Continuous-Time Stochastic Dynamic Programming

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1 Dynamic programming

1.1 Motivations of learning continuous-time stochastic dynamic programming

Why do we learn continuous-time stochastic dynamic programming? There are three reasons. The first one is that you have learned how to solve the other 3 cases of dynamic problems: discrete-time deterministic dynamic programming, discrete-time stochastic dynamic programming, and continuous-time deterministic optimization, and you may want to learn the fourth case: continuous-time stochastic dynamic programming. Here is a table representing this idea.

	Deterministic	Stochastic
Discrete time	✓	✓
Continuous time	✓	To learn

The second reason is that some literatures in macroeconomics and financial economics use this technique. The third reason is that this method has advantages of solving the optimal stopping time problem such as the optimal investment timing.

1.2 Using dynamic programming to solve a continuous-time deterministic problem

We first look at a continuous-time deterministic problem,

$$\max \int_0^T f(t,x,u)dt + \phi(x(T),T)$$

$$s.t.\;\; dx = g(t,x,u)dt, \quad x(0) = \bar{x}.$$

Here you can view $\phi(x(T),T)$ as a utility function of bequest. x is a state variable and u is a control variable.

Usually we use calculus of variations or optimal control to solve this problem. Here we use dynamic programming to solve this problem. The aim is to show the idea of the recursive method which plays a crucial role in solving the stochastic dynamic optimization.

The optimal value function is

$$J(t_0, x_0) = \max_u \int_{t_0}^{T} f(t, x, u)dt + \phi(x(T), T)$$

$$s.t. \quad dx = g(t, x, u)dt.$$

Now let's write the value function recursively,

$$\begin{aligned}
J(t,x) &= \max_u \int_t^T f(s,x,u)ds + \phi(x(T),T) \\
&= \max_u \left(\int_t^{t+\Delta t} f(s,x,u)ds + \int_{t+\Delta t}^T f(s,x,u)ds + \phi(x(T),T) \right) \\
&= \max_u \left(\int_t^{t+\Delta t} f(s,x,u)ds + \max \left(\int_{t+\Delta t}^T f(s,x,u)ds + \phi(x(T),T) \right) \right) \\
&= \max_u \left(\int_t^{t+\Delta t} f(s,x,u)ds + J(t+\Delta t, x+\Delta x) \right) \\
&= \max_u \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J(t+\Delta t, x+\Delta x) \right) \\
&= \max_u \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J(t,x) + J_t \Delta t + J_x \Delta x \right) \\
&= \max_u \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J(t,x) + J_t \Delta t + J_x g \Delta t \right).
\end{aligned}$$

Thus we have

$$0 = \max_u \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J_t \Delta t + J_x g \Delta t \right).$$

Dividing Δt on both sides and letting $\Delta t \to 0$, we have

$$0 = \max_u \left(f(t,x,u) + J_t + J_x g \right).$$

This equation is called the Hamilton-Jacobi-Bellman (HJB) equation.

For an infinite horizon problem with $f(t,x,u) = e^{-\beta t} h(x,u)$, we assume $J(t,x) = e^{-\beta t} V(x)$. The HJB equation becomes

$$0 = \max_u \left(e^{-\beta t} h(x,u) - \beta e^{-\beta t} V(x) + e^{-\beta t} V'(x) g \right).$$

Thus

$$\beta V(x) = \max_u \left(h(x,u) + V'(x) g \right).$$

1.3 An example

Consider the problem

$$\min \int_0^{\infty} e^{-rt} \left(ax^2 + bu^2\right) dt$$
$$s.t.\ dx = udt, \quad x(0) = \bar{x} > 0,$$

where $a > 0$, $b > 0$.

Let

$$V(x(t_0)) = \min_u \int_{t_0}^{\infty} e^{-r(t-t_0)} \left(ax^2 + bu^2\right) dt$$
$$s.t.\ dx = udt.$$

The HJB equation is

$$rV(x) = \min_u \left(ax^2 + bu^2 + V'(x)u\right). \tag{1}$$

The first-order condition is

$$2bu + V'(x) = 0.$$

Thus we have

$$u = -\frac{1}{2b} V'(x).$$

Guess that

$$V(x) = Cx^2. \tag{2}$$

Thus we have

$$V'(x) = 2Cx. \tag{3}$$

We have

$$u = -\frac{C}{b} x. \tag{4}$$

Plugging equations (2), (3), and (4) into equation (1), we have

$$\frac{C^2}{b} + rC - a = 0. \tag{5}$$

Then we can solve C from equation (5). We are finished.

2 Stochastic dynamic optimization

We want to solve an optimization problem

$$\max\ E_0 \int_0^T f(t, x, u) dt + \phi(x(T), T)$$
$$s.t.\ \ dx = g(t, x, u)dt + \sigma(t, x, u)dz, \quad x(0) = \bar{x},$$

where $z(t)$ is a standard Brownian motion. dz is the differential of a standard Brownian motion. Thus we need some basic knowledge of stochastic calculus.

Before we solve this problem, we review some preliminary knowledge of stochastic calculus.

2.1 Preliminary knowledge of stochastic calculus

2.1.1 Brownian motion

We first state the definition of Brownian motion. I copy the definition from Shreve (2004)'s textbook, *Stochastic Calculus for Finance II: Continuous-Time Models*.

Definition 1 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. For each* $\omega \in \Omega$*, suppose there is a continuous function* $W(t)$ *of* $t \geq 0$ *that satisfies* $W(0) = 0$ *and that depends on* ω*. Then* $W(t)$*,* $t \geq 0$*, is a Brownian motion if for all* $0 = t_0 < t_1 < \cdots < t_m$ *the increments*

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1),\ \cdots,\ W(t_m) - W(t_{m-1}),$$

are independent and each of these increments is normally distributed with

$$E[W(t_{i+1}) - W(t_i)] = 0,$$

and

$$Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

We could use Brownian motions to construct a diffusion process, $x(t)$,

$$dx = g(t, x)dt + \sigma(t, x)dz.$$

A diffusion process has continuous paths. The property of continuous paths is an advantage for the continuous-time stochastic model to study optimal stopping rules.

A Brownian motion has continuous paths. But a path of a Brownian motion is nowhere differentiable. But we have the definition of stochastic differential of a Brownian motion. Intuitively the differential of a Brownian motion is a stochastic variable which is different from the differential of the path of a Brownian motion. Again, a path of a Brownian motion is nowhere differentiable.

