq \varepsilon$. Then $P\left(\sum_{j=1}^{+\infty} \chi(\phi, e_{t+j-1})\beta^j > \alpha + \varepsilon | e_t\right) \geq P(D_1|e_t) \geq \varepsilon$. We have

$$
P\left(\alpha \leq \sum_{j=1}^{+\infty} \chi(\phi, e_{t+j-1})\beta^j \leq \alpha + \varepsilon | e_t\right) \leq 1 - \varepsilon.
$$

Case (ii) $\alpha > \bar{\alpha}$

Pick event $D_2 = \{e_t,\, e_{t+j-1} = \check{e} \text{ for } j = 2, \cdots, T+1\}$. On D_2 we have

$$\begin{aligned}
& \sum_{j=1}^{+\infty} \chi(\phi, e_{t+j-1})\beta^j \\
= \;& \beta\chi(\phi, e_t) + \sum_{j=2}^{+\infty} \chi(\phi, \check{e})\beta^j + \sum_{j=T+2}^{+\infty} (\chi(\phi, e_{t+j-1}) - \chi(\phi, \check{e}))\beta^j \\
\leq \;& \beta\chi(\phi, e_t) + \sum_{j=2}^{+\infty} \chi(\phi, \check{e})\beta^j + \sum_{j=T+2}^{+\infty} (\chi(\phi, \hat{e}) - \chi(\phi, \check{e}))\beta^j \\
= \;& \beta\chi(\phi, e_t) + \frac{\beta^2}{1-\beta}\chi(\phi, \check{e}) + \frac{\beta^2\beta^T}{1-\beta}(\chi(\phi, \hat{e}) - \chi(\phi, \check{e})) \\
\leq \;& \beta\chi(\phi, e_t) + \frac{\beta^2}{1-\beta}\chi(\phi, \check{e}) + \frac{\beta^2}{1-\beta}\frac{1}{2}(\chi(\phi, \hat{e}) - \chi(\phi, \check{e})) \\
= \;& \bar{\alpha} \\
< \;& \alpha.
\end{aligned}$$

We know $P(D_2|e_t) = P\left(e_{t+j-1} = \check{e} \text{ for } j = 2, \cdots, T+1|e_t\right) \geq \bar{P}^T \geq \varepsilon$. Then $P\left(\sum_{j=1}^{+\infty} \chi(\phi, e_{t+j-1})\beta^j < \alpha|e_t\right) \geq P(D_2|e_t) \geq \varepsilon$. We have

$$P\left(\alpha \leq \sum_{j=1}^{+\infty} \chi(\phi, e_{t+j-1})\beta^j \leq \alpha + \varepsilon|e_t\right) \leq 1 - \varepsilon.$$

■

5.7 Proof of Theorem 2

Proof: Suppose that $P(\lim_{t\to+\infty} V_1(a_t, e_t) \leq \bar{\lambda}) < 1$. Then there exists a $\psi > \bar{\lambda}$. For any $\delta > 0$. $P(\lim_{t\to+\infty} V_1(a_t, e_t) \in [\psi, \psi + \delta]) > 0$. For $\forall\varepsilon > 0$, let $\eta = \frac{1-\beta}{2\beta}\varepsilon$. Since $\xi(\lambda, e)$ and $g(\lambda, e)$ are uniformly continuous on any interval bounded away from zero. We may choose ϕ and δ, $\bar{\lambda} < \phi < \psi < \phi + \delta$, so that $P(\lim_{t\to+\infty} V_1(a_t, e_t) \in [\phi, \phi+\delta]) > 0$ and $P(\lim_{t\to+\infty} V_1(a_t, e_t) = \phi) = P(\lim_{t\to+\infty} V_1(a_t, e_t) = \phi + \delta) = 0$. At the same time $|\xi(\phi, e) - \xi(\phi + \delta, e)| < \frac{\eta}{2}$ and $|g(\phi, e) - g(\phi + \delta, e)|ew < \frac{\eta}{2}$ for $\forall e \in E$. Define $B = \{\lim_{t\to+\infty} V_1(a_t, e_t) \in [\phi, \phi+\delta]\}$. For $\tau \geq 0$. Define $A_\tau = \{V_1(a_\tau, e_\tau) \in [\phi, \phi+\delta]\}$ and $B_\tau = \{V_1(a_t, e_t) \in [\phi, \phi+\delta], t \geq \tau\}$. Then $\lim_{\tau\to+\infty} P(A_\tau) = P(B) > 0$ and $\lim_{\tau\to+\infty} P(B_\tau) = P(B) > 0$. We may choose $\tau < +\infty$ such that $P(B_\tau) > (1-\varepsilon)P(A_\tau) > 0$. We have