A Brownian motion is a mathematical model describing jiggling of pollen grains in water. I copy and past the following materials from website: http://www.einsteinyear.org/facts/brownian_motion/

Einstein Year - a year celebrating physics - Brownian Motion

did you know... Brownian Motion

In 1827 the biologist Robert Brown noticed that if you looked at pollen grains in water through a microscope, the pollen jiggles about. He called this jiggling 'Brownian motion', but Brown couldn't work out what was causing it. The first of the three papers that Einstein published in 1905 finally came up with an explanation.

Everything around us is made up of atoms and molecules: the chair you're sitting on, the food you eat, the air you're breathing. The idea of atoms has been around since the time of the ancient Greeks, and a century before Einstein, the great chemist John Dalton had suggested that all chemicals were made of tiny *invisible molecules, which in turn were made of even tinier atoms. The problem was that there was no proof of their existence, until Einstein looked into the problem of Brownian motion.*

Einstein realised that the jiggling of the pollen grains seen in Brownian motion was due to molecules of water hitting the tiny pollen grains, like players kicking the ball in a game of football. The pollen grains were visible but the water molecules weren't, so it looked like the grains were bouncing around on their own.

Einstein also showed that it was possible to work out how many molecules were hitting a single pollen grain and how fast the water molecules were moving - all by looking at the pollen grains.

Importantly, Einstein's paper also made predictions about the properties of atoms that could be tested. The French physicist Jean Perrin used Einstein's predictions to work out the size of atoms and remove any remaining doubts about the existence of atoms.

2.1.2 Stochastic calculus

For our application purposes, we only concentrate on Itô's formula.

Itô's formula Intuitively Itô's formula is a Taylor expansion formula for an expression with stochastic variables.

The following table plays crucial roles in deriving Itô's formula. I call it the magic table.

	dt	dz
dt	0	0
dz	0	dt

where dz is the differential of a standard Brownian motion. The table means

$$dt \times dt = 0,$$

$$dz \times dt = 0,$$

and

$$dz \times dz = dt.$$

The proper way to dealing with the magic table is to remember it as you remembered the multiplication table in your preliminary school.

The quadratic variation of a Brownian motion is t. Thus intuitively we have $dz(t)dz(t) = dt$. Another way to understand this relationship is to think that $dz(t)$ is a normal distribution with a mean of 0 and a variance of dt. Thus $var(dz(t)) = dt$ and $E(dz(t)) = 0$. Then we have $E\left((dz(t))^2\right) = var(dz(t)) = dt$. Thus we can think that $dz(t)$ has the order of $\sqrt{dt}$. So $dz(t)dt = 0$, since its order is higher than dt.

We learn Itô's formula from simple cases to complicated cases.

$y = F(t,\ z)$ The differential of y is

$$\begin{aligned} dy &= F_t dt + F_z dz + \frac{1}{2} F_{tt} (dt)^2 + F_{tz} dt dz + \frac{1}{2} F_{zz} (dz)^2 \\ &= F_t dt + F_z dz + \frac{1}{2} F_{zz} dt \\ &= (F_t + \frac{1}{2} F_{zz}) dt + F_z dz. \end{aligned}$$

Here we applied the magic table to derive the differential of y.

$y = F(t,\ x)$ Remember that

$$dx = g(t, x) dt + \sigma(t, x) dz.$$

The differential of y is

$$\begin{aligned} dy &= F_t dt + F_x dx + \frac{1}{2} F_{tt} (dt)^2 + F_{tx} dt dx + \frac{1}{2} F_{xx} (dx)^2 \\ &= F_t dt + F_x dx + F_{tx} dt dx + \frac{1}{2} F_{xx} (dx)^2 \\ &= F_t dt + F_x \left(g dt + \sigma dz \right) + F_{tx} dt \left(g dt + \sigma dz \right) + \frac{1}{2} F_{xx} \left(g dt + \sigma dz \right)^2 \\ &= F_t dt + F_x g dt + F_x \sigma dz + \frac{1}{2} F_{xx} \sigma^2 dt \\ &= \left(F_t + F_x g + \frac{1}{2} F_{xx} \sigma^2 \right) dt + F_x \sigma dz. \end{aligned}$$

We also applied the magic table to derive the differential of y.

$y = F(t,\ x_1,\ x_2)$ Suppose that $y = F(t,\ x_1,\ x_2)$ and

$$\begin{aligned} dx_1 &= g_1(t, x_1) dt + \sigma_1(t, x_1) dz_1, \\ dx_2 &= g_2(t, x_2) dt + \sigma_2(t, x_2) dz_2, \end{aligned}$$

and

$$dz_1 dz_2 = \rho dt.$$

The differential of y is

$$\begin{aligned} dy &= F_t dt + F_{x_1} dx_1 + F_{x_2} dx_2 \\ &\quad + \frac{1}{2} \left(F_{tt} (dt)^2 + F_{x_1 x_1} \left(dx_1 \right)^2 + F_{x_2 x_2} \left(dx_2 \right)^2 + 2 F_{t x_1} dt dx_1 + 2 F_{t x_2} dt dx_2 + 2 F_{x_1 x_2} dx_1 dx_2 \right) \\ &= F_t dt + F_{x_1} dx_1 + F_{x_2} dx_2 + \frac{1}{2} \left(F_{x_1 x_1} \left(dx_1 \right)^2 + F_{x_2 x_2} \left(dx_2 \right)^2 + 2 F_{x_1 x_2} dx_1 dx_2 \right) \\ &= F_t dt + F_{x_1} \left(g_1 dt + \sigma_1 dz_1 \right) + F_{x_2} \left(g_2 dt + \sigma_2 dz_2 \right) \\ &\quad + \frac{1}{2} \left(F_{x_1 x_1} \sigma_1^2 dt + F_{x_2 x_2} \sigma_2^2 dt + 2 F_{x_1 x_2} \sigma_1 \sigma_2 \rho dt \right) \\ &= \left(F_t + F_{x_1} g_1 + F_{x_2} g_2 + \frac{1}{2} \left(F_{x_1 x_1} \sigma_1^2 + F_{x_2 x_2} \sigma_2^2 + 2 F_{x_1 x_2} \sigma_1 \sigma_2 \rho \right) \right) dt \\ &\quad + F_{x_1} \sigma_1 dz_1 + F_{x_2} \sigma_2 dz_2. \end{aligned}$$