$$\begin{aligned} a_\tau &= \sum_{j=1}^{+\infty}[c_{\tau+j-1} - \xi(\phi, e_{\tau+j-1}) + [h_{\tau+j-1} - g(\phi, e_{\tau+j-1})]e_{\tau+j-1}w]R^{-j} \\ &\quad + \sum_{j=1}^{+\infty}[\xi(\phi, e_{\tau+j-1}) - (1 - g(\phi, e_{\tau+j-1}))e_{\tau+j-1}w]R^{-j}. \end{aligned}$$

Thus we have

$$P\left(\begin{array}{c} \sum_{j=1}^{+\infty}[c_{\tau+j-1} - \xi(\phi, e_{\tau+j-1}) + [h_{\tau+j-1} - g(\phi, e_{\tau+j-1})]e_{\tau+j-1}w]R^{-j} \\ + \sum_{j=1}^{+\infty}[\xi(\phi, e_{\tau+j-1}) - (1 - g(\phi, e_{\tau+j-1}))e_{\tau+j-1}w]R^{-j} - a_\tau = 0|B_\tau \end{array}\right) = 1.$$

and

$$P\left(|\sum_{j=1}^{+\infty}[c_{\tau+j-1} - \xi(\phi, e_{\tau+j-1}) + [h_{\tau+j-1} - g(\phi, e_{\tau+j-1})]e_{\tau+j-1}w]R^{-j}| < \frac{\beta}{1-\beta}\eta = \frac{\varepsilon}{2}|B_\tau\right) = 1.$$

Thus

$$P\left(|\sum_{j=1}^{+\infty}[\xi(\phi, e_{\tau+j-1}) - (1 - g(\phi, e_{\tau+j-1}))e_{\tau+j-1}w]R^{-j} - a_\tau| < \frac{\varepsilon}{2}|B_\tau\right) = 1.$$

Let $\alpha = a_\tau - \frac{\varepsilon}{2}$. Since $B_\tau \subseteq A_\tau$ and $P(B_\tau) > (1-\varepsilon)P(A_\tau)$, it follows that

$$P\left(\alpha < \sum_{j=1}^{+\infty}[\xi(\phi, e_{\tau+j-1}) - (1 - g(\phi, e_{\tau+j-1}))e_{\tau+j-1}w]R^{-j} < \alpha + \varepsilon|A_\tau\right) > 1-\varepsilon.$$

But then A_τ is measurable with respect to z^τ such that the event,

$$P\left(\alpha < \sum_{j=1}^{+\infty}[\xi(\phi, e_{\tau+j-1}) - (1 - g(\phi, e_{\tau+j-1}))e_{\tau+j-1}w]R^{-j} < \alpha + \varepsilon|z^\tau\right) > 1-\varepsilon,$$

has a positive probability. Since $\beta R = 1$ we have

$$P\left(\alpha < \sum_{j=1}^{+\infty}[\xi(\phi, e_{\tau+j-1}) - (1 - g(\phi, e_{\tau+j-1}))e_{\tau+j-1}w]\beta^{j} < \alpha + \varepsilon|z^\tau\right) > 1-\varepsilon.$$

Note that $\{e_t\}_{t=1}^{\infty}$ follows a Markov chain. Thus $\exists e_\tau$ such that

$$P\left(\alpha < \sum_{j=1}^{+\infty}[\xi(\phi, e_{\tau+j-1}) - (1 - g(\phi, e_{\tau+j-1}))e_{\tau+j-1}w]\beta^j < \alpha + \varepsilon | e_\tau\right) > 1-\varepsilon.$$

This contradicts with Lemma 3. ■

5.8 Proof of Corollary 2.1

Proof: From the proof of Theorem 2, we know that $V_1(a_t, e_t) \to R\bar{\lambda}$, if $a_0 \leq \bar{k}$. And $V_1(\bar{k}, e) = R\bar{\lambda}$, $\forall e \in E$. Then $a_t \to \bar{k}$, if $a_0 \leq \bar{k}$. ■

5.9 Proof of Lemma 4

Proof: For $a > 0$, suppose that $a(a, e) \geq a > 0$ for all $e \in E$, then $V_1(a, e) = \beta RE[V_1(a(a, e), e')|e]$ for all $e \in E$. By the strict concavity of $V(a, e)$ in a, we have $V_1(a, e) = \beta RE[V_1(a(a, e), e')|e] \leq \beta RE[V_1(a, e')|e] < E[V_1(a, e')|e]$. This is impossible since $0 < \pi(e'|e) < 1$ for all $(e, e') \in E \times E$. ■

5.10 Proof of Lemma 5

Proof: Suppose that $a(a, \hat{e}(a)) = \max_{e \in E}\{a(a, e)\} \geq a$, $\forall a > 0$. Then $V_1(a, \hat{e}) = \beta RE[V_1(a(a, \hat{e}), e')|\hat{e}] \leq \beta RE[V_1(a, e')|\hat{e}] < E[V_1(a, e')|\hat{e}]$.

When a is sufficiently large, $h(a, e) = 1$, $\forall e \in E$, by Proposition 3. The budget constraint becomes $a(a, e) + c(a, e) = Ra$. We have $c(a, e) \geq c(a, \hat{e})$. since $a(a, \hat{e}) = \max_{e \in E}\{a(a, e)\} \geq a(a, e)$. By Lemma 4 there exists at least one $\tilde{e} \in E$ such that $a(a, \tilde{e}) < a$. We have $c(a, \tilde{e}) > c(a, \hat{e})$ since $a(a, \hat{e}) \geq a > a(a, \tilde{e})$. This implies $V_1(a, \hat{e}) = Ru_1(c(a, \hat{e}), 1) > E[Ru_1(c(a, e'), 1)|\hat{e}] = E[V_1(a, e')|\hat{e}]$. We obtain a contradiction. ■

5.11 Proof of Lemma 6

Proof: Suppose that $a(a, e) > 0$ for $\forall e \in E$ and $a > 0$. Then $\forall t > 0$ we have

$$V_1(a_0, e_0) = (\beta R)^t E_0 V_1(a_t, e_t),$$

for $a_0 > 0$ and $e \in E$. This can not hold for all t since $V_1(a,e)$ is bounded and $\beta R < 1$.

Then $\exists \tilde{e} \in E$ and $\tilde{a} > 0$, $a(\tilde{a}, \tilde{e}) = 0$. By the monotonicity of $a(a, \tilde{e})$ from Proposition 2.2, we have $a(a, \tilde{e}) = 0$, for $a \in [0, \tilde{a}]$. ■

5.12 Proof of Theorem 3

Proof: By Theorem 16.0.2 of Meyn and Tweedie (1993), $\{(a_t, e_t)\}_{t=0}^{\infty}$ will be uniformly ergodic whenever the state space S is v_m-small for some m.

Definition 2 *Small sets*

A set $C \in \mathbf{B}(S)$ is called a small set if there exists an $m > 0$, and a non-trivial measure v_m on $\mathbf{B}(S)$, such that for all $s \in C$, $B \in \mathbf{B}(S)$, $P^m(s, B) \geq v_m(B)$.