$y = F(t,\ x)$ **and** $dx = g(t,x)dt + \phi_1(t,x)dz_1 + \phi_2(t,x)dz_2$ The differential of y is

$$\begin{aligned}
dy &= F_t dt + F_x dx + \frac{1}{2}F_{tt}(dt)^2 + F_{tx}dtdx + \frac{1}{2}F_{xx}(dx)^2 \\
&= F_t dt + F_x dx + \frac{1}{2}F_{xx}(dx)^2 \\
&= F_t dt + F_x\left(gdt + \phi_1 dz_1 + \phi_2 dz_2\right) + \frac{1}{2}F_{xx}\left(gdt + \phi_1 dz_1 + \phi_2 dz_2\right)^2 \\
&= F_t dt + F_x\left(gdt + \phi_1 dz_1 + \phi_2 dz_2\right) + \frac{1}{2}F_{xx}\left(\phi_1^2 dt + \phi_2^2 dt + 2\phi_1\phi_2\rho dt\right) \\
&= \left(F_t + F_x g + \frac{1}{2}F_{xx}\left(\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2\rho\right)\right)dt + F_x\phi_1 dz_1 + F_x\phi_2 dz_2.
\end{aligned}$$

2.1.3 Solving stochastic differential equations

There are few stochastic differential equations having explicit solutions. Here is one. Usually we use a stochastic differential equation,

$$\frac{dp(t)}{p(t)} = \alpha dt + \sigma dz(t),$$

to model the stock price $p(t)$. This equation has an explicit solution,

$$p(t) = \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma z(t)\right).$$

We can use Itô's formula to verify it,

$$\begin{aligned}
dp(t) &= p(t)\left(\alpha - \frac{\sigma^2}{2}\right)dt + p(t)\sigma dz(t) + \frac{1}{2}p(t)\sigma^2 dt \\
&= p(t)\alpha dt + p(t)\sigma dz(t).
\end{aligned}$$

We call $p(t)$ a geometric Brownian motion.

2.2 Stochastic dynamic optimization

We now start solving a continuous-time stochastic problem.

The optimal value function is

$$J(t_0, x_0) = \max\ E_{t_0}\int_{t_0}^{T} f(t,x,u)dt + \phi(x(T),T)$$

$$s.t.\ \ dx = g(t,x,u)dt + \sigma(t,x,u)dz.$$

Now let's write the value function recursively,

$$\begin{aligned}
J(t,x) &= \max_u \ E_t \int_t^T f(s,x,u)ds + \phi(x(T),T) \\
&= \max_u \ E_t \left(\int_t^{t+\Delta t} f(s,x,u)ds + \int_{t+\Delta t}^T f(s,x,u)ds + \phi(x(T),T) \right) \\
&= \max_u \ E_t \left(\int_t^{t+\Delta t} f(s,x,u)ds + \max \ \left(E_{t+\Delta t} \int_{t+\Delta t}^T f(s,x,u)ds + \phi(x(T),T) \right) \right) \\
&= \max_u \ E_t \left(\int_t^{t+\Delta t} f(s,x,u)ds + J(t+\Delta t, x+\Delta x) \right) \\
&= \max_u \ E_t \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J(t+\Delta t, x+\Delta x) \right) \\
&= \max_u \ E_t \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J(t,x) + J_t \Delta t + J_x \Delta x + \frac{1}{2} J_{xx} (\Delta x)^2 \right) \\
&= \max_u \ E_t \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J(t,x) + J_t \Delta t + J_x g \Delta t + J_x \sigma \Delta z + \frac{1}{2} J_{xx} \sigma^2 \Delta t \right).
\end{aligned}$$

Thus we have

$$0 = \max_u \ E_t \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J_t \Delta t + J_x g \Delta t + J_x \sigma \Delta z + \frac{1}{2} J_{xx} \sigma^2 \Delta t \right).$$

First take expectation operator. Note that $E_t \triangle z = 0$. Then dividing Δt on both sides and letting $\Delta t \to 0$, we have

$$0 = \max_u \ \left(f(t,x,u) + J_t + J_x g + \frac{1}{2} J_{xx} \sigma^2 \right).$$

This equation is called the HJB equation. Using first order condition, we can solve optimal u^* as a function of J_x and J_{xx}. Then plugging the expression of optimal u^* into the HJB equation, we can obtain a partial differential equation of $J(t,x)$,

$$-J_t = \ f(t,x,u^*) + J_x g(t,x,u^*) + \frac{1}{2} J_{xx} \left(\sigma(t,x,u^*) \right)^2.$$

This partial differential equation plus its boundary condition,

$$J(T, x(T)) = \phi(x(T), T),$$

should have a solution of $J(t,x)$. For general functions of $f(t,x,u)$ and $\phi(x(T),T)$, we could use a computer to solve this partial differential equation. But there are some cases in which we could obtain closed-form solution.

For an infinite horizon problem with $f(t,x,u) = e^{-\beta t} h(x,u)$, we assume $J(t,x) = e^{-\beta t} V(x)$. The HJB equation becomes

$$0 = \max_u \ \left(e^{-\beta t} h(x,u) - \beta e^{-\beta t} V(x) + e^{-\beta t} V'(x) g + \frac{1}{2} e^{-\beta t} V''(x) \sigma^2 \right).$$

Thus we have

$$\beta V(x) = \max_{u} \left(h(x,u) + V'(x)g + \frac{1}{2}V''(x)\sigma^2 \right).$$

3 The Merton model

We solve a Merton's portfolio problem by using stochastic dynamic optimization method.

There are two assets in the economy. One asset is a risk-free bond. Its value $Q(t)$ follows

$$dQ(t) = rQ(t)dt.$$

Thus the risk-free bond has rate of return, r. The other asset is a risky stock. Its value $S(t)$ follows

$$dS(t) = \alpha S(t)dt + \sigma S(t)dz(t).$$

Thus the risky stock has mean return α and standard deviation of return σ. We assume $\alpha > r$.