Let $\check{a}(a) = \min_{e \in E}\{a(a,e)\}$. Then $\check{a}(a)$ is continuous in a, since $a(a,e)$ is continuous in a by Proposition 2.2. By Lemma 4, $\check{a}(a) < a$ for $\forall a > 0$. By Lemma 6, $\exists \tilde{a} > 0$, $\check{a}(a) = 0$, when $a \in [0, \tilde{a}]$. Let $\kappa = \min\{a - \check{a}(a) : a \in [\tilde{a}, \bar{a}]\}$. Then $\kappa > 0$. Let

$$m = \frac{\bar{a}}{\kappa} + 1$$

Let $\bar{P} = \min_{(e,e') \in E \times E} \pi(e'|e)$. Define a non-trivial measure v_m on $\mathbf{B}(S)$ as, for $\forall B \in \mathbf{B}(S)$,

$$v_m(B) \begin{cases} = \bar{P}^m & \text{if } (0, \tilde{e}) \in B \\ = 0 & \text{otherwise} \end{cases},$$

where $\tilde{e}$ is defined in Lemma 6.

For $\forall s \in S$, we can pick the realization sequence of work efficiency shocks e's such that (a,e) moves along $(\check{a}(a), e)$ to reach the state $s^* = (0, \tilde{e})$ in at most m steps. Then it is easy to see that for all $s \in S$, $B \in \mathbf{B}(S)$, $P^m(s, B) \geq v_m(B)$. We conclude that S is v_m-small.

Let

$$\rho = [1 - v_m(S)]^{1/m}.$$

Thus we obtain the results of Theorem 3 from Theorem 16.0.2 of Meyn and Tweedie (2005). ■

5.13 Proof of Proposition 4

Proof: By Theorem 17.0.1 of Meyn and Tweedie (1993), LLN holds whenever $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a positive Harris chain. Theorem 18.0.2 of Meyn and Tweedie (1993) says that $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a positive Harris chain if it satisfies the following three conditions,

1) $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a T-Chain,

2) There exists a reachable state s^*, and

3) $\{(a_t, e_t)\}_{t=0}^{\infty}$ is bounded.[16]

Condition 3) is obviously satisfied since S is compact. From the proof of of Theorem 3, we know that $s^* = (0, \tilde{e})$, where $\tilde{e}$ is defined in Lemma 6, is a reachable state. Then condition 2) is satisfied. We only need to verify condition 1), $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a T-Chain. Theorem 6.2.5 of Meyn and Tweedie (1993) says that if every compact set is petite, $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a T-Chain. A slight change of the proof for Lemma 6 can show that every compact set of S is a small set. By Proposition 5.5.3 of Meyn and Tweedie (1993), every small set is a petite set. Then $\{(a_t, e_t)\}_{t=0}^{\infty}$ is a T-Chain. ■

5.14 Proof of Proposition 5

Proof: Let $s^* = (0, \tilde{e})$, where $\tilde{e}$ is defined in Lemma 6. And let $\tau_{s^*} = \min\{t \geq 1 : (a_t, e_t) = s^*\}$. By Theorem 10.2.2 (Kac's theorem) of Meyn and Tweedie (1993), $E_{s^*}[\tau_{s^*}] < +\infty$, and $\mu(s^*) = (E_{s^*}[\tau_{s^*}])^{-1}$ whenever $\{(a_t, e_t)\}_{t=0}^{\infty}$ is positive recurrent. From the proof of Proposition 4 we know that $\{(a_t, e_t)\}_{t=0}^{\infty}$ is positive Harris recurrent which implies that it is positive recurrent. Thus $\mu(\{s = (a, e) : a = 0\}) \geq \mu(s^*) = (E_{s^*}[\tau_{s^*}])^{-1} > 0$. ■

5.15 Proof of Lemma 7

Proof: $\forall \varepsilon > 0$, there exists a subdivision of $[b, d]$, $b = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m(\varepsilon)} = d$ such that $0 \leq f(\xi_{i+1}) - f(\xi_i) < \frac{\varepsilon}{2}$, $i = 0, 1 \cdots\ m(\varepsilon)$. $\forall x \in [b, d]$, $\exists i(x)$, $0 \leq i(x) < m(\varepsilon)$ such that $\xi_{i(x)} \leq x \leq \xi_{i(x)+1}$. $f_n(\xi_{i(x)}) - f(x) \leq f_n(x) - f(x) \leq f_n(\xi_{i(x)+1}) - f(x)$. Then $|f_n(x) - f(x)| \leq \max\{|f_n(\xi_{i(x)}) - f(x)|,\ |f_n(\xi_{i(x)+1}) - f(x)|\}$. $\forall i$, $0 \leq i \leq m(\varepsilon)$, $\exists N_i$, such that $n > N_i$, $|f_n(\xi_i) - f(\xi_i)| < \frac{\varepsilon}{2}$. Let $N = \max\{N_0,\ N_1,\ \cdots,\ N_{m(\varepsilon)}\}$, $n > N$, $|f_n(\xi_i) -$

[16] Actually the theorem only requires bounded in probability.

$f(\xi_i)| < \frac{\varepsilon}{2}$ for $\forall i$, $0 \le i \le m(\varepsilon)$. $|f_n(\xi_{i(x)}) - f(x)| \le |f_n(\xi_{i(x)}) - f(\xi_{i(x)})| + |f(\xi_{i(x)}) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Similarly, $|f_n(\xi_{i(x)+1}) - f(x)| \le \varepsilon$. Then we have $|f_n(x) - f(x)| < \varepsilon$ for $\forall x \in [b, d]$. ■

5.16 Proof of Lemma 8

Proof: Note that

$$\begin{aligned} L(r_n) &= \int_S e(1 - h(s; r_n)) d\mu(r_n) \\ &= \int_S e d\mu(r_n) - \int_S eh(s; r_n) d\mu(r_n). \end{aligned}$$

The first term converges to $\int_S e d\mu(r)$, as $r_n \to r$. We only need to show that $\int_S eh(a, e; r_n) d\mu(r_n) \to \int_S eh(a, e; r) d\mu(r)$ as $r_n \to r$. Note that

$$\begin{aligned} & |\int_S eh(a, e; r_n) d\mu(r_n) - \int_S eh(a, e; r) d\mu(r)| \\ \le & |\int_S eh(a, e; r_n) d\mu(r_n) - \int_S eh(a, e; r) d\mu(r_n)| \\ & + |\int_S eh(a, e; r) d\mu(r_n) - \int_S eh(a, e; r) d\mu(r)| \\ \le & \int_S e|h(a, e; r_n) - h(a, e; r)| d\mu(r_n) \\ & + |\int_S eh(a, e; r) d\mu(r_n) - \int_S eh(a, e_l; r) d\mu(r)|. \end{aligned}$$

By Lemma 7, $h(a, e; r_n)$ uniformly converges to $h(a, e; r)$, as $r_n \to r$. Then the first part converges to zero as $r_n \to r$. It follows from the definition of weak convergence of probability that the last part converges to zero as $r_n \to r$. ■

5.17 Proof of Lemma 9

Proof: By Theorem 12.13 of Stockey and Lucas (1989), the unique stationary distribution on S, μ behaves continuously with respect to r and w. By Lemma 8 and the remarks following it, we know that the capital-labor ratio $\zeta(r)$ is a continuous function of r.

When $r_n \uparrow (\frac{1}{\beta} - 1)$, we know that there exist a $\hat{k}$ such that $\bar{a}(r_n) < \hat{k}$

$\forall n \geq 1$ and $\bar{k} < \hat{k}$. Then for each r_n there exist a unique stationary distribution on $[0, \hat{k}]$ by extension of $\mu(r_n)$ on $[0, \bar{a}(r_n)]$. The unique stationary distribution on $[0, \hat{k}]$ is constructed by putting zero measure on $(\bar{a}(r_n), \hat{k}]$ and combining stationary distribution $\mu(r_n)$ on $[0, \bar{a}(r_n)]$. We still use $\mu(r_n)$ for the extended stationary distribution. A slight change of the proof for Theorem 12.13 of Stockey and Lucas (1989) can show that there exists a subsequence of $\{r_{n_k}\}_{k=1}^{\infty}$ such that $\mu(r_{n_k})$ weakly converges to a distribution μ_0. And μ_0 is a stationary distribution of the dynamic system of $r = \frac{1}{\beta} - 1$. From Theorem 2 and Corollary 2.1, we know that in any stationary of distribution for $r = \frac{1}{\beta} - 1$, the asset-labor ratio is infinity. Then $\limsup \zeta(r_n) = +\infty$ as $r_n \uparrow (\frac{1}{\beta} - 1)$. ■

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