The agent has wealth $w(t)$ and invests wealth into these two assets. The agent consumes $c(t)$ at time t and invests a fraction $\omega(t)$ of wealth into risky asset. Thus the agent's wealth movement equation is

$$\begin{aligned} dw(t) &= r(1-\omega(t))w(t)dt + \alpha\omega(t)w(t)dt + \sigma\omega(t)w(t)dz(t) - c(t)dt \\ &= \left(r(1-\omega(t))w(t) + \alpha\omega(t)w(t) - c(t)\right)dt + \sigma\omega(t)w(t)dz(t) \\ &= \left(rw(t) + (\alpha - r)\omega(t)w(t) - c(t)\right)dt + \sigma\omega(t)w(t)dz(t). \end{aligned}$$

The agent chooses optimal consumption and investment rules, $c(t)$ and $\omega(t)$ to maximize his utility,

$$\max_{c(t),\omega(t)} E_0 \int_0^\infty e^{-\beta t}\frac{(c(t))^{1-\gamma}}{1-\gamma}dt$$

$$s.t.\ \ dw(t) = \left(rw(t) + (\alpha - r)\omega(t)w(t) - c(t)\right)dt + \sigma\omega(t)w(t)dz(t).$$

Let

$$V(w(t)) = \max_{c(s),\omega(s)} E_t \int_t^\infty e^{-\beta(s-t)}\frac{(c(s))^{1-\gamma}}{1-\gamma}ds.$$

The HJB is

$$\beta V(w) = \max_{c(t),\omega(t)} \left(\begin{array}{c} \frac{(c(t))^{1-\gamma}}{1-\gamma} + V'(w)\left(rw(t) + (\alpha - r)\omega(t)w(t) - c(t)\right) \\ +\frac{1}{2}V''(w)\sigma^2\omega^2(t)w^2(t) \end{array} \right). \quad (6)$$

The first-order conditions are

$$(c(t))^{-\gamma} - V'(w) = 0,$$

and

$$V'(w)(\alpha - r)w(t) + V''(w)\sigma^2\omega(t)w^2(t) = 0.$$

Thus we have

$$c(t) = (V'(w))^{-\frac{1}{\gamma}}, \tag{7}$$

and

$$\omega(t) = -\frac{V'(w)(\alpha - r)}{V''(w)\sigma^2 w(t)}. \tag{8}$$

Guess that

$$V(w) = \frac{A}{1-\gamma}(w(t))^{1-\gamma}. \tag{9}$$

Thus we have

$$V'(w) = A(w(t))^{-\gamma}, \tag{10}$$

and

$$V''(w) = -\gamma A(w(t))^{-\gamma-1}. \tag{11}$$

Plugging equations (10) and (11) into equations (7) and (8), we have

$$c(t) = A^{-\frac{1}{\gamma}}w(t), \tag{12}$$

and

$$\omega(t) = \frac{\alpha - r}{\gamma\sigma^2}. \tag{13}$$

The last step is to determine A. Plugging equations (9), (10), (11), (12) and (13) into equation (6), we have

$$\frac{\beta}{1-\gamma} = \frac{\gamma}{1-\gamma}A^{-\frac{1}{\gamma}} + r + \frac{(\alpha - r)^2}{2\gamma\sigma^2}.$$

Thus

$$A = \left(\frac{\beta - (1-\gamma)\left(r + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)}{\gamma}\right)^{-\gamma}.$$

We are done.

Here are some comments for equation (13). This optimal investment rule is intuitive economically. The higher the risk premium, $\alpha - r$, the more fraction the agent invests in risky asset. The higher risk aversion, γ, the lower $\omega(t)$. The higher the volatility of risky asset, the lower $\omega(t)$.

Note that the wealth accumulation process under optimal policies is a geometric Brownian motion,

$$dw(t) = \left(r + \frac{(\alpha - r)^2}{\gamma\sigma^2} - A^{-\frac{1}{\gamma}}\right)w(t)dt + \frac{\alpha - r}{\gamma\sigma}w(t)dz(t).$$

The transversality condition for this problem is

$$\lim_{t\to\infty} e^{-\beta t}V(w(t)) = 0.$$

4 The Richard model

Now the agent has uncertain lifetime and there is a complete life insurance market to cover this uncertainty. The lifetime of the agent follows an exponential distribution, $\pi(t) = pe^{-pt}$. In a short period of time $\triangle t$, the death probability is $p\triangle t$. The price of life insurance is μ. At time t agent pays $P(t)$ to life insurance company. If the agent dies the life insurance company pays $\frac{P(t)}{\mu}$ to the agent. $P(t)$ could be negative. Then we call this life insurance annuity. Essentially the agent receives payment $-P(t)$ if the agent is alive. If the agent dies, the insurance (annuity) company collects $-\frac{P(t)}{\mu}$ from the agent. In the short period of time $\triangle t$. The insurance company receives $P(t)\triangle t$ and pays $p\frac{P(t)}{\mu}\triangle t$. If we assume that the life insurance company earns zero profit, then

$$P(t)\triangle t = p\frac{P(t)}{\mu}\triangle t.$$

This implies

$$\mu = p.$$

Now the agent can invest wealth in a risk-free asset and a risky asset. The agent can also buy life insurance. The agent draws utility from consumption and leaving bequest, $Z(t)$, to his child. We have

$$Z(t) = w(t) + \frac{P(t)}{p},$$

if the agent dies at time t. The agent's problem is

$$\max_{c(s),\omega(s),P(s)} E_0 \int_0^\infty pe^{-pt}\left(\int_0^t e^{-\beta s}\frac{(c(s))^{1-\gamma}}{1-\gamma}ds + e^{-\beta t}\chi\frac{(Z(t))^{1-\gamma}}{1-\gamma}\right)dt$$

$$\begin{aligned} s.t.\ \ dw(s) &= (rw(s) + (\alpha - r)\omega(s)w(s) - P(s) - c(s))\,ds + \sigma\omega(s)w(s)dz(s) \\ Z(t) &= w(t) + \frac{P(t)}{p}, \end{aligned}$$

where β is a time discount rate and χ governs the bequest motive.

By integration by parts, we rewrite the agent's problem,

$$\max_{c(t),\omega(t),P(t)} E_0 \int_0^\infty e^{-(p+\beta)t}\left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi\frac{(Z(t))^{1-\gamma}}{1-\gamma}\right)dt$$

$$\begin{aligned} s.t.\ \ dw(t) &= (rw(t) + (\alpha - r)\omega(t)w(t) - P(t) - c(t))\,dt + \sigma\omega(t)w(t)dz(t) \\ Z(t) &= w(t) + \frac{P(t)}{p}. \end{aligned}$$

Let

$$V(w(v)) = \max_{c(t),\omega(t),P(t)} E_v \int_v^\infty e^{-(p+\beta)(t-v)} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt.$$

Now let's derive HJB

$$\begin{aligned}
& V(w(v)) \\
= & \max_{c(t),\omega(t),P(t)} E_v \int_v^\infty e^{-(p+\beta)(t-v)} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \\
= & \max_{c(t),\omega(t),P(t)} E_v \left(\begin{array}{c} \int_v^{v+\triangle v} e^{-(p+\beta)(t-v)} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \\ + e^{-(p+\beta)\triangle v} \int_{v+\triangle v}^\infty e^{-(p+\beta)(t-(v+\triangle v))} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \end{array} \right) \\
= & \max_{c(t),\omega(t),P(t)} E_v \left(\begin{array}{c} \int_v^{v+\triangle v} e^{-(p+\beta)(t-v)} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \\ + E_{v+\triangle v} \left(e^{-(p+\beta)\triangle v} \int_{v+\triangle v}^\infty e^{-(p+\beta)(t-(v+\triangle v))} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \right) \end{array} \right) \\
= & \max_{c(t),\omega(t),P(t)} E_v \left(\begin{array}{c} \int_v^{v+\triangle v} e^{-(p+\beta)(t-v)} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \\ + e^{-(p+\beta)\triangle v} E_{v+\triangle v} \left(\int_{v+\triangle v}^\infty e^{-(p+\beta)(t-(v+\triangle v))} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \right) \end{array} \right) \\
= & \max_{c(t),\omega(t),P(t)} E_v \left(\begin{array}{c} \int_v^{v+\triangle v} e^{-(p+\beta)(t-v)} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \\ + \max \left(e^{-(p+\beta)\triangle v} E_{v+\triangle v} \left(\int_{v+\triangle v}^\infty e^{-(p+\beta)(t-(v+\triangle v))} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \right) \right) \end{array} \right) \\
= & \max_{c(t),\omega(t),P(t)} E_v \left(\begin{array}{c} \int_v^{v+\triangle v} e^{-(p+\beta)(t-v)} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \\ + e^{-(p+\beta)\triangle v} \max \left(E_{v+\triangle v} \left(\int_{v+\triangle v}^\infty e^{-(p+\beta)(t-(v+\triangle v))} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt \right) \right) \end{array} \right) \\
= & \max_{c(t),\omega(t),P(t)} E_v \left(\int_v^{v+\triangle v} e^{-(p+\beta)(t-v)} \left(\frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(t))^{1-\gamma}}{1-\gamma} \right) dt + e^{-(p+\beta)\triangle v} V(w(v+\triangle v)) \right) \\
= & \max_{c(v),\omega(v),P(v)} E_v \left(\begin{array}{c} \left(\frac{(c(v))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(v))^{1-\gamma}}{1-\gamma} \right) \triangle v \\ + (1-(p+\beta)\triangle v) \left(V(w(v)) + V'(w(v))\triangle w + \frac{1}{2} V''(w(v)) (\triangle w)^2 \right) \end{array} \right).
\end{aligned}$$

We then use Itô's formula

$$\begin{aligned}
& V(w) \\
= & \max E_v \left(\begin{array}{c} \left(\frac{(c(v))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(v))^{1-\gamma}}{1-\gamma} \right) \triangle v \\ + (1-(p+\beta)\triangle v) \left(\begin{array}{c} V(w(v)) + V'(w(v))\triangle w \\ + \frac{1}{2} V''(w(v)) (\triangle w)^2 \end{array} \right) \end{array} \right) \\
= & \max E_v \left(\begin{array}{c} \left(\frac{(c(v))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(v))^{1-\gamma}}{1-\gamma} \right) \triangle v \\ + (1-(p+\beta)\triangle v) \left(\begin{array}{c} V(w(v)) + V'(w(v))\triangle w \\ + \frac{1}{2} V''(w(v)) \sigma^2 \omega^2(v) w^2(v) \triangle v \end{array} \right) \end{array} \right).
\end{aligned}$$

Note that $E_v \triangle z = 0$. Taking expectation operator, we have

$$
\begin{aligned}
& V(w) \\
= \quad & \max \left(\begin{array}{c} \left(\frac{(c(v))^{1-\gamma}}{1-\gamma} + p\chi\frac{(Z(v))^{1-\gamma}}{1-\gamma}\right)\triangle v \\ +(1-(p+\beta)\triangle v) \left(\begin{array}{c} V(w(v)) \\ +V'(w(v))\left(rw(v) + (\alpha - r)\omega(v)w(v) - P(v) - c(v)\right)\triangle v \\ +\frac{1}{2}V''(w(v))\sigma^2\omega^2(v)w^2(v)\triangle v \end{array} \right) \end{array} \right).
\end{aligned}
$$

Dividing by $\triangle v$ on both sides and letting $\triangle v \to 0$, we have

$$
(p+\beta)V(w) = \max \left(\begin{array}{c} \frac{(c(v))^{1-\gamma}}{1-\gamma} + p\chi\frac{(Z(v))^{1-\gamma}}{1-\gamma} \\ +V'(w)\left(rw(v) + (\alpha - r)\omega(v)w(v) - P(v) - c(v)\right) \\ +\frac{1}{2}V''(w)\sigma^2\omega^2(v)w^2(v) \end{array} \right).
$$

This is equivalent to

$$
(p+\beta)V(w) = \max \left(\begin{array}{c} \frac{(c(t))^{1-\gamma}}{1-\gamma} + p\chi\frac{(Z(t))^{1-\gamma}}{1-\gamma} \\ +V'(w)\left(rw(t) + (\alpha - r)\omega(t)w(t) - P(t) - c(t)\right) \\ +\frac{1}{2}V''(w)\sigma^2\omega^2(t)w^2(t) \end{array} \right).
$$

Using the relationship,

$$
Z(t) = w(t) + \frac{P(t)}{p},
$$

we have the first-order conditions,

$$
(c(t))^{-\gamma} - V'(w) = 0,
$$

$$
\chi\left(Z(t)\right)^{-\gamma} - V'(w) = 0,
$$

and

$$
V'(w)(\alpha - r)w(t) + V''(w)\sigma^2\omega(t)w^2(t) = 0.
$$

Thus we have

$$
c(t) = \left(V'(w)\right)^{-\frac{1}{\gamma}},
$$

$$
Z(t) = \chi^{\frac{1}{\gamma}}\left(V'(w)\right)^{-\frac{1}{\gamma}},
$$

and

$$
\omega(t) = -\frac{V'(w)(\alpha - r)}{V''(w)\sigma^2 w(t)}.
$$

Guess that

$$
V(w) = \frac{A}{1-\gamma}w^{1-\gamma}.
$$

Thus we have

$$
c(t) = A^{-\frac{1}{\gamma}}w(t),
$$

$$
Z(t) = \left(\frac{\chi}{A}\right)^{\frac{1}{\gamma}}w(t),
$$

and

$$\omega(t) = \frac{\alpha - r}{\gamma \sigma^2}.$$

From HJB we can determine A,

$$A = \left(\frac{p + \beta - (1-\gamma)\left(r + p + \frac{(\alpha - r)^2}{2\gamma\sigma^2}\right)}{\gamma\left(1 + p\chi^{\frac{1}{\gamma}}\right)} \right)^{-\gamma}.$$

5 Dynamic optimization with a jump process

We want to solve an optimization problem

$$\max \ E_0 \int_0^T f(t,x,u)dt + \phi(x(T),T)$$

$$s.t. \ \ dx = g(t,x,u)dt + \sigma(t,x,u)dz + m(t,x,u,\omega)dq, \quad x(0) = \bar{x},$$

where $z(t)$ is a standard Brownian motion and $q(t)$ is a Poisson process. The term of $m(t,x,u,\omega)dq$ means that when a jump happens, x changes from $x(t)$ to $x(t) + m(t,x,u,\omega)$, i.e. the jump part dominates the diffusion part. If no jump happens, x follows $dx = g(t,x,u)dt + \sigma(t,x,u)dz$. Here the jump size $m(t,x,u,\omega)$ is a random variable. I use ω to represent a sample path.

We usually use a Poisson process to represent a jump process.

5.1 Preliminary knowledge of a Poisson process

To state the definition of a Poisson process we use the definition of a counting process. I copy the following two concepts from Ross (1983)'s textbook, *Stochastic Processes.*

Definition 2 *A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of 'events' that have occurred up to time t. Hence, a counting process $N(t)$ must satisfy:*

(i) $N(t) \geq 0$.

(ii) $N(t)$ is integer valued.

(iii) If $s < t$, then $N(s) \leq N(t)$.

(iv) For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval $(s, t]$.

Definition 3 *The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if:*

(i) $N(0) = 0$.

(ii) The process has independent increments.

(iii) The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$,

$$\Pr\{N(t+s) - N(t) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2 \cdots.$$

I find that the materials of Poisson process on Wikipedia are helpful. I copy the following materials from website: http://en.wikipedia.org/wiki/Poisson_process

A Poisson process, named after the French mathematician Siméon-Denis Poisson (1781–1840), is a stochastic process in which events occur continuously and independently of one another (the word event used here is not an instance of the concept of event frequently used in probability theory). Examples that are well-modeled as Poisson processes include the radioactive decay of atoms, telephone calls arriving at a switchboard, page view requests to a website, and rainfall.

The Poisson process is a collection $\{N(t) : t \geq 0\}$ *of random variables, where* $N(t)$ *is the number of events that have occurred up to time* t *(starting from time* 0*). The number of events between time* a *and time* b *is given as* $N(b) - N(a)$ *and has a Poisson distribution. Each realization of the process* $\{N(t)\}$ *is a non-negative integer-valued step function that is non-decreasing, but for intuitive purposes it is usually easier to think of it as a point pattern on* $[0, \infty)$ *(the points in time where the step function jumps, i.e. the points in time where an event occurs).*

5.2 Dynamic optimization

To solve the dynamic optimization problem, we could use Itô's formula for stochastic process with jumps. But now we will the recursive structure of the stochastic process with jumps directly to derive HJB. The optimal value function is

$$J(t_0, x_0) = \max\ E_{t_0} \int_{t_0}^{T} f(t, x, u) dt + \phi(x(T), T)$$

$$s.t.\ \ dx = g(t, x, u)dt + \sigma(t, x, u)dz + m(t, x, u, \omega)dq.$$

Let's derive HJB

$$\begin{aligned}
J(t, x) &= \max_u\ E_t \int_t^T f(s, x, u) ds + \phi(x(T), T) \\
&= \max_u\ E_t \left(\int_t^{t+\Delta t} f(s, x, u) ds + \int_{t+\Delta t}^T f(s, x, u) ds + \phi(x(T), T) \right) \\
&= \max_u\ E_t \left(\int_t^{t+\Delta t} f(s, x, u) ds + \max\ \left(E_{t+\Delta t} \int_{t+\Delta t}^T f(s, x, u) ds + \phi(x(T), T) \right) \right) \\
&= \max_u\ E_t \left(\int_t^{t+\Delta t} f(s, x, u) ds + J(t + \Delta t, x + \Delta x) \right) \\
&= \max_u\ E_t \left(f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + J(t + \Delta t, x + \Delta x) \right) \\
&= \max_u\ E_t \left(\begin{array}{c} f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + \lambda \triangle t J(t + \triangle t, x + m) \\ + (1 - \lambda \triangle t) \left(J(t, x) + J_t \Delta t + J_x \Delta x + \frac{1}{2} J_{xx} (\Delta x)^2 \right) \end{array} \right) \\
&= \max_u\ E_t \left(\begin{array}{c} f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + \lambda J(t + \triangle t, x + m) \triangle t \\ + (1 - \lambda \triangle t) \left(J(t, x) + J_t \Delta t + J_x g \Delta t + J_x \sigma \Delta z + \frac{1}{2} J_{xx} \sigma^2 \Delta t \right) \end{array} \right).
\end{aligned}$$

Note that $E_t \triangle z = 0$. Taking expectation operator, we have

$$J(t,x) = \max_u \left(\begin{array}{c} f(\tilde{t}, x(\tilde{t}), u(\tilde{t}))\Delta t + \lambda E_t J(t + \triangle t, x + m) \triangle t \\ + (1 - \lambda \triangle t) \left(J(t,x) + J_t \Delta t + J_x g \Delta t + \frac{1}{2} J_{xx} \sigma^2 \Delta t \right) \end{array} \right).$$

We have the term of $E_t J(t + \triangle t, x + m)$ in the equation, because $m(t, x, u, \omega)$ is a random variable.

Dividing by $\triangle t$ on both sides and letting $\triangle t \to 0$, we have

$$\lambda J - J_t = \max_u \left(f(t,x,u) + \lambda E_t J(t, x+m) + J_x g + \frac{1}{2} J_{xx} \sigma^2 \right).$$

In Section 5.3 we will discuss an example of investment return with a jump process. You can find another example in Wang (2007, JME).

5.3 Investment returns with a jump process

Now the agent can access a risky asset: a stock with possibility to receive dividend. The value of the stock follows

$$dS(t) = \alpha S(t) dt + \sigma S(t) dz(t) + \phi S(t) dq(t).$$

The jump size ϕ could be a random variable. Here for simplicity, I assume ϕ is a constant. When a jump happens, the value of stock changes from $S(t)$ to $(1 + \phi) S(t)$.

The agent chooses optimal consumption and investment rules, $c(t)$ and $\omega(t)$ to maximize his utility,

$$\max_{c(t), \omega(t)} E_0 \int_0^\infty e^{-\beta t} \frac{(c(t))^{1-\gamma}}{1-\gamma} dt$$

$$s.t.\ \ dw(t) = (rw(t) + (\alpha - r)\omega(t) w(t) - c(t))\, dt + \sigma \omega(t) w(t) dz(t) + \phi \omega(t) w(t) dq(t).$$

Let

$$V(w(t)) = \max_{c(s), \omega(s)} E_t \int_t^\infty e^{-\beta(s-t)} \frac{(c(s))^{1-\gamma}}{1-\gamma} ds.$$

The HJB is

$$(\lambda + \beta) V(w) = \max_{c(t), \omega(t)} \left(\begin{array}{c} \frac{(c(t))^{1-\gamma}}{1-\gamma} + \lambda V((1 + \phi\omega(t)) w(t)) \\ + V'(w) \left(rw(t) + (\alpha - r) \omega(t) w(t) - c(t) \right) \\ + \frac{1}{2} V''(w) \sigma^2 \omega^2(t) w^2(t) \end{array} \right).$$

We have the first-order conditions,

$$(c(t))^{-\gamma} = V'(w),$$

and

$$\lambda V'((1 + \phi \omega(t)) w(t)) \phi w(t) + V'(w)(\alpha - r) w(t) + V''(w) \sigma^2 \omega(t) w^2(t) = 0.$$

Guess that

$$V(w(t)) = \frac{A}{1-\gamma}(w(t))^{1-\gamma}.$$

Thus we have

$$c(t) = A^{-\frac{1}{\gamma}} w(t),$$

and

$$\omega(t) = \bar{\omega},$$

where $\bar{\omega}$ solves

$$\gamma\sigma^2\bar{\omega} - \lambda\phi(1+\phi\bar{\omega})^{-\gamma} - (\alpha - r) = 0.$$

From HJB we find

$$A = \left(\frac{\lambda + \beta - \lambda(1+\phi\bar{\omega})^{1-\gamma} - (1-\gamma)\left(r + (\alpha - r)\bar{\omega} - \frac{1}{2}\gamma\sigma^2\bar{\omega}^2\right)}{\gamma}\right)^{-\gamma}.$$

6 The optimal stopping problem

The problem we want to solve is

$$\max_{u(t),T} E_0 \int_0^T f(t,x,u)dt + \phi(x(T),T)$$

$$s.t.\ \ dx = g(t,x,u)dt + \sigma(t,x,u)dz, \quad x(0) = \bar{x}.$$

Before the stopping time T, the optimal value function is

$$J(t_0, x(t_0)) = \max_{u(t),T} E_{t_0} \int_{t_0}^T f(t,x,u)dt + \phi(x(T),T)$$

$$s.t.\ \ dx = g(t,x,u)dt + \sigma(t,x,u)dz.$$

At the stopping time T, we have the "value-matching condition,"

$$J(T, x(T)) = \phi(x(T),T). \tag{14}$$

Here the stopping time T is a random variable since $x(t)$ is a stochastic process.

Before the stopping time T, repeating the procedure in section 2.2, we should have the same HJB

$$-J_t = \max_u \left(f(t,x,u) + J_x g + \frac{1}{2}J_{xx}\sigma^2\right). \tag{15}$$

Using first order condition, we can solve optimal u^* as a function of J_x and J_{xx}. Then plugging the expression of optimal u^* into the HJB equation (15), we can obtain a partial differential equation of $J(t,x)$,

$$-J_t = f(t,x,u^*) + J_x g(t,x,u^*) + \frac{1}{2}J_{xx}\left(\sigma(t,x,u^*)\right)^2. \tag{16}$$

Partial differential equation (16) plus its boundary condition of equation (14) should have a solution of $J(t,x)$. This is the same as that of the standard problem. The extra problem here is that the HJB equation (15) only holds before $x(t)$ hits $x(T)$. But we do not know T. Thus the boundary of the region, given by the curve $x(T)$ is called a "free boundary". The problem to solve this partial differential equation and its valid region is called a free-boundary problem.

The extra boundary condition, which is called "smooth-pasting condition" helps us to determine T

$$J_x(T,x(T)) = \phi_x(x(T),T).$$

6.1 The real option method of investment

Now we learn the technique to solving an optimal stopping problem. We then can solve a class of investment problems, relating to American options. We should interpret this American option generally and use the technique flexibly. We interpret any once-in-life optimal investment timing decision as an American option problem. And we should not view $\phi(x(T),T)$ as a bequest function in the narrow sense. We should view $\phi(x(T),T)$ as an exit payoff, which sometimes is a value function derived from another dynamic programming problem.

I copy an example from Miao and Wang (2007). Let's obtain some sense of using the "smooth-pasting condition" in a skillful way.

The agent has an investment option of an risky asset. The cost of investment is $I > 0$. The value of the risky asset follows a Brownian motion

$$dx(t) = \alpha dt + \sigma dz(t),\ x(0) \text{ given.}$$

The agent can choose the investment time τ. At time τ the agent pays fixed investment cost I to obtain the risky asset and then he sells the risky asset at price $x(\tau)$. The agent's profit of this investment is $x(\tau) - I$. There is no flow income from this investment project afterwards. The agent can also borrow or save money in a risk-free account with interest rate r.

The agent choose consumption path $c(t)$ and investment time τ to maximize his utility. The agent's problem is

$$\max_{c(t),\tau} E_0 \int_0^\infty e^{-\beta t} U(c(t)) dt$$

$$\begin{aligned} s.t.\ dw(t) &= (rw(t) - c(t))dt \\ dx(t) &= \alpha dt + \sigma dz(t). \end{aligned}$$

For simplicity we assume that $\beta = r$. But this assumption is not crucial for the economic interpretation and the mathematical derivation.

We solve this problem backwards.

After the agent exercises the investment option, the agent's problem is a deterministic dynamic problem. Let $V^0(w(t))$ be the agent's value function after exercising the investment option. We have

$$V^0(w(t)) = \max_{c(s)} \; E_t \int_t^{\infty} e^{-\beta(s-t)} U(c(s)) ds.$$

The HJB after investment is

$$rV^0(w(t)) = \max_{c(t)} \; \left(U(c(t)) + (rw(t) - c(t)) V_w^0(w(t)) \right).$$

Using a guess-and-verify procedure, we find that

$$V^0(w(t)) = \frac{U(rw(t))}{r}.$$

This particular solution is due to $\beta = r$.

Now let's solve the agent's problem before exercising the investment option. Let $V(w(t), x(t))$ be the agent's value function before investment. We have

$$V(w(t), x(t)) = \max_{c(s),\tau} \; E_t \int_t^{\infty} e^{-\beta(s-t)} U(c(s)) ds.$$

Thus

$$\begin{aligned}
& V(w(t), x(t)) \\
= & \max_{c(s),\tau \geq t} \; E_t \int_t^{\infty} e^{-\beta(s-t)} U(c(s)) ds \\
= & \max \left(\begin{array}{c} \max_{c(s)} \; E_t \int_t^{\infty} e^{-\beta(s-t)} U(c(s)) ds, \\ \max_{c(t)} \; E_t \left(\begin{array}{c} \int_t^{t+\triangle t} e^{-\beta(s-t)} U(c(s)) ds \\ +e^{-\beta \triangle t} \max_{c(s),\tau \geq t+\triangle t} \; E_{t+\triangle t} \int_{t+\triangle t}^{\infty} e^{-\beta(s-(t+\triangle t))} U(c(s)) ds \end{array} \right) \end{array} \right) \\
= & \max \left(V^0(w(t) + x(t) - I), \; \max_{c(t)} \; E_t \left(U(c(t)) \triangle t + (1 - \beta \triangle t) V(w(t + \triangle t), x(t + \triangle t)) \right) \right).
\end{aligned}$$

If $\tau = t$, i.e. the agent chooses to undertake investment at time t, we have

$$V(w(t), x(t)) = V^0(w(t) + x(t) - I).$$

This is the 'value-matching condition'. This condition defines an investment boundary $x = \bar{x}(w)$. When x hits this boundary, the agent exercises the investment option.

If $\tau > t$, i.e. the agent does not undertake the investment at time t, we have

$$V(w(t), x(t)) = \max_{c(t)} \; E_t \left(U(c(t)) \triangle t + (1 - \beta \triangle t) V(w(t + \triangle t), x(t + \triangle t)) \right).$$

Using the assumption $\beta = r$, we have HJB

$$rV(w(t), x(t)) = \max_{c(t)} \; \left(\begin{array}{c} U(c(t)) + (rw(t) - c(t)) V_w(w(t), x(t)) \\ +\alpha V_x(w(t), x(t)) + \frac{\sigma^2}{2} V_{xx}(w(t), x(t)) \end{array} \right).$$

Now we impose two 'smooth pasting conditions,'

$$\left.\frac{\partial V(w,x)}{\partial w}\right|_{x=\bar{x}(w)} = \left.\frac{\partial V^0(w+x-I)}{\partial w}\right|_{x=\bar{x}(w)},$$

and

$$\left.\frac{\partial V(w,x)}{\partial x}\right|_{x=\bar{x}(w)} = \left.\frac{\partial V^0(w+x-I)}{\partial x}\right|_{x=\bar{x}(w)}.$$

Now we suppose that

$$U(c(t)) = -\frac{1}{\gamma}e^{-\gamma c(t)}.$$

Thus we know that

$$V^0(w) = -\frac{1}{\gamma r}e^{-\gamma r w}.$$

Now we solve the HJB

$$rV(w,x) = \max_c \begin{pmatrix} -\frac{1}{\gamma}e^{-\gamma c} + (rw-c)V_w(w,x) \\ +\alpha V_x(w,x) + \frac{\sigma^2}{2}V_{xx}(w,x) \end{pmatrix}.$$

F.O.C.

$$e^{-\gamma c} = V_w(w,x).$$

Guess that

$$V(w,x) = -\frac{1}{B}e^{-B(w+G(x))}.$$

Thus we have

$$c = \frac{B}{\gamma}\left(w+G(x)\right).$$

From HJB we have

$$B = \gamma r,$$

and

$$rG(x) = \alpha G'(x) + \frac{\sigma^2}{2}\left(G''(x) - \gamma r\left(G'(x)\right)^2\right).$$

From the value-matching condition we have

$$G(\bar{x}) = \bar{x} - I.$$

Then the smooth pasting condition,

$$\left.\frac{\partial V(w,x)}{\partial w}\right|_{x=\bar{x}(w)} = \left.\frac{\partial V^0(w+x-I)}{\partial w}\right|_{x=\bar{x}(w)},$$

holds.

From the smooth pasting condition,

$$\left.\frac{\partial V(w,x)}{\partial x}\right|_{x=\bar{x}(w)} = \left.\frac{\partial V^0(w+x-I)}{\partial x}\right|_{x=\bar{x}(w)},$$

we have

$$G'(\bar{x}) = 1.$$

The last condition is the no-bubble condition,

$$\lim_{x \to -\infty} G(x) = 0.$$

6.2 Some comments on the optimal stopping problem

The last example gives us some sense of using dynamic programming to solve a finance problem. But this optimal stopping technique has a broad scope of applications. The optimal stopping problem in an uncertain environment is especially useful when we discuss the impacts of risk on timing decisions. We can apply this technique to household decision problems, for example, endogenous retirement problems, education timing problems, marriage timing problems, the time to giving birth to a child, etc. Note that the timing choice is a random variable. It depends on the different realizations of shocks. Thus in a cross section there is a distribution of this timing choice. This is useful when you try to match some distributions in data